will cause an error, since it is more sensitive to the density than the two-body force. From simple estimates it seems that calculations of this kind should be corrected for the three-body force by reducing the binding by 1 or 2 MeV. At the moment this number is of the same order as both the experimental and theorteical uncertainties in the binding of a Λ hyperon in nuclear matter, and does not account for the apparent disagreement between theory and experiment. Reasons for introducing suppression of the odd-state Λ-N force therefore still remain.

The main part of the three-body Λ-nucleon force was due to the coupling to the Σ hyperon, and would therefore not be present in the nuclear case.

ACKNOWLEDGMENTS

The author wishes to thank Professor G. E. Brown and Professor J. D. Walecka for many stimulating discussions. The hospitality and financial support of Princeton University, Princeton, New Jersey, and NORDITA, Copenhagen, Denmark is gratefully acknowledged.

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Comparison of Moderate-Energy Proton-Proton Models. III*

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(Received 20 February 1967)

The predictions of 12 proton-proton models and phase-shift representations are compared to a selected but comprehensive set of 9–330-MeV scattering data. The best fit was found to be produced by a quadratic interpolation of Arndt and MacGregor's phase-shift table, with a ratio of χ^2 to its expected value of 1.4. The best potential is that of Hamada and Johnston, with a ratio of 3.1. The ratio for the Tabakin potential is 28.

I. INTRODUCTION

IN this paper we bring up to date the comparison[1–3] of published proton-proton models and energy-dependent phase-shift analyses with data that, in our opinion, represent the most accurate experimental information currently available on proton-proton scattering between 9 and 330 MeV. The models we consider were constructed for a variety of purposes and were fitted to various other selections of the data, so that a simple χ^2 listing does not necessarily serve as a figure of merit as to how well the original authors' purposes were served. However, these models are often used for other purposes, accompanied by some statement to the effect that "this model agrees with existing scattering data." To the best of our knowledge, this is not true for existing models according to the usually accepted statistical criteria[4]; on the other hand the discrepancies may be irrelevant for some applications. This point obviously should be investigated in each case. For example, a small adjustment of the parameters might improve the fit to the data without affecting the calculation at hand; but it also might be extremely significant. In other cases, the model does provide a good fit over some energy ranges, but might be applied in an energy range where it is in violent disagreement with the data. Clearly this point should also always be investigated, and one of our purposes is to make this more obvious. Another is to give some indication of where the best existing models should be corrected. A third is to point up the fact that many popular models which are often

* Supported in part by the U. S. Atomic Energy Commission.
[1] P. Signell and N. R. Yoder, Phys. Rev. **132**, 1707 (1963).
[2] P. Signell and N. R. Yoder, Phys. Rev. **134**, B100 (1964).
[3] A preliminary version of this paper was presented at the New York meeting of the American Physical Society in January, 1967 [N. R. Yoder and P. Signell, Bull. Am. Phys. Soc. **12**, 50 (1967)]; numerical results given here supercede that report.

[4] Since this work was completed, work by both the Yale and the Livermore groups as reported at the Special Topics Conference on the Nucleon-Nucleon Interaction, Gainsville, Florida, March, 1967 (unpublished) is in much closer agreement with the data and with each other than any phase representation discussed in this paper. Also a new potential model was reported from La Jolla, and a new revision of the boundary condition model from M.I.T. For this recent work the reader should consult the abstracts in the Bull. Am. Phys. Soc. (to be published), and the appropriate papers in the July, 1967 issue of the Rev. Mod. Phys. (to be published).

used as if they were accurate representations of *p-p* scattering simply do not agree with existing information, and to urge caution in their application.

II. THE DATA

The data set used here was made by combining the independent sets maintained at the several institutions represented by the authors, with a closing date of September 1, 1966. Old data were dropped when they were clearly outclassed by more recent data on the basis of smallness of experimental standard deviations. The point here is that in any adjustment of model parameters for a *least-squares fit*, data with large errors are effectively ignored if similar data with small errors are also present in the data set. In addition, some data have been eliminated on the basis of energy-independent phase-shift analyses.[5,6] Such analyses provide the best possible test for inconsistencies in the data groups used, since they are virtually model-independent: They use only the short range of the strong force, the usual symmetries, and the one-pion-exchange (OPE) interaction for the longer-range part of the strong force.

Our final data set contained 648 individual values in the energy range 9–330-MeV laboratory energy. The upper limit was raised slightly from the value of 320 MeV used in the previous comparisons[1,2] in order to include a new group of data[7]; we believe it is still low enough to avoid complications due to pion production. The lower limit allows us to avoid examination of effects due to vacuum polarization, since a recent analysis[8] of the 9.69-MeV differential cross section[9] together with the ratio of A_{yy} to A_{xx} at 11.4 Mev [10] has shown that these effects are negligible at this energy. This analysis also shows that the well-established fact that the longest-range contribution to the strong interaction between two protons can be accurately computed from one-pion exchange (OPE) allows one to fit all *p-p* scattering data at 10 MeV and below in terms of only two phenomenological parameters at each energy, namely the ${}^1S_0$ phase shift and the J-weighted average of the ${}^3P_{0,1,2}$ phase shifts.[11] It also shows that the energy variation of the former can be completely described by the scattering length and effective range, once the OPE effect is included, while the energy variation of the latter can be described by a single parameter which measures the ratio of the strength of the effective intermediate-range central P-wave interaction to OPE. We therefore assume that anyone who wishes to use the *p-p* model for any application which requires good agreement with data at 10 MeV and below will first check to see whether it is in agreement with these three numbers, and confine our attention here to the agreement of the models with data at higher energy.

Where experiments quote both relative and absolute error, the normalization factor was included in the χ^2 calculation and χ^2 minimized with respect to all such parameters. Normalizations with experimental errors were not counted in the number of data, while those lacking errors were counted against the number of data. The data set was considered too extensive to list here. It is, however, available[12] in its entirety along with references.

III. MODELS AND COMPARISONS

The models of our previous communications which gave the poorest fit to the data of that time have been dropped.[13] An exception is the Brueckner-Gammel-Thaler (BGT)[14] hard-core potential, which continues to be mentioned in occasional textbooks and calculations.

Table I lists the models considered, along with the goodness-of-fit parameter χ^2 for each of them. At the time of our last communication, the best-fitting model was the 21-parameter phase-shift representation CR21.[2] Arndt and MacGregor have adopted the procedures used by the CR21 authors,[2] except for a change of representation, and have made a least-squares fit to a recent data set. No record of Arndt and MacGregor's

TABLE I. The goodness-of-fit parameter χ^2 for various model and phase-shift representation predictions compared to 648 proton-proton scattering data in the energy range 9–330 MeV.

No.	Model	Year	Origin of phases	$\chi^2/648$	Ref.
1	Livermore: AMIV	1966	table	1.38	15
2	Yale: YRB1(K_0)	1966	table	1.94	16
3	CR21	1964	parameters	2.08	2
4	Scotti-Wong-2-σ	1965	table	2.53	17
5	Scotti-Wong-2-$\pi\pi$	1965	table	2.70	17
6	Yale: YLAM	1960	table	2.77	19
7	Hamada-Johnston	1962	potential	3.08	20
8	HJM	1965	potential	3.73	21
9	Bryan-Scott ($+{}^1S_0$)	1964	table	3.90	23
10	Yale potential	1962	potential	3.91	22
11	Tabakin	1964	potential	28.0	26
12	BGT	1958	potential	106.0	14

[5] P. Signell, N. R. Yoder, and J. E. Matos, Phys. Rev. **135**, B1128 (1964).
[6] P. Signell and D. L. Marker, Phys. Rev. **134**, B365 (1964).
[7] F. Betz, J. Arens, O. Chamberlain, C. Schultz, and G. Shapiro, Phys. Rev. **148**, 1289 (1966); D. Fischer and G. Goldhaber, *ibid.* **95**, 1350 (1954); as listed in W. N. Hess, Rev. Mod. Phys. **30**, 368 (1958); Chamberlain, Pettengill, Segre, and Wiegand, Phys. Rev. **93**, 1414 (1954).
[8] H. P. Noyes and H. M. Lipinski, Stanford Linear Accelerator Center Report No. SLAC-PUB-268; Phys. Rev. (to be published).
[9] L. H. Johnston and D. E. Young, Phys. Rev. **116**, 989 (1959).
[10] P. Catillon, M. Chapellier, and D. Garreta, in Proceedings of the Conference on Nuclear Physics, Gatlinburg, Tennessee, 1966 (unpublished), paper 6.3.
[11] H. P. Noyes, Phys. Rev. Letters **12**, 171 (1964).
[12] The actual data used are included as an Appendix to the Stanford Linear Accelerator Center Report No. (SLAC-PUB-269) version of this paper (unpublished).
[13] As reported at Gainsville by Lomon (cf. Ref. 4), a new adjustment of the parameters of the boundary-condition model [H. Feshbach, E. Lomon, and A. Tubis, Phys. Rev. Letters **6**, 635 (1961)] gives a χ^2 value comparable to that obtained by the model of Scotti and Wong (Ref. 17) against the same data table.
[14] K. A. Brueckner, J. A. Gammel, and R. M. Thaler, Phys. Rev. **109**, 1023 (1958).

35 $T=1$ representation parameters was saved by them,[15] so we have used a quadratic interpolation to the phases in their published table.[15] The resulting phase shifts produced the best fit to our data set of any of the models tested, as can be seen in Table I. The next best fit is that of the Yale group's phase-shift–versus–energy curves[16] labeled YRB1(K_0). The old 21-parameter phase-shift representation[2] CR21 is third, and the 12-parameter one-boson-exchange model of Scotti and Wong[17] is fourth. We note, however, that the phase-shift table supplied to us by Scotti and Wong does not correspond precisely to the true predictions of their model parameters, since they used an incorrect Coulomb correction in making their pp phase-shift predictions.[18] The earlier fit from the Yale group[19] (YLAM) is not as good as their lastest work,[16] but is included because it has been used in a number of calculations.

The 15-parameter hard-core Hamada-Johnston potential[20] is followed by the more recent version[21] HJM, which is identical to the Hamada-Johnston potential except for a short range cutoff on the quadratic spin-orbit component. These potentials are followed by the similar 31-parameter Yale potential,[22] and by the Bryan-Scott potential.

The one-boson-exchange Bryan-Scott[23] potential posed something of a problem since it was not intended to be used for S states. The Livermore group has added the Bryan-Scott $L\geq 1$ phases to their latest $^1S_0$ energy-dependent representation[24] and have then adjusted the parameters of the latter for a least-squares fit to their data set. We note that the more recent Bryan-Arndt[25] one-boson-exchange amplitude model uses *effective* scalar- and vector-boson-exchange coupling constants which differ by a factor of 10 from those of the corresponding Bryan-Scott proton-proton potentials.

TABLE II. χ^2 contributions from various data energy ranges, for the three leading representations. Each value quoted is χ^2 divided by the number of data in the energy range (column 2).

Energy range (MeV)	No. data	AMIV	YRB1(K_0)	CR21
9.68– 20.0	38	4.34	1.09	1.28
25.62– 36.9	40	0.93	6.25	1.97
39.4 – 69.5	118	1.84	2.12	1.80
70.0 –122.0	116	1.74	1.67	2.24
127.0 –155.0	210	1.37	1.50	2.12
210.0 –276.0	50	1.10	1.86	1.94
310.0 –330.0	76	1.26	1.46	2.68

The Tabakin potential[26] is nonlocal, with different parameters in each partial-wave state. The partial-wave potentials are separable for the case of chargeless particles, but in order to include the local Coulomb potential, one must solve an integrodifferential equation. This has been done for the current calculation. It is likely that a change in the published parameters would improve the fit to p-p data thus obtained; this question should obviously be investigated before the published model is used in other calculations. The well-known Brueckner-Gammel-Thaler (BGT)[14] potential is identical to the Gammel-Thaler[27] potential for proton-proton scattering and consists of hard cores with single-range Yukawa tails. The ranges of the latter were free parameters and so do not correspond to one-pion or one-boson exchange.

IV. DETAILS OF THE FITS

The partial χ^2 contributions to the leading three models from various energy ranges are shown in Table II. There are three obvious misfits: AMIV below 20 MeV, YRBI(K_0) in the 25–35-MeV range, and CR21 in the 310–330-MeV range. That the AMIV fit should

TABLE III. Data with a χ^2 contribution of 20 or more for any one of the three leading representations.

Energy (MeV)	Angle	Type	χ^2:AMIV	χ^2:YRB1(K_0)	χ^2:CR21
9.68	···	N_σ	38	10	17
25.7	90°	A_{yy}	0	96	0
25.7	90°	A_{xx}	0	39	1
34.2	90°	σ	5	24	6
45.04	90°	σ	4	48	8
49.7	45°	P	8	12	23
50.02	90°	σ	0	30	2
68.3	13.2°	σ	22	5	0
70.0	···	$\sigma_{\mathrm{int.}}$	22	8	9
98.0	···	N_P	21	22	27
108.0	···	$\sigma_{\mathrm{int.}}$	19	12	28
310.0	6.5°	σ_{rel}	1	1	22
315.0	21.7°	σ_{rel}	1	1	37

[15] R. A. Arndt and M. H. MacGregor, Phys. Rev. **141**, 873 (1966); and private communication from R. A. Arndt to P. S. Signell.

[16] G. Breit and R. D. Haracz, in *High-Energy Physics*, edited by E. H. S. Burhop (Academic Press Inc., New York, 1967), Vol. I. This fit was reported in *Proceedings of the Dubna Conference on High-Energy Physics, 1964* (Atomizdat, Moscow, 1965) but not in a form which allowed direct numerical comparison with the data. We are indebted to G. Breit for kindly supplying us with copies of this manuscript prior to publication.

[17] A. Scotti and D. Y. Wong, Phys. Rev. **138**, B145 (1965).

[18] L. Heller and M. Rich, Phys. Rev. **144**, 1324 (1966). Scotti and Wong (Ref. 17) used N/D equations for the amplitude $e^{i\delta}\sin\delta/e^2q$, but included only the Coulomb modification of the left cut computed from Coulomb plus OPE in the first Born approximation; this ignored the Coulomb modification of the higher-mass one-boson-exchange cuts. The effect of the latter modification has been estimated by the use of a single scalar-boson exchange with constants adjusted to fit the low-energy $^1S_0$ phase shift. The full Coulomb modification shifts the chargeless value of the $^1S_0$ phase shift at 10 MeV by $-2.5°$, while the Scotti-Wong-type modification shifts it by $-5.3°$. We are indebted to L. Heller and M. Rich for the discussion and numbers in this footnote (private communication from L. Heller to P. Signell).

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[23] R. A. Bryan and B. L. Scott, Phys. Rev. **135**, B434 (1964).

[24] This representation goes to the correct scattering length and effective range at low energy; it has now been published [R. A. Arndt and M. H. MacGregor, Phys. Rev. **154**, 1549 (1967)].

[25] R. A. Bryan and R. A. Arndt, Phys. Rev. **150**, 1299 (1966).

[26] F. Tabakin, Ann. Phys. (N. Y.) **30**, 51 (1964).

[27] J. A. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957).

TABLE IV. Phase shifts at 27.6 MeV from the three leading models and an energy-independent phase shift analysis (EIPSA).

Model	$^1S_0$	$^3P_0$	$^3P_1$	$^3P_2$
AMIV	48.0	8.2	−4.8	2.7
YRB1(K_0)	49.2	12.1	−5.8	2.6
CR21	46.6	8.7	−5.7	3.2
EIPSA[a]	48.6±0.4	7.6±0.6	−4.1±0.5	2.4±0.2

[a] P. Signell, Phys. Rev. 139, B315 (1965).

not be extended below 24 MeV was known to the authors and was indicated in their paper. One way they could achieve a good low-energy fit would be to add the effective-range contributions to their representation,[24] as was done in the CR21 representation.

Data which give large χ^2 contributions to any one of the three representations are shown in Table III. Nearly half the high χ^2 contribution to YRB1(K_0) in the second energy range is seen to result from the 25.7-MeV measurements of A_{xx} and A_{yy}. Since these are determined primarily by the ^3P phase parameters, we compare these parameters to the other models in Table IV. It is seen that $^3P_0$ is high and $^1S_0$ is low both compared to the other models and to single-energy phase-shift analyses at that energy. Again, a correlated adjustment of parameters should remove this difficulty.

V. CONCLUSION

Examination of the existing fits to the best proton-proton scattering data reveals discrepancies in the fits which should be taken account of in any application where these discrepancies are potentially important. If this paper encourages more care to be taken in applying these models in specific cases, we will have accomplished our purpose.

ACKNOWLEDGMENTS

The calculations of χ^2 were carried out in the computer laboratory of Michigan State University. We wish to thank our many colleagues in the various accelerator laboratories who have helped us in the compilation and treatment of the data. Receipt of the manuscript of the article by Breit and Haracz on nucleon-nucleon scattering[16] prior to publication is gratefully acknowledged.

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Eigenstates of the $L=0$, Charge- and Spin-Independent Pairing Hamiltonian. I. Seniority-Zero States*

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(Received 28 February 1967)

Equations for all the seniority-zero eigenstates of $2N$ nucleons in an arbitrary charge- and spin-independent potential well and interacting through charge- and spin-independent pairing forces are derived. These equations are solved exactly for a large number of states of this system. The interaction in this Hamiltonian is effective in the $L=0$ states of the two-nucleon system, and its strength is independent of the remaining quantum numbers of the two nucleons. We solve our equations exactly for those states whose wave functions are totally symmetric functions of the spin-isospin coordinates of the N $L=0$ pairs of nucleons in the state. The wave functions of these states factor into a spin-isospin-dependent part and a spatially dependent part. The spin-isospin-dependent part of one of these wave functions is an eigenvector of three tridiagonal matrices which insure that the state is a spin, isospin, and supermultiplet eigenstate, respectively. Explicit expressions are given for the eigenvalues of these three matrices in terms of the quantum numbers of the state. The spatially-dependent part of one of these wave functions is given explicitly in terms of N parameters which we call pair energies. These pair energies are shown to satisfy N coupled algebraic equations which depend parametrically upon the supermultiplet quantum numbers of the state. An expression for the occupation probabilities of the levels of the single-particle well is given. Throughout this work, an arbitrary splitting of the single-particle levels is treated exactly.

I. INTRODUCTION

THE simplicity of the pairing Hamiltonian has made it a fruitful model for the study of the approximation techniques used in nuclear-structure calculations.[1] In addition to this, it has also proven to be a useful model for the calculation of specific nuclear properties.[1] In its most elementary form, the pairing model uses jj coupling single-particle states and a pairing interaction that is effective between any two identical nucleons that are coupled to $J=0$. The neutrons and protons therefore act as two independent systems. In this form, the model has been very successful in representing the properties of heavy nuclei. These properties have generally been calculated using

* This research was supported in part by the National Science Foundation.

[1] A. M. Lane, *Nuclear Theory* (W. A. Benjamin, Inc., New York, 1964), Part I and the references cited therein.