Fluctuations of the Order Parameter in Type-II Superconductors. I. Dirty Limit*

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We study the time dependence of fluctuations of the order parameter in dirty type-II superconductors in the high-field region ($H\simeq H_{c2}$). In this domain we can calculate their dispersion relation without taking into account the other collective modes (i.e., density and current oscillations), since the coupling of these oscillations with the order-parameter fluctuations is small (i.e., of the order of Δ, which vanishes at $H=H_{c2}$). Among many modes the following two are of special interest: the longitudinal mode, which couples to the density oscillations, and the transverse mode, which couples to the current oscillations. The longitudinal mode is of diffusion type at all temperatures. The transverse one is essentially damped, except in the vicinity of T_{c0} (critical temperature in zero field), where it is also diffusionlike. The thermal conductivity, ultrasonic attenuation, and spin susceptibility are not affected by the existence of these fluctuations. We calculate the transverse conductivity, which is strongly modified, showing an important anisotropy, which should be easily seen in surface-impedance measurements.

I. INTRODUCTION

THE properties of type-II superconductors have been extensively studied in the last few years.[1] In particular we are able to describe not only equilibrium, but also nonequilibrium properties of dirty type-II superconductors in the high-field region (i.e., in the vicinity of their upper critical field) in terms of gapless superconductivity.[2]

The previous calculations of thermal[3] and electrical[4] conductivity show that, at least as far as we are concerned with terms of order $|\Delta|^2$ (Δ being the order parameter), these properties are given by expressