Fluctuations of the Order Parameter in Type-II Superconductors. II. Pure Limit*

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We study here the fluctuations of the order parameter in pure type-II superconductors in the vicinity of the upper critical field H_{c2}, where the equilibrium value of the order parameter $\Delta(\mathbf{r})$ is described in terms of the Abrikosov solution. Among many fluctuations we concentrate on two modes: (1) The longitudinal mode, which couples to the density fluctuations. This mode is described by an equation of diffusion type at all temperatures. The diffusion coefficient is weakly temperature-dependent and given by $D \simeq \xi_0 v^2$, where ξ_0 is the coherence distance and v the Fermi velocity. (2) The transverse mode, which couples to the current fluctuations. At high temperature (i.e., $T \sim T_{c0}$, where T_{c0} is the critical temperature in zero field) this mode is of diffusion type, while at low temperature, it becomes a strongly damped oscillation. We show that the transverse modes cause an important modification of the electromagnetic response of the type-II superconductor. When the oscillating current flows in the plane perpendicular to the static magnetic field, the reactive part of the resulting response vanishes identically at low frequency (i.e., $\omega < T_{c0}$). This modifies strongly both the surface impedance and the attenuation coefficient of a transverse ultrasonic wave in the corresponding geometry.

I. INTRODUCTION

In a previous paper[1] (to which we will refer as I in the following) we have studied the collective oscillations of the order parameter in dirty type-II superconductors. It is shown there that those collective modes have a significant contribution to the transverse response function, giving rise to an anisotropic surface impedance in dirty type-II superconductors in the vicinity of the upper critical field ($H \sim H_{c2}$).

Since the equilibrium as well as the nonequilibrium properties of a pure type-II superconductor are quite different from those of the dirty superconductor, it is worthwhile to study these collective oscillations in a pure type-II superconductor.

In the following we adopt the general formalism developed in I, which applies as well to the present problem.

In Sec. II, we will discuss the various collective modes associated with the fluctuations of the order parameter. Among many modes the following two are most important for their physical relevance:

(1) longitudinal modes: which couple to the density fluctuations,

(2) transverse modes: which couple to the current fluctuations.

The longitudinal mode is of diffusion type at all temperatures. The diffusion coefficient is always of order v^2/T_{c0} (v being the Fermi velocity and T_{c0} the transition temperature in zero field). This mode is very similar, close to T_{c0}, to the one discussed by Abrahams and Tsuneto.[2] However, there is a significant difference in the fact that they study the situation of zero magnetic field and constant order parameter, while we look at the fluctuations of Δ in high field ($H \sim H_{c2}$), where the order parameter at equilibrium is given in terms of the Abrikosov solution.[3,4]

At low temperature our mode is still diffusionlike (contrary to the one of Abrahams and Tsuneto) because we are dealing with a gapless superconductor.

The transverse mode is, at low temperatures, a strongly damped oscillation, the real and imaginary part of its frequency are both of the order of Δ_{00} at $T=0$ (Δ_{00} is the BCS gap at $T=0$ in zero field). This mode can be interpreted as a resonance in the scattering amplitude of particle-particle (or hole-hole) pairs due to the pair interaction. At temperatures close to T_{c0}, this mode becomes of diffusion type like the longitudinal one. At $T=0$ the two types of modes obey nonanalytic dispersion relations. The contribution of these modes to the various transport properties can be discussed as in I. We find that, in the present case, similar to the case of dirty type-II superconductors, only the electromagnetic (transverse) response couples strongly to these modes (i.e., the transverse modes). The transverse conductivity is strongly anisotropic, which will be discussed in Sec. III. In particular, when the microwave current flows in the plane perpendicular to the direction of the static field $\boldsymbol{H}$, the reactive part of the conductivity coming from $\langle [j_\mu, j_\mu] \rangle$ is exactly canceled up to order $\langle |\Delta|^2 \rangle$ and at all temperatures by the one coming from the collective modes. This result is strictly valid only in the low-frequency limit ($\omega \ll T_{c0}$). Generally speaking, in type-II superconductors the conductivity has a finite reactive part

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[1] C. Caroli and K. Maki, preceding paper, Phys. Rev. 159, 306 (1967).
[2] E. Abrahams and T. Tsuneto, Phys. Rev. **152**, 416 (1966).
[3] E. Helfand and N. R. Werthamer, Phys. Rev. Letters **13**, 686 (1964); Phys. Rev. **147**, 288 (1966).
[4] K. Maki and T. Tsuzuki, Phys. Rev. **139**, A868 (1965).

which is associated with the Meissner effect (i.e., the oscillating field cannot penetrate into the bulk type-II superconductor indefinitely). The above result indicates that for a pure type-II superconductor in the vicinity of H_{c2} there is no Meissner effect in the above sense if the microwave current flows perpendicularly to the static magnetic field. A similar result has been obtained already in I for dirty type-II superconductors at temperatures immediately below T_{c0}. The explicit form of the surface impedance is given, which can be directly measured experimentally.

II. COLLECTIVE MODES

We shall concentrate here on the discussion of various collective modes associated with the fluctuations of the order parameter in a pure type-II superconductor in the high-field region (i.e., $H\sim H_{c2}$). Since the general formalism we use to deal with the above problem has already been developed in I in detail, we shall recapitulate the results, which we need for the following discussions.

Generally speaking, the fluctuations of the order parameter couple closely with the other fluctuations (e.g., density and current fluctuations). In the field region close to H_{c2}, these coupling terms are of the order of $|\Delta|^2$, and we can neglect this coupling in the study of the dispersion relation of the collective modes.

It is important to point out here that the dispersion relation that we obtain in this approximation is strictly valid only at $H=H_{c2}$. In fact, we can show that this approximation is sufficient to obtain exactly the terms of order $\langle|\Delta|^2\rangle$ in the various low-frequency ($\omega\ll T_{c0}$) response functions. This no longer holds if we are interested in higher-frequency phenomena (e.g., far-infrared region, $\omega\sim T_{c0}$). In that region we would need to know the corrections of order $\langle|\Delta|^2\rangle$ to the dispersion relation, which are not necessarily negligible even when Δ is small.

This can be qualitatively understood in the following way: In a dirty superconductor the pair-pair correlations decrease exponentially with time. This loss of correlation is isotropic in space because of the high rate of collisions on impurities. In the pure superconductor these correlations are strongly anisotropic and remain finite even when the time difference increases indefinitely, if the correlation is taken in the direction parallel to the field.[5] This gives rise to nonanalytic behaviors of some response functions similar to the irregularity associated with the BCS state, in spite of the fact that we are dealing with a gapless situation. For this reason, we are not able to calculate the precise position of the bump in the infrared absorption, although we expect that it is only slightly shifted with respect to its position at $H=H_{c2}$.

To lowest order in Δ, the dispersion relation is thus given by

$$\int \frac{d^3q'}{(2\pi)^3}\{\delta_{\mathbf{q}\mathbf{q}'}-|g|\langle[\Psi^\dagger,\Psi]\rangle_{\mathbf{q}\mathbf{q}'\omega}\}\delta\Delta_{\mathbf{q}'\omega}=0, \quad (1)$$

where $|g|$ is the pair interaction and $\langle[\ ]\rangle$ indicates the retarded products on the Gibbs ensemble associated with the reduced Hamiltonian (see I). Here $\Psi^\dagger$ and Ψ are the pair-creation and -annihilation operators;

$$\Psi^\dagger(\mathbf{r})=\psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r}),$$
$$\Psi(\mathbf{r})=\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r}), \quad (2)$$

where $\Psi^\dagger(\mathbf{r})$ is the electron-field operator.

In the present situation, where the external field H is directed along the z axis, the eigensolutions of Eq. (1) are the functions

$$\phi_{nkp_z}(\mathbf{r})=C_{nkp_z}\exp(ip_zz+iky)(\Pi^\dagger)^n \times\exp[-eH(x-k/2eH)^2],$$

where $\Pi^\pm=i(\partial/\partial x)\mp[\partial/\partial y-2eHxi]$ and C_{nkp_z} is a normalization constant. We show in the Appendix using a method due to Cyrot and one of us[6] that, in terms of these solutions, Eq. (1) reduces to

$$0=1-|g|N(0)\int_{-\infty}^{\infty}d\alpha\int\frac{d\Omega}{4\pi}\rho_n(\alpha,\Omega)\left\{\ln\left(\frac{\omega_D}{2\pi T}\right)-\psi\left(\frac{1}{2}+\frac{-i\omega+ivp_z\cos\theta}{4\pi T}+\frac{i\alpha}{2\pi T}\right)\right\} \quad (3)$$

$$0=|g|N(0)\int_{-\infty}^{\infty}d\alpha\int\frac{d\Omega}{4\pi}\left[\rho_n(\alpha,\Omega)\psi\left(\frac{1}{2}+\frac{-i\omega+ivp_z\cos\theta}{4\pi T}+\frac{i\alpha}{2\pi T}\right)-\rho_0(\alpha,\Omega)\psi\left(\frac{1}{2}+\frac{i\alpha}{2\pi T}\right)\right], \quad (4)$$

where

$$\rho_n(\alpha,\Omega_v)=2\langle\phi_{n00}^*(\mathbf{r}')\delta(\mathbf{v}\cdot\mathbf{q}+2\alpha)\phi_{n00}(\mathbf{r})\rangle$$
$$=2\int d^3r\int d^3r'\int\frac{d^3q}{(2\pi)^3}\left[\exp\left\{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')+2ie\int_{\mathbf{r}'}^{\mathbf{r}}\mathbf{A}(\mathbf{l})\,d\mathbf{l}\right\}\delta(\mathbf{v}\cdot\mathbf{q}+2\alpha)\phi_{n00}^*(\mathbf{r})\phi_{n00}(\mathbf{r}')\right], \quad (5)$$

and $\psi(z)$ is the digamma function, ω_D the Debye frequency, $N(0)=mp_0/2\pi^2$ the density of states at the Fermi surface, and v the Fermi velocity. In the above derivation we have made use of the identity (which is the implicit equation for H_{c2})[3,4] given by

$$0=1-|g|N(0)\int_{-\infty}^{\infty}d\alpha\int\frac{d\Omega}{4\pi}\rho_0(\alpha,\Omega)\left[\ln\left(\frac{\omega_D}{2\pi T}\right)-\psi\left(\frac{1}{2}+\frac{i\alpha}{2\pi T}\right)\right]. \quad (6)$$

[5] The Orsay group of superconductivity, Physik Kondensierten Materie 5, 141 (1966).
[6] M. Cyrot and K. Maki, Physik Rev. 156, 433 (1967).

The explicit forms of the important $\rho_n(\alpha,\Omega)$ are given as

$$\rho_0(\alpha,\Omega)=(\sqrt{\pi}\epsilon\sin\theta)^{-1}\exp[-(\alpha/\epsilon\sin\theta)^2], \quad (7)$$

$$\rho_1(\alpha,\Omega)=\frac{2}{\sqrt{\pi}\epsilon\sin\theta}\left(\frac{\alpha}{\epsilon\sin\theta}\right)^2\exp\left[-\left(\frac{\alpha}{\epsilon\sin\theta}\right)^2\right], \quad (8)$$

where

$$\epsilon=v(eH_{c2}/2)^{1/2}.$$

As in the case of dirty type-II superconductors, the nth mode (associated with functions ϕ_{n,k,p_z}) has the symmetry of an nth-order tensor, so that in the following we will limit ourselves to the discussion of the two modes $n=0$ and $n=1$, which are the most important in the various response functions.

A. Longitudinal Modes

As in I, we designate the mode with $n=0$ as longitudinal. In this case, Eq. (4) reduces further, and we have

$$0=\int_{-\infty}^{\infty}d\alpha\int\frac{d\Omega}{4\pi}\rho_0(\alpha,\Omega)\times\left\{\psi\left(\frac{1}{2}+\frac{-i\omega+ivp_z\cos\theta}{4\pi T}+\frac{i\alpha}{2\pi T}\right)-\psi\left(\frac{1}{2}+\frac{i\alpha}{2\pi T}\right)\right\}$$
$$=(2\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}dx\int_{-1}^{1}\frac{dz}{(1-z^2)^{1/2}}e^{-x^2/(1-z^2)}\times\left\{\psi\left(\frac{1}{2}+\frac{-i\omega+ivp_z z}{4\pi T}+i\rho x\right)-\psi(\tfrac{1}{2}+i\rho x)\right\}, \quad (9)$$

where $\rho=\epsilon/2\pi T$.

The asymptotic expansion of Eq. (9) at high temperature is

$$0=-\frac{i\omega}{4\pi T}(\tfrac{1}{2}\pi^2-\tfrac{1}{6}\pi^4\rho^2)+\left(\frac{\omega}{4\pi T}\right)^2 7\zeta(3)-\left(\frac{vp_z}{4\pi T}\right)^2\tfrac{7}{3}\zeta(3),\quad \text{for}\quad T_{c0}-T\ll T_{c0}. \quad (10)$$

At $T=0$, the dispersion relation becomes

$$0=(2\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}dx\int_{-1}^{1}\frac{dz}{(1-z^2)^{1/2}}e^{-x^2/(1-z^2)}\times\ln\left(\frac{-\omega+vp_z z+2\epsilon x}{2\epsilon x}\right), \quad (11)$$

which reduces, for small frequency and wave vector (ω, $vp\ll T_{c0}$), to

$$i\omega=(2\pi\sqrt{\pi})^{-1}\frac{(vp_z)^2}{2\epsilon}\left[1-\ln\left\{\gamma\left(\frac{vp_z}{2\epsilon}\right)^2\right\}\right],\quad \gamma=1.78. \quad (11')$$

The above expressions indicate that, for small wave vectors ($p_z\xi_0\ll 1$, where ξ_0 is the BCS coherence length) the dispersion relation is of diffusion type at $T=0$ and $T\simeq T_{c0}$. In fact, starting from Eq. (9), we can show that this is the case throughout the whole temperature region $0\leq T\leq T_{c0}$, the diffusion coefficient being of order v^2/T_{c0}. But it is worthwhile to emphasize that Eq. (11′) cannot be obtained by a simple power-series expansion. This suggests that one cannot use a time-dependent Ginzburg-Landau type of expansion in the region $T<vp_z$ in the pure limit. Using this result, and the analogous one obtained in the dirty limit (see I), we can infer that the longitudinal mode is of diffusion type for any value of the mean free path l, the diffusion coefficient being roughly of the order[7] of $(1/\xi_0+1/l)^{-1}v$. A result similar to ours has been derived, close to T_{c0}, by Abrahams and Tsuneto,[2] in the case where $H=0$ and Δ is a constant. However, they find that, at low temperature, the collective mode is wavelike. (It is essentially the Anderson-Bogoliubov mode.)[8] The difference between this result and ours comes from the fact that we are concerned with a gapless superconductor, while they deal with the case of a finite gap.

We shall now discuss the coupling of this mode to the density fluctuation. The induced change in the order parameter in the presence of a scalar potential $\Phi_{q\omega}$ is given by[1]

$$\delta\Delta_{q\omega}=-\left\{\frac{|g|}{1-|g|\langle[\Psi^\dagger,\Psi]\rangle^L}\langle[\Psi^\dagger,n]\rangle\Phi\right\}_{q\omega}. \quad (12)$$

$\langle[\Psi^\dagger,\Psi]\rangle^L$ is the longitudinal part of $\langle[\Psi^\dagger,\Psi]\rangle$ and $n=\sum_\sigma\psi_\sigma^\dagger(rt)\psi_\sigma(rt)$ is the density operator. Therefore, we need only to compute $\langle[\Psi^\dagger,n]\rangle_{qq'\omega}$ for this purpose. The details of the evaluation of $\langle[\Psi^\dagger,n]\rangle$ are given in the Appendix. We have

$$\langle[\Psi^\dagger,n]\rangle_{qq'\omega}=N(0)\int_{-\infty}^{\infty}d\alpha\int\frac{d\Omega}{4\pi}\rho_0(\alpha,\Omega)\frac{2\omega}{\omega^2-(\mathbf{v}\cdot\mathbf{q}')^2}\times\left[\psi\left(\frac{1}{2}+\frac{i\alpha}{2\pi T}-\frac{-i\omega-i\mathbf{v}\cdot\mathbf{q}'}{4\pi T}\right)-\psi\left(\frac{1}{2}+\frac{-i\omega+i\alpha}{2\pi T}\right)\right]\Delta_{q-q'}. \quad (13)$$

We note here that this retarded product vanishes like ω at low frequency. Therefore, the contribution of the longitudinal collective oscillation to the longitudinal ultrasonic attenuation is always negligible and we can always use the expression given by one of us (K.M.)[9]

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[7] This order of magnitude can be easily checked, close to T_{c0}, using the Ginzburg-Landau equation obtained by Gor'kov for superconducting alloys {see L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **37**, 1407 (1959) [English transl.: Soviet Phys.—JETP **10**, 998 (1960)]}.

[8] N. N. Bogoliubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau Enterprises, Inc., New York, 1959); P. W. Anderson, Phys. Rev. **112**, 1900 (1958).

[9] K. Maki, Phys. Rev. **156**, 437 (1967).

B. Transverse Modes

We consider here the modes belonging to $n=1$. The basic dispersion relation is given by

$$0=\int_{-\infty}^{\infty}d\alpha\int\frac{d\Omega}{4\pi}\left\{\rho_1(\alpha,\Omega)\psi\left(\frac{1}{2}+\frac{i(-\omega+vp_z\cos\theta)}{4\pi T}+\frac{i\alpha}{2\pi T}\right)-\rho_0(\alpha,\Omega)\psi\left(\frac{1}{2}+\frac{i\alpha}{2\pi T}\right)\right\}. \tag{14}$$

In the high-temperature region $T\sim T_{c0}$, the above equation reduces to

$$0=\frac{-i\omega}{4\pi T}\left(\tfrac{1}{2}\pi^2-\tfrac{1}{2}\pi^4\rho^2\right)+7\zeta(3)\left[\left(\frac{\omega}{4\pi T}\right)^2-\frac{1}{3}\left(\frac{vp_z}{4\pi T}\right)^2\right]+\frac{14}{3}\zeta(3)\rho^2, \tag{15}$$

which is diffusionlike (for $p_z\xi_0\ll1$).

At low temperature (in particular at $T=0$) we have

$$0=\int_{-\infty}^{\infty}d\alpha\int\frac{d\Omega}{4\pi}\{\rho_1(\alpha,\Omega)\ln[i(-\omega+vp_z\cos\theta+2\alpha)]-\rho_0(\alpha,\Omega)\ln(2\alpha i)\}$$

$$=\sqrt{\pi}^{-1}\int_{-\infty}^{\infty}dx\int_{-1}^{1}\frac{dz}{2(1-z^2)^{1/2}}e^{-x^2/(1-z^2)}\left[\frac{2x^2}{1-z^2}\ln(-\omega+vp_zz+2\epsilon x)-\ln(2\epsilon x)\right]. \tag{16}$$

Neglecting p_zv in Eq. (16), we have

$$0=(2\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}dx\int_{-1}^{1}\frac{dz}{(1-z^2)^{1/2}}e^{-x^2/(1-z^2)}\left[\frac{2x^2}{1-z^2}\ln(-\omega+2\epsilon x)-\ln(2\epsilon x)\right]$$

$$=1+\tfrac{1}{2}[\ln(\zeta^2\gamma)-e^{-\zeta^2}Ei(\zeta^2)], \tag{17}$$

where

$$\zeta=\frac{\omega}{2\epsilon},\qquad e^{-\zeta^2}E_i(\zeta^2)=\int_0^{\infty}\frac{e^{-t}}{\zeta^2-t}dt$$

and we consider ζ as a complex variable.

One can show easily that this equation has no solution on the first Riemann sheet (i.e., physical sheet), but that it has a complex solution in the second sheet:

$$|\zeta|\cong0.66\qquad \text{Arg}\zeta\cong-(\tfrac{1}{4}\pi-0.14). \tag{17'}$$

This is very analogous to a resonance pole in potential scattering. Here a well-defined collective mode could be understood as a particle-particle (and hole-hole) bound state due to the pair interaction. The finite width of the resonance state (17′) is related to the fact that, since the superconductor is gapless, it has a finite density of pair excitations at zero energy. The collective mode can then decay into incoherent particle-particle or hole-hole states.

This should be a general feature for pure type-II superconductors in the gapless region, and should apply to all the higher-order excited states (i.e., $n>1$).

This resonance should show up as a broad bump superimposed on the usual infrared absorption when the polarization vector is perpendicular to the static magnetic field.

Finally, the difference in the structure of the dispersion equations in the longitudinal and the transverse case comes from the fact that the former ($n=0$) is connected to the zero-azimuthal-momentum component of the scattering amplitude, while the latter is connected to the component with an azimuthal number equal to 1. It is interesting to note that the transition in the transverse mode from the diffusionlike to the damped propagation behavior occurs when ω becomes comparable to T (i.e., $\epsilon\sim T$). The effect of finite temperature in the damped oscillation region is to add a term proportional to T^2 to the damping coefficient.

We shall now give an explicit expression for expression (14) at zero frequency ($\omega=0$), which will be useful in the discussion of surface impedance.

$$1-|g|\langle[\Psi^\dagger,\Psi]\rangle_{\mathbf{q}\mathbf{q}',\omega=0}{}^T=-|g|N(0)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}d\alpha\int\frac{d\Omega_v}{4\pi}$$

$$\times\left[\rho_1(\alpha,\Omega_v)\psi\left(\frac{1}{2}+\frac{iq_zv\cos\theta}{4\pi T}+\frac{i\alpha}{2\pi T}\right)-\rho_0(\alpha,\Omega_v)\psi\left(\frac{1}{2}+\frac{i\alpha}{2\pi T}\right)\right]\phi_{1k0}{}^*(\mathbf{q})\phi_{1k0}(\mathbf{q}')\delta_{q_zq_z'}$$

$$=\left[|g|N(0)\sqrt{\pi}^{-1}\int_{-\infty}^{\infty}dx\int_0^1\frac{dz}{(1-z^2)^{1/2}}e^{-x^2/(1-z^2)}\left\{1-\frac{2x^2}{1-z^2}\right\}\psi(\tfrac{1}{2}+i\rho x)+O(q_zv)^2\right]\delta_{q_zq_z'}$$

$$\times\int_{-\infty}^{\infty}\frac{dk}{2\pi}\phi_{1k0}{}^*(\mathbf{q})\phi_{1k0}(\mathbf{q}'), \tag{18}$$

where $\rho=\epsilon/2\pi T$.

The coupling of the fluctuations of Δ to a transverse electromagnetic wave is given by $\langle[j_\mu,\Psi]\rangle$. The explicit form of this product is computed in the Appendix (where we also discuss the convergence of its development in powers of Δ), and we have for $\mu=x, y$

$$\langle[j_\mu,\Psi^\dagger]\rangle_{qq'\omega}=\frac{ip_0}{\pi^2}\int\frac{d\Omega_v}{4\pi}mv_\mu\int_{-\infty}^{\infty}d\alpha\rho_{10}(\alpha,\Omega_v)\frac{2\mathbf{v}\cdot\mathbf{q}}{\omega^2+(\mathbf{v}\cdot\mathbf{q})^2}$$
$$\times\left\{\psi\left(\frac{1}{2}+\frac{i\alpha}{2\pi T}\right)-\psi\left(\frac{1}{2}+\frac{-i\omega+i\mathbf{v}\cdot\mathbf{q}}{4\pi T}+\frac{i\alpha}{2\pi T}\right)\right\}\frac{\Pi^+}{(4eH)^{1/2}}\Delta_{q'-q}$$
$$=-\frac{p_0^2}{4\pi^2\sqrt{\pi}}\int_{-1}^{1}\frac{dz}{(1-z^2)^{1/2}}\int_{-\infty}^{\infty}xdx\,e^{-x^2/(1-z^2)}\frac{\mathbf{q}\cdot\mathbf{v}}{\omega^2+(\mathbf{q}\cdot\mathbf{v})^2}$$
$$\times\left\{\psi(\tfrac{1}{2}+i\rho x)-\psi\left(\frac{1}{2}+\frac{i(-\omega+\mathbf{v}\cdot\mathbf{q})}{4\pi T}+i\rho x\right)\right\}\frac{\Pi^+}{(4eH)^{1/2}}\Delta_{q'-q},\qquad(19)$$

where, using the same convention as in Eq. (5)

$$\rho_{10}(\alpha,\Omega_v)=2\langle\phi_{100}^*(\mathbf{r}')\delta(\mathbf{q}\cdot\mathbf{v}+2\alpha)\phi_{000}(\mathbf{r})\rangle$$
$$=-\frac{i\alpha e^{i\phi}}{(2\pi)^{1/2}(\epsilon\sin\theta)^2}\exp\left[-\left(\frac{\alpha}{\epsilon\sin\theta}\right)^2\right].\qquad(20)$$

Furthermore, in the limit $\omega\to0$, $vq\to0$ [keeping $\omega/qv\to0$] we obtain

$$\langle[j_{x,y},\Psi^\dagger]\rangle_{0q'0}=\frac{p_0^2}{(2\pi)^2}(\pi T)^{-1}\int_0^1dz\frac{1}{(1-z^2)^{1/2}}\int_{-\infty}^{\infty}xdx\,e^{-x^2/(1-z^2)}\psi'(\tfrac{1}{2}+i\rho x)\frac{\Pi^+}{(4eH)^{1/2}}\Delta_{q'},$$

$$\langle[j_{x,y},\Psi^\dagger]\rangle_{0q'0}=\frac{p_0^2}{(2\pi)^2}(\pi T)^{-1}\int_0^1dz\frac{1}{(1-z^2)^{1/2}}\int_{-\infty}^{\infty}xdx\,e^{-x^2/(1-z^2)}\psi'(\tfrac{1}{2}+i\rho x)\frac{\Pi^+}{(4eH)^{1/2}}\Delta_{q'}$$
$$=\frac{2p_0^2}{(2\pi)^3T(2\pi)^{1/2}}\int_0^1\frac{dz}{(1-z^2)^{1/2}}\int_{-\infty}^{\infty}\frac{idx}{\rho}\left(1-\frac{2x^2}{1-z^2}\right)e^{-x^2/(1-z^2)}\psi(\tfrac{1}{2}+i\rho x)\frac{\Pi^+}{(4eH)^{1/2}}\Delta_{q'},$$
$$\langle[j_z,\Psi^\dagger]\rangle_{0q'0}=0.\qquad(21)$$

Using the above expressions, we find the change in $\Delta(\mathbf{r})$ induced by an oscillating transverse field in the same limit[1] [ω, q', $\omega/vq'\to0$]

$$\delta\Delta_{q0}=-\sum_\mu\left\{\frac{|g|}{1-|g|\langle[\Psi^\dagger,\Psi]\rangle^T}\langle[\Psi^\dagger,j_\mu]\rangle\right\}_{q00}\delta A_\mu\qquad(q'=\omega=0)$$
$$=\frac{2p_0^2}{(2\pi)^3T}\frac{i}{\rho}\frac{2\pi^2}{mp_0}\frac{\Pi^+}{(4eH)^{1/2}}\Delta_q\delta A_-\qquad(\mathbf{q}'=\omega=0).\qquad(22)$$

In the above derivation we have substituted the expressions for $\langle[\Psi^\dagger,\Psi]\rangle$ and $\langle[\Psi^\dagger,j_\mu]\rangle$ given by Eqs. (18) and (21), respectively. We can take the wave vector of the microwave field to be zero since in the situation of interest the penetration depth is much larger than ξ_0.

It is quite easy to show that the coupling of the above mode to the heat current vanishes like ω for low frequency, and hence these modes do not contribute to the thermal conductivity at all.

III. COMPLEX CONDUCTIVITY AND SURFACE IMPEDANCE

We have seen in the foregoing sections that, in a pure type-II superconductor and in high field, the most remarkable effect of the collective oscillations of the order parameter appears in the electromagnetic response. We shall discuss in the present section the complex conductivity in the mixed state in high field ($H\sim H_{c2}$) and the surface impedance, which we find to be strongly anisotropic. In the last section we shall discuss the effect of this modification of electrical conductivity on the attenuation coefficient of a transverse ultrasonic wave.

The response of the transverse electromagnetic field is described by introducing a quantity $Q_{\mu\nu}(\omega,\mathbf{q})$:

$$j_\mu(\mathbf{q},\omega)=\sum_\nu Q_{\mu\nu}(\mathbf{q},\omega)A_\nu(\mathbf{q},\omega).\qquad(23)$$

We take $Q_{\mu\nu}$ to be diagonal in momentum, because on the scale of $\mathbf{q}$ the system can be considered as homogeneous, since $q\xi_0 \ll 1$. $Q_{\mu\nu}(\mathbf{q}, \omega)$ is given by

$$Q_{\mu\nu}(\mathbf{q}, \omega) = \langle [j_\mu, j_\nu] \rangle_{\mathbf{q},\mathbf{q},\omega} + R_{\mu\nu}(\mathbf{q}, \omega) + R_{\nu\mu}(\mathbf{q}, \omega), \quad (24)$$

and[1]

$$R_{\mu\nu}(\mathbf{q}, \omega) = -|g| \{ \langle [\Psi^\dagger, j_\mu] \rangle \times (1 - |g| \langle [\Psi^\dagger, \Psi] \rangle^T)^{-1} \langle [\Psi, j_\nu] \rangle \}_{\mathbf{q}\mathbf{q}\omega}. \quad (25)$$

We can calculate $R_{\mu\nu}(\omega, \mathbf{q})$ very easily, using the expressions of $\langle [\Psi^\dagger, \Psi] \rangle^T$, $\langle [\Psi^\dagger, j_\mu] \rangle$, etc., already obtained. We note here that $R_{\mu\nu}(\omega, \mathbf{q})$ gives rise to a nonvanishing contribution only when the microwave current flows in the plane perpendicular to the external static field.

$\langle [j_\mu, j_\nu] \rangle$ is obtained by making use of the technique developed by one of us (K.M.)[9] and we have

$$\langle [j_\mu, j_\nu] \rangle_{\mathbf{q}\mathbf{q}\omega} = 0 \quad \text{if} \quad \mu \neq \nu,$$

$$\langle [j_\mu, j_\mu] \rangle_{\mathbf{q},\mathbf{q},\omega} = Q_{\mu\mu}^0(\mathbf{q}, \omega) = \frac{Ne^2}{m} 3\pi T \sum_{n=-\infty}^{\infty} \int \frac{d\Omega_v}{4\pi} x_\mu^2 \int_{-\infty}^{\infty} d\alpha\, \rho_0(\alpha, \Omega_v)$$

$$\times \left\{ \frac{1 - \{(\omega_n - i\alpha)(\omega_n' - i\alpha) - \Delta^2 / [(\omega_n - i\alpha)^2 + \Delta^2]^{1/2} \times [(\omega_n' - i\alpha)^2]^{1/2} + \Delta^2\}}{[(\omega_n - i\alpha)^2 + \Delta^2]^{1/2} + [(\omega_n' - i\alpha)^2 + \Delta^2]^{1/2} - \mathbf{v} \cdot \mathbf{q}} \right\} \Bigg|_{\omega_\nu \to i\omega} - \frac{Ne^2}{m},$$

$$\omega_n = (2n+1)\pi T, \qquad \omega_n' = \omega_n - \omega_\nu.$$

For small Δ, the above expression reduces to

$$Q_{\mu\mu}^0 = -\frac{Ne^2}{m} \left\{ 3\pi T \sum_{n=-\infty}^{\infty} \int \frac{d\Omega_v}{4\pi} x_\mu^2 \int_{-\infty}^{\infty} d\alpha\, \rho_0(\alpha, \Omega_v) \frac{\omega_n}{|\omega_r|} \frac{\Delta^2}{(\omega_n - i\alpha)^3} - \frac{3\pi\omega i}{4vq} + \frac{\pi i \omega \Delta}{2T} \left[1 - \ln\left(\frac{8\Delta}{|\omega|}\right) \right] \times \int \frac{d\Omega_v}{4\pi} 3x_\mu^2 \delta(\mathbf{v} \cdot \mathbf{q}) \int_{-\infty}^{\infty} d\alpha\, \rho_0(\alpha, \Omega_v) \cosh^{-2}\left(\frac{\alpha}{2T}\right) \right\}, \quad (26)$$

where x_μ is the direction cosine, $\rho_0(\alpha, \Omega)$ has been already defined, and $\Delta \equiv [\langle |\Delta(\mathbf{r})|^2 \rangle_{\text{Av}}]^{1/2}$. The second term in Eq. (26) gives the electromagnetic response in the normal state, which is the usual anomalous skin-limit expression.

On the other hand, it is easy to see that

$$Q_{\mu\nu}(\mathbf{q}, \omega) = 0 \qquad \text{for } \mu \neq \nu.$$

Finally, Q^0 depends not only on the direction of the microwave current, but also on the direction of the propagation vector $\mathbf{q}$. Because of the cylindrical symmetry of the problem,[10] it is necessary to distinguish between the two situations $\mathbf{q} \parallel Oz$ and $\mathbf{q} \perp Oz$.

Using Eqs. (18), (21), and (25), one obtains the correction $R_{\mu\nu}$ to $Q_{\mu\nu}$ from the collective oscillations. It follows from these equations that

$$Q_{\mu\nu} = 0, \qquad \text{if } \mu \neq \nu$$
$$Q_{zz} = Q_{zz}^0 = Q_{\parallel},$$
$$Q_{xx} = Q_{yy} = Q_\perp. \quad (27)$$

As in the dirty case, we do not find any "gyrotropic" effect (i.e., we find $Q_{xy} = Q_{yx} = 0$). This result neglects the Landau-splitting effects, which should give only a negligible contribution (i.e., $Q_{xy}/Q_{xx} \sim \Delta_{00}/E_F$ is the Fermi energy of the metal). Finally, we have considered only the case of a spherical Fermi surface, but the result still holds for cubic metals, like Nb or V. We then must consider three distinct geometries:

(a) $\mathbf{J} \parallel Oz$. In that case $\mathbf{q}$ is perpendicular to the static field, so that Q does not depend on its precise orientation in the (xy) plane

$$\text{Re}Q_{\parallel} = -\frac{3Ne^2\Delta^2}{8m(\pi T)^2} G_{\parallel}(\rho),$$

$$\text{Im}Q_{\parallel} = \frac{3\pi Ne^2\omega}{4mvq} \left\{ 1 - \frac{\Delta}{2T} \left[1 - \ln\left(\frac{8\Delta}{|\omega|}\right) \right] f_{\parallel}(\rho) \right\}, \quad (28)$$

where

$$G_{\parallel}(\rho) = -\int_0^1 \frac{z^2 dz}{(1-z^2)^{1/2}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2/(1-z^2)} \psi^{(2)}(\tfrac{1}{2} + i\rho x),$$

$$f_{\parallel}(\rho) = \frac{4}{\pi} \int_0^1 dz \frac{z^2}{1-z^2} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2/(1-z^2)} \cosh^{-2}(\pi\rho x). \quad (29)$$

and $\psi^{(2)}(z)$ is the tetra-gamma function. In Eqs. (28), Δ is given by

$$\Delta^2 = \frac{2\pi Tm}{3\pi eNG(\rho)} \frac{H_{c2} - H}{\beta_A [2\kappa_2^2(T) - 1]}, \quad \beta_A = 1.16$$

with

$$G(\rho) = -\int_0^1 \frac{dz}{2} (1-z^2)^{1/2} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2/(1-z^2)} \psi^{(2)}(\tfrac{1}{2} + i\rho x). \quad (30)$$

[10] The various directions in the (xy) plane are equivalent as long as the external momentum q is small compared with the scale $1/\xi_0$ characteristic of the Abrikosov structure.

The reactive part of the parallel component of the conductivity is proportional to Δ^2, so that the penetration depth along the magnetic field decreases like $(H_{c2}-H)^{-1/2}$, when H is decreased below H_{c2}. The leading correction to the normal-state absorptive part is proportional to $\Delta\ln(2\Delta/\omega)$; this increases as $(H_{c2}-H)^{1/2}$ in decreasing field. It seems that this conductivity could be measured by surface-impedance experiments only for pure materials with $\kappa\gg1$. In that geometrical situation, the surface impedance can be measured only on a sample face parallel to the external field. Thus, if $\kappa\sim1$, the contribution of the surface sheath will be significant and will obscure the interpretation of the measurements. The only way to circumvent that difficulty would be to coat the sample with a very thin magnetic layer which would destroy the surface sheath.[11]

However, measurements of transverse ultrasonic attenuation with a relatively high wave vector ($l^{-1}\ll q\ll\xi_0^{-1}$) should give an indirect measurement of Q_{zz}. This is discussed in Sec. IV.

(b) $\mathbf{J}\perp Oz$, $\mathbf{q}\parallel Oz$. In this case, the collective oscillations give rise to a finite correction to Q. This correction is of order Δ^2, thus we can neglect the correction to the absorptive part of the conductivity, since its leading term is of order Δ. From Eq. (26) we obtain

$$\mathrm{Re}Q_\perp{}^0=-[3Ne^2\Delta^2/8m(\pi T)^2]G(\rho),$$

$$\mathrm{Im}Q_\perp{}^0=\frac{3\pi Ne^2\omega}{4mvq}\left\{1-\frac{\Delta}{2T}\left[1-\ln\left(\frac{8\Delta}{|\omega|}\right)\right]f_\perp{}^{(1)}(\rho)\right\},\tag{31}$$

where

$$f_\perp{}^{(1)}(\rho)=\int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}e^{-x^2}\cosh^{-2}(\pi\rho x)$$

and $G(\rho)$ is given by Eq. (30).

The collective oscillations contribution is given by

$$\mathrm{Re}(2R_\perp)=[3Ne^2\Delta^2/8m(\pi T)^2]G(\rho),\tag{32}$$

where we have made use of Eqs. (18) and (21). Then we obtain

$$\mathrm{Re}Q_\perp=0,$$

$$\mathrm{Im}Q_\perp\cong\mathrm{Im}Q_\perp{}^0.\tag{33}$$

(c) $\mathbf{J}\perp Oz$, $\mathbf{q}\perp Oz$. As one can see in Eq. (26), $\mathrm{Re}Q_\perp{}^0$ does not depend on the direction of $\mathbf{q}$, the same holds for $R_\perp(\omega)$. Thus we obtain again

$$\mathrm{Re}Q_\perp=0.$$

Proceeding as in case (b) we find

$$\mathrm{Im}Q_\perp(\omega)\cong\mathrm{Im}Q_\perp{}^0(\omega)=\frac{3\pi Ne^2\omega}{4mvq}\left\{1-\frac{\Delta}{2T}\left[1-\ln\left(\frac{8\Delta}{|\omega|}\right)\right]f_\perp{}^{(2)}(\rho)\right\},\tag{34}$$

with

$$f_\perp{}^{(2)}(\rho)=\frac{4}{\pi}\int_0^1dz\int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}e^{-x^2(1-z^2)}\cosh^{-2}(\pi\rho x).$$

Thus we find that when the microwave current flows in the xy plane the reactive part of the conductivity due to direct current-current correlations is exactly canceled out (up to order Δ^2) by the contribution of the collective oscillations. Contrary to the case of dirty superconductors, this is true at all temperatures.

This means that the penetration depth of a microwave current perpendicular to the static field is much larger than in geometry (a) (it is now essentially controlled by the anomalous skin depth of the normal metal).

The surface impedance can be easily measured in geometry (b) and is given in terms of $Q_\perp$ and of the surface resistance in the normal state R_n by[12]

$$Z_\perp{}^{(b)}(\omega)/R_n=-2e^{2i\pi/3}\times\{1-(\Delta/2T)[1-\ln(8\Delta/|\omega|)]f_\perp{}^{(1)}(\rho)\}^{-1/3},\tag{35}$$

while in geometry (c), it seems again that the most simple way of measuring $Q_\perp$ would be by transverse ultrasonic attenuation experiments.

IV. TRANSVERSE ULTRASONIC ATTENUATION

The general expression for the attenuation of a transverse ultrasonic wave with a polarization vector $\mathbf{e}$ in the μ direction and a propagation vector $\mathbf{q}$ in the ν direction is given by[13]

$$\alpha^T(\mathbf{q},\mu,\nu)=\mathrm{Re}\left\{\frac{q^2}{i\omega\rho_{\mathrm{ion}}v_s}\left(\langle[\tau_{\mu\nu},\tau_{\nu\mu}]\rangle+\frac{(4\pi e^2/q^2)\langle[\tau_{\mu\nu},j_\mu]\rangle\langle[j_\mu,\tau_{\nu\mu}]\rangle}{1-(4\pi e^2/q^2)\langle[j_\mu,j_\mu]\rangle}\right)\right\},\tag{36}$$

where $\tau_{\mu\nu}$ is the $\mu\nu$ component of the stress tensor, j the current operator. This expression is valid insofar as one neglects the effects of collective oscillations.

The first term in Eq. (36) is the "collision-drag" term originally discussed by Tsuneto[14] and Kadanoff and Falko.[15] Let us note that the stress-tensor operator commutes with the time-reversal operator so that one can show that the correction to the collision-drag term due to collective effects (which is proportional to $\langle[\tau_{\mu\nu},\Psi]\rangle^2$) vanishes at low frequency as does ω^2. We can thus neglect that correction.

The second term in Eq. (36) arises from the electromagnetic current induced by the transverse wave. The collective modes modify only its denominator. This is

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due to the fact that $\tau_{\mu\nu}$ is a tensor of second order, while j_μ is a vector. Thus no collective oscillation can couple with both of them.

Therefore, in order to include in Eq. (36) the collective effects, it is sufficient to make the replacement

$$\langle[j_\mu, j_\mu]\rangle \to \langle[j_\mu, j_\mu]\rangle + 2R_{\mu\mu} = Q_{\mu\mu},$$

where $Q_{\mu\mu}$ has been calculated in Sec. III for the three different geometries.

The collision-drag term for pure type-II superconductors has been already discussed by one of us (K.M.),[9] so we will concentrate only on the electromagnetic term. It is important in actual experimental situations to take into account the fact that the electron mean free path l is finite although much bigger than ξ_0. As in the dirty case,[13] we can show that $\langle[\tau_{\mu\nu}, j_\nu]\rangle$ has exactly the same value as in the normal state[16]:

$$\langle[\tau_{\nu\mu}, j_\mu]\rangle_{q\omega} = -(p_0^3\omega/3\pi q)[1-g(ql)],$$

with

$$g(z) = \tfrac{3}{2}z^{-3}\{-z+(z^2+1)\arctan z\}. \qquad (37)$$

When $ql \ll 1$, $g(ql) \sim 1$, hence in order that the electromagnetic term be nonnegligible one needs $ql \gtrsim 1$. In order that we can still apply the results of Sec. III to the evaluation of $Q_{\mu\mu}$, we also need $q\xi_0 \ll 1$. This means that the measurements have to be done on very clean materials (in Nb, l should be at least of the order of 5000 Å). The expression of the electromagnetic term simplifies very much if q also satisfies the condition

$$q\lambda_L(v_F/v_s)^{1/2} \ll 1,$$

where $\lambda_L = (4\pi Ne^2/m)^{-1/2}$ is the London penetration depth and v_s the velocity of sound. (This condition can be satisfied only with very long mean free paths.) In this case the electromagnetic term reduces to

$$\alpha_{\text{el.m.}}^T = -\text{Re}\left\{\frac{q^2}{i\omega\rho_{\text{ion}}v_s}\frac{\langle[\tau_{\nu\mu}, j_\mu]\rangle_{q,\omega}^2}{Q_{\mu\mu}}\right\}. \qquad (38)$$

This can be expressed using the total attenuation coefficient for the transverse wave in the normal metal

$$\alpha_n^T = \frac{p_0^4}{3\pi}(l\rho_{\text{ion}}v_s)^{-1}\frac{1-g(ql)}{g(ql)}, \qquad (39)$$

$$\frac{\alpha_{\text{el.m.}}^T}{\alpha_n^T} = \frac{p_0^2}{3\pi}\omega l g(ql)[1-g(ql)]\frac{\text{Im}Q_{\mu\mu}}{|Q_{\mu\mu}|^2}. \qquad (40)$$

Again there are three different geometries:

(a) $\mathbf{e} \parallel Oz$.

$$\frac{\alpha_{\text{el.m.}}^T}{\alpha_n^T} = [1-g(ql)]\frac{1-(\Delta/2T)[1-\ln(8\Delta/|\omega|)]f_{\parallel}(\rho)}{\{1-(\Delta/2T)[1-\ln(8\Delta/|\omega|)]f_{\parallel}(\rho)\}^2 + [3\Delta^2 g^{-1}(ql)/8\pi(\pi T)^2\tau\omega]G_{\parallel}(\rho)}. \qquad (41)$$

In order to take into account the finite mean-free-path effects, one has to modify slightly the definition of $f_{\parallel}$ and $G_{\parallel}$. This amounts to making, in expressions like Eq. (26), the transformation $\mathbf{v}\cdot\mathbf{q} \to \mathbf{v}\cdot\mathbf{q} - i/\tau_{\text{tr}}$ (see for instance, Ref. 9). It is easy to see that such a replacement only affects the imaginary part of $Q_{\mu\mu}$ [and consequently $f_{\parallel}(\rho)$], since we are interested in the region $l/\xi_0 \gg 1$ [i.e., the correction on the reactive part is of the order of ξ_0/l]. Therefore $G_{\parallel}(\rho)$ is still given by (29) while we now have for $f_{\parallel}(\rho)$

$$f_{\parallel}(\rho) = \frac{3}{g(y)}\int_0^1 dz \frac{z^2}{(1-z^2)^{1/2}[1+y^2(1-z^2)]^{1/2}} \times \int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}e^{-x^2/(1-z^2)}\cosh^{-2}(\pi\rho x), \qquad y = ql, \qquad (42)$$

and $g(y)$ has been already defined in Eq. (37).

In the denominator of Eq. (41), the second term increases rapidly in the superconducting region, so that the electromagnetic term vanishes almost abruptly below H_{c2}. This appears as a discontinuity in the attenuation coefficient if ql is sufficiently large (i.e., $ql \gtrsim 1$).

(b) $\mathbf{e} \perp Oz$. In this case the attenuation coefficient is given by

$$\alpha_{\text{el.m.}}^{T(i)}/\alpha_n^T = [1-g(ql)] \times \{1-(\Delta/2T)[1-\ln(8\Delta/|\omega|)]f_\perp^{(i)}(\rho)\}^{-1}, \qquad (43)$$

where the $f_\perp^{(i)}(\rho)$ are now given by

$$f_\perp^{(1)}(\rho) = \frac{3}{2g(y)}\int_0^1 \frac{dz(1-z^2)^{1/2}}{1+(yz)^2} \times \int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}e^{-x^2/(1-z^2)}\cosh^{-2}(\pi\rho x),$$

$$f_\perp^{(2)}(\rho) = \frac{3}{2g(y)}\int_0^1 \frac{dz(1-z^2)^{1/2}}{[1+y^2(1-z^2)]^{1/2}} \times \int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}e^{-x^2/(1-z^2)}\cosh^{-2}(\pi\rho x). \qquad (44)$$

We have to choose the appropriate $f_\perp^{(i)}$ depending on the direction of the propagation vector $\mathbf{q}$. Thus the electromagnetic contribution to the attenuation coefficient is not screened out in the superconducting region, hence the total attenuation should change continuously through the transition point.

V. CONCLUSION

We have studied above the two collective modes, longitudinal and transverse, in detail. The longitudinal

[16] This is a consequence of the fact that $\tau_{\mu\nu}$ and j_ν have different transformation properties with respect to the time-reversal operation.

modes are found to be of diffusion type at all temperatures. The transverse modes are diffusionlike at high temperatures (i.e., $T\sim T_{c0}$), while they obey a damped propagation law at low temperatures. It is interesting to point out that their dispersion equations are nonanalytic in frequency and in momentum at $T=0°\text{K}$, which cannot be obtained by any power expansion.

It is shown that the transverse mode plays an important role in the electrical conductivity. In particular, we have seen that the reactive part of the complex conductivity vanishes identically in the region $H\sim H_{c2}$ independently of temperature if the microwave current flows in the plane perpendicular to the static magnetic field. This fact is closely connected with the degeneracy of the Abrikosov solutions at $H=H_{c2}$. Thus this exact cancellation of the reactive part of Q cannot occur in situations (like the surface sheath region) where such a degeneracy does not exist.

In the ideal situation where the quasiparticle has an infinite lifetime, the Abrikosov state can adjust itself to the low-frequency external magnetic field without absorption of energy. This can happen only if the external electric field is perpendicular to the static magnetic field so that the corresponding adjustment involves a real motion of the vortex lines (i.e., in the plane perpendicular to H). This takes place by means of virtual excitations of the transverse collective mode. Consequently, in the pure limit, we have obtained a complete cancellation of the reactive part of the complex conductivity at low frequency. In the dirty limit, on the other hand, we know that this cancellation happens only at temperatures close to T_{c0}. This is clear from the fact that, because of the impurity scattering, the quasiparticles cannot follow the motion of vortex lines instantaneously, which gives rise to the incomplete cancellation at low temperatures. This cancellation of the reactive part has significant consequences on the surface impedance and the transverse ultrasonic attenuation coefficients as discussed in the text, which can be easily verified experimentally.

Finally, the infrared absorption would probably reveal an anisotropy at low temperatures and for frequencies comparable to T_{c0}, due to real excitations of the transverse mode. Unfortunately we are not able to make a precise prediction about the position of the corresponding bump for a finite value of $H_{c2}-H$. It may be very difficult to observe this anisotropy for two reasons. First, the transverse mode is strongly damped. Secondly, as in the case of the surface impedance, it is difficult to achieve the geometry where the propagation vector $\mathbf{q}$ is perpendicular to the static magnetic field, since in that case the surface sheath might play an important role, especially if κ is small (as is the case for Nb and V). Therefore, it is probably necessary to destroy the surface sheath by depositing a thin layer of magnetic material in order to achieve this situation.

APPENDIX

We want to calculate retarded products of the type $\langle[\Psi^\dagger, \Psi]\rangle_{\mathbf{q}\mathbf{q}'\omega}$, $\langle[\Psi^\dagger, j_\mu]\rangle_{\mathbf{q}\mathbf{q}'\omega}$, $\cdots$, etc. In order to do such calculations, one first expresses products like $\Psi^\dagger\Psi$ in terms of the Gor'kov G and F Green's functions. It is then tempting to expand these functions in powers of the order parameter Δ and to keep the lowest-order terms, since Δ is small in the region of interest ($H\simeq H_{c2}$). However, it has been shown[6] that such an expansion does not always converge at low frequency ($\omega<T_{c0}$). In order to circumvent that difficulty a systematic procedure has been proposed by one of us (K.M.)[9]:

(1) One first calculates the corresponding retarded thermal product $R(\mathbf{q},\omega)$ for a BCS superconductor in zero field. This is usually given[17] by

$$R(\mathbf{q},\omega_0)=\text{const}\sum_{n=-\infty}^{\infty}\int\frac{d\Omega_v}{4\pi}\frac{h_\pm(\tilde{\omega}_n,\tilde{\omega}_n',\tilde{\Delta})Y_{lm}(\Omega_v)Y_{l'm'}(\Omega_v)}{(\tilde{\omega}_n{}^2+\tilde{\Delta}^2)^{1/2}+[(\tilde{\omega}_n')^2+\tilde{\Delta}^2]^{1/2}+\mathbf{v}\cdot\mathbf{q}},$$
$$\omega_n=(2n+1)\pi T,\qquad \omega_n'=\omega_n+\omega_0,\qquad \omega_0=4n_0\pi T,\tag{A1}$$

where n and n' are integers. $\tilde{\omega}$ and $\tilde{\Delta}$ are renormalized by the impurity scattering

$$\tilde{\omega}=\omega\eta_\omega,\qquad \tilde{\Delta}=\Delta\eta\tilde{\omega}_\omega,$$
$$\eta_\omega=\left[1+\frac{1}{2\tau(\omega^2+\Delta^2)^{1/2}}\right].\tag{A2}$$

In the pure limit this effect is always negligible, except in the case of ultrasonic attenuation and thermal conductivity.[9] In order to simplify the expressions, we will neglect it in the following. $h_\pm(\omega_n,\omega_n+\omega_0)$ is the relevant coherence factor

$$h_\pm(\omega_n,\omega_n+\omega_0)=1-\frac{\omega_n(\omega_n+\omega_0)\pm\Delta^2}{(\omega_n{}^2+\Delta^2)^{1/2}[(\omega_n+\omega_0)^2+\Delta^2]^{1/2}},$$

and $Y_{lm}(\Omega_v)$ is a spherical harmonic. l, m, l', m' are given by the transformation properties of the retarded product of interest. For example, $\langle[n,n]\rangle$ corresponds to $l=l'=0$, $\langle[j_z,j_z]\rangle$ corresponds to $l=l'=1$, $m=m'=0$, $\langle[j_+,j_-]\rangle$ to $l=l'=1$, $m=-m'=1$ (where $j_\pm=j_x\pm ij_y$), etc.

(2) Then one generalizes expression (A1) to the case of a current-carrying state by doing the transformation $\omega\to\omega-i\alpha$ with $\alpha=\mathbf{v}\cdot\mathbf{q}_s$, where $\mathbf{q}_s$ is the momentum of each pair, associated with the supercurrent.

(3) In the type-II superconductor (where the current is spatially varying) $\mathbf{q}_s$ no longer has a unique

[17] This expression is valid for all retarded products except the one relevant to the calculation of the nuclear-spin-relaxation time. This is due to the fact that nuclear relaxation involves spin operators which are completely localized. This corresponds to large momentum transfers.

value, but can take all values. So, an average has to be taken on the distribution of values of $\mathbf{q}_s$, which is given by the spectral function $\rho_0(\alpha, \Omega)$, so that

$$R(\mathbf{q}, \omega_0) = \text{const} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\alpha \int \frac{d\Omega_v}{4\pi} \rho_0(\alpha, \Omega_v) \frac{h_\pm(\omega_n - i\alpha, \omega_n + \omega_0 - i\alpha) Y_{lm}(\Omega_v) Y_{l'm'}(\Omega_v)}{[(\omega_n - i\alpha)^2 + \Delta^2]^{1/2} + [(\omega_n + \omega_0 - i\alpha) + \Delta^2]^{1/2} + \mathbf{q} \cdot \mathbf{v}}, \quad (A3)$$

$\Delta^2 = \langle | \Delta(\mathbf{r}) |^2 \rangle_{\text{Av}}$ and $\rho_0(\alpha, \Omega)$ is a normalized function defined, as in Eq. (5), by

$$\rho_0(\alpha, \Omega_v) = 2 \int d^3r \int d^3r' \int \frac{d^3q}{(2\pi)^3} \left\{ \exp\left[i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}') + 2ie \int_{\mathbf{r}'}^{\mathbf{r}} \mathbf{A}(\mathbf{l}) \cdot d\mathbf{l} \right] \delta(\mathbf{v} \cdot \mathbf{q} + 2\alpha) \phi_0^*(x) \phi_0(x') \right\}, \quad (A4)$$

where

$$\phi_0(x) = \left(\frac{2eH}{\pi}\right)^{1/4} (L_y L_z)^{-1/2} \exp(-eHx^2). \quad (A5)$$

L_y and L_z are the dimensions of the sample.

We can easily carry out the integration in (A4) and find,[6]

$$\rho_0(\alpha, \Omega) = [(\sqrt{\pi}) \epsilon \sin\theta]^{-1} \exp[-(\alpha/\epsilon \sin\theta)^2]. \quad (A6)$$

Expression (A3) evaluates only the $\mathbf{q} = \mathbf{q}'$ elements of the general-response function $R(\mathbf{q}, \mathbf{q}', \omega)$. This is sufficient if the external perturbation is slowly varying in space, namely, if $q\xi_0 \ll 1$. Equation (A3) can then be used to obtain the behavior of $R(\mathbf{q}, \omega_0)$ for small values of Δ. Two cases can occur:

(a) If, for small Δ, $R(q, \omega_0)$ can be expanded in powers of Δ^2, and if the development is convergent, the first term (coefficient of Δ^2) is equivalent to the one that would be obtained by a direct development of the Green's functions. This is, for example, the case for the reactive part of the electrical conductivity.

(b) If $R(q, \omega_0)$ cannot be expanded in powers of Δ^2, expression (A3) gives, to lowest order in Δ, a formal summation of the divergent-power series one would obtain by an expansion of the Green's functions. [Equation (A3) then gives a correct description of the singular part of R considered as a function of Δ.]

Let us now calculate, for instance, $\langle [j_z, j_z] \rangle_{\mathbf{q}, \omega}$. Since the coherence factor for the current operator is h_-, we have

$$\langle [j_z, j_z] \rangle_{\mathbf{q}, \omega_0} = -\frac{Ne^2}{m} \left\{ 1 + 3\pi T \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{\infty} d\alpha \int \frac{d\Omega_v}{4\pi} \rho_0(\alpha, \Omega_v) \cos^2\theta_v \frac{h_-(\omega_n - i\alpha, \omega_n + \omega_0 - i\alpha)}{[(\omega_n - i\alpha)^2 + \Delta^2]^{1/2} + [(\omega_n + \omega_0 - i\alpha)^2 + \Delta^2]^{1/2} + \mathbf{q} \cdot \mathbf{v}} \right\}. \quad (A7)$$

After analytical continuation we have (replacing ω_0 by $i\omega$)

$$\langle [j_z, j_z] \rangle_{\mathbf{q}, \omega} = \frac{Ne^2}{m} \left\{ 3\pi T \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{\infty} d\alpha \int \frac{d\Omega_v}{4\pi} \frac{\Delta^2 \rho_0(\alpha, \Omega_v) \cos^2\theta_v}{[(\omega_n - i\alpha)^2 + \Delta^2]^{3/2}} - 3\pi i \int_{-\infty}^{\infty} d\alpha \int \frac{d\Omega_v}{4\pi} \rho_0(\alpha, \Omega_v) \cos^2\theta_v \, \delta(\mathbf{v} \cdot \mathbf{q}) \right.$$
$$\left. \times \int_{\Delta+\alpha}^{\infty} d\omega' \left(\tanh\frac{\omega + \omega'}{2T} - \tanh\frac{\omega'}{2T} \right) \frac{(\omega' - \alpha)(\omega + \omega' - \alpha) + \Delta^2}{[(\omega' - \alpha)^2 - \Delta^2]^{1/2} \times [(\omega + \omega' - \alpha)^2 - \Delta^2]^{1/2}} \right\}, \quad (A8)$$

where we have assumed $\omega \ll T_{c0}$.

Finally, for small Δ (i.e., $\Delta \ll T_{c0}$) we can further simplify (A7) and obtain (we put here $\langle [j_z, j_z] \rangle_{\mathbf{q}\omega} = Q_{zz}$)

$$\text{Re}Q_{zz} = -\frac{Ne^2}{m} \frac{3\Delta^2}{(2\pi T)^2} \int_{-\infty}^{\infty} d\alpha \int \frac{d\Omega_v}{4\pi} \cos^2\theta_v \rho_0(\alpha, \Omega_v) \psi^{(2)}\left(\frac{1}{2} - \frac{i\alpha}{2\pi T}\right),$$

$$\text{Im}Q_{zz} = \frac{Ne^2}{m} \left\{ \frac{3\pi\omega}{4qv} - \frac{3\pi\omega\Delta}{2T} \left[1 - \ln\left(\frac{8\Delta}{|\omega|}\right) \right] \int_{-\infty}^{\infty} d\alpha \int \frac{d\Omega_v}{4\pi} \cos^2\theta_v \delta(\mathbf{v} \cdot \mathbf{q}) \rho_0(\alpha, \Omega_v) \cosh^{-2}\left(\frac{\alpha}{2T}\right) \right\}, \quad (A9)$$

where $\psi^{(2)}(z)$ is a tetra-gamma function. We note here that $\text{Re}Q_{zz}$ is of the order of Δ^2. An identical expression can be obtained for $\text{Re}Q_{zz}$ by a direct expansion of the Green's functions in powers of $\Delta(\mathbf{r})$. On the other hand, the superconducting correction to $\text{Im}Q_{zz}$ begins with $\Delta \ln(\Delta/\omega)$, which cannot be obtained by an expansion of Green's function in powers of $\Delta(\mathbf{r})$. This singular behavior in Δ is a consequence of a coalescence of two singularities in the densities of states,[9] which also appears in the BCS case.[12] The above prescription does not apply if one calculates the retarded products, which are relevant to the fluctuations of the order parameter (which involves the operator Ψ or $\Psi^\dagger$). They are of two types.

(1) $\langle [\Psi^\dagger, \Psi] \rangle_{\mathbf{q}\omega}$ needs to be calculated only to zeroth order in Δ (i.e., we can evaluate the above average over the normal state). In order to compute this average, it is convenient to decompose into the complete set of eigenfunctions of the operator $\Pi^2 = (\mathbf{q} - 2e\mathbf{A})^2$

$$\bar{\Psi}^\dagger(\mathbf{r}) = \sum_{n,k,p_z} \Psi_{n,k,p_z}{}^\dagger \phi_{nkp_z}{}^*(\mathbf{r}), \quad (A10)$$

where $\phi_{nkp_z}(\mathbf{r})$ has been already defined in Sec. II. In terms of $\Psi_{nkp_z}{}^*$ the above retarded product is evaluated as

$$\langle[\Psi^\dagger(\mathbf{r}),\Psi(\mathbf{r}')]\rangle_\omega=\sum_{nkp_z}\phi_{nkp_z}(\mathbf{r})\phi_{nkp_z}{}^*(\mathbf{r}')I_{nkp_z}(\omega), \tag{A11}$$

$$\begin{aligned}I_{nkp_z}(\omega_0)&=\langle[\Psi_{nkp_z}{}^\dagger,\Psi_{nkp_z}]\rangle_{\omega_0}\\&=T\sum_{m=-\infty}^{\infty}\int\frac{d^3p'}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\{[i\omega_m-\xi_{p'}][i(\omega_m+\omega_0)+\xi_{p'}-\mathbf{v}_{p'}\cdot\mathbf{q}]\}^{-1}\\&\quad\times\int d^3r\int d^3r'\exp\left[i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')+2ie\int_{\mathbf{r}}^{\mathbf{r}'}\mathbf{A}(1)\cdot d\mathbf{l}\right]\phi_{nkp_z}{}^*(\mathbf{r}')\phi_{nkp_z}(r)\\&=N(0)\pi T\sum_{m=-\infty}^{\infty}\int\frac{d\Omega_v}{4\pi}\int_{-\infty}^{\infty}d\alpha\,\rho_{nn}(\alpha,\Omega_v)\,\frac{\theta[\omega_m(\omega_m+\omega_0)]}{\omega_m+\frac{1}{2}\omega_0-i\alpha-\frac{1}{2}iv_zp_z},\end{aligned}$$

$$I_{nkp_z}(\omega)=N(0)\int_{-\infty}^{\infty}d\alpha\int\frac{d\Omega_v}{4\pi}\rho_{nn}(\alpha,\Omega_v)\left\{\ln\frac{\omega_D}{2\pi T}-\psi\left(\frac{1}{2}+\frac{i\omega+ivp_z\cos\theta_v}{4\pi T}+\frac{i\alpha}{2\pi T}\right)\right\}, \tag{A12}$$

$$\rho_{nn'}(\alpha,\Omega_v)=\int\frac{d^3q}{(2\pi)^3}\int d^3r\int d^3r'\left\{\exp\left(i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')+2ie\int_{\mathbf{r}}^{\mathbf{r}'}\mathbf{A}(1)\cdot d\mathbf{l}\right)\phi_{n00}(\mathbf{r})\phi_{n'00}{}^*(\mathbf{r}')\right\}2\delta(\mathbf{v}\cdot\mathbf{q}+2\alpha), \tag{A13}$$

and the $\Phi_{nk0}(\mathbf{r})$ are the normalized eigenfunctions.

In the above derivation we have made use of the orthogonality

$$\int\frac{d\Omega}{4\pi}\int_{-\infty}^{\infty}d\alpha\,\rho_{nn'}(\alpha,\Omega)=\delta_{nn'}. \tag{A14}$$

We note also here that the $\rho_n(\alpha,\Omega)$ defined in Sec. II are given by

$$\rho_n(\alpha,\Omega)\equiv\rho_{nn}(\alpha,\Omega). \tag{A15}$$

(2) $\langle[\Psi,n]\rangle$ and $\langle[\Psi,j_\mu]\rangle$ have lowest-order terms of the order of Δ. First, we shall consider $\langle[\Psi,n]\rangle$. Taking only the first-order term in the development of the Green's functions in powers of $\Delta(\mathbf{r})$ we have

$$\begin{aligned}&\langle[\Psi,n]\rangle_{\mathbf{q},\mathbf{q}',\omega(=-i\omega_0)}\\&=T\sum_k\sum_{n=-\infty}^{\infty}\int\frac{d^3p'}{(2\pi)^3}\int\frac{d^3q''}{(2\pi)^3}\int d^3r\int d^3r'\,\{[i\omega_n-\xi_{p'}][i\omega_n+\xi_{p'}+\mathbf{v}'\cdot\mathbf{q}][i(\omega_n+\omega_0)+\xi_{p'}+\mathbf{v}'\cdot(\mathbf{q}+\mathbf{q}')]\}^{-1}\\&\quad\times\exp\left[i\mathbf{q}''\cdot(\mathbf{r}-\mathbf{r}')+2ie\int_{\mathbf{r}}^{\mathbf{r}'}\mathbf{A}(1)\cdot d\mathbf{l}\right]\Delta^*(\mathbf{r})\phi_{0k0}(\mathbf{r}')\phi_{0kq'_z}{}^*(\mathbf{q})\\&=-N(0)\Delta_{\mathbf{q}-\mathbf{q}'}{}^\dagger\int\frac{d\Omega_v}{4\pi}\int_{-\infty}^{\infty}d\alpha\,\rho_0(\alpha,\Omega_v)\left\{\frac{2\omega}{\omega^2-(\mathbf{v}\cdot\mathbf{q}')^2}\left[\psi\left(\frac{1}{2}+\frac{-i\omega+i\alpha}{2\pi T}\right)-\psi\left(\frac{1}{2}-\frac{i(\omega+\mathbf{v}\cdot\mathbf{q}')}{4\pi T}+\frac{i\alpha}{2\pi T}\right)\right]\right\},\end{aligned} \tag{A16}$$

where we have restricted Ψ to the longitudinal modes (i.e., to its projection on the eigenstates $n=0$).

In the derivation of the above expression, we have made use of the reflection invariance of the spectral function

$$\rho_0(\alpha,\Omega)=\rho_0(-\alpha,\Omega). \tag{A17}$$

We note that $\langle[\Psi,n]\rangle$ vanishes like ω for small frequency. It is easily shown that this low-frequency behavior is common to any operator which conserves the time-reversal symmetry (e.g., $\tau_{\mu\nu}$ the stress-tensor operator and $j_{E\mu}$ the energy-current operator). We can calculate $\langle[\Psi,j_\mu]\rangle$ in a similar way. The only difference is that we now have to take the projection of Ψ on $n=1$ eigenstate (i.e., the transverse mode), which introduces the spectral function

$$\rho_{10}(\alpha,\Omega)=-\frac{2i\alpha e^{i\phi}}{(2\pi)^{1/2}(\epsilon\sin\theta)^2}\exp\left[-\left(\frac{\alpha}{\epsilon\sin\theta}\right)^2\right]. \tag{A18}$$

$\langle[\Psi,j_\mu]\rangle$ is given by

$$\begin{aligned}\langle[\Psi,j_-]\rangle_{\mathbf{q}\mathbf{q}'\omega}&=-N(0)\int\frac{d\Omega_v}{4\pi}\int_{-\infty}^{\infty}d\alpha\,v_-\rho_{10}(\alpha,\Omega_v)\\&\quad\times\frac{2\mathbf{v}\cdot\mathbf{q}'}{\omega^2-(\mathbf{v}\cdot\mathbf{q}')^2}\left[\psi\left(\frac{1}{2}+\frac{-i\omega+i\alpha}{2\pi T}\right)\right.\\&\quad\left.-\psi\left(\frac{1}{2}-\frac{i(\omega+\mathbf{v}\cdot\mathbf{q}')}{4\pi T}+\frac{i\alpha}{2\pi T}\right)\right]\left[\left(\frac{\Pi^-}{(4eH)^{1/2}}\right)\Delta^\dagger\right]_{\mathbf{q}-\mathbf{q}'},\\\langle[\Psi,j_+]\rangle_{\mathbf{q}\mathbf{q}'\omega}&=0,\\\langle[\Psi,j_z]\rangle_{\mathbf{q},\mathbf{q}',\omega}&=O(q_z).\end{aligned} \tag{A19}$$

In the limit $\omega\to0$, the above expression gives rise to a finite value of $\langle[\Psi,j_-]\rangle$, which is a consequence of the odd parity of the spectral function involved [i.e., $\rho_{10}(-\alpha,\Omega)=-\rho_{10}(\alpha,\Omega)$].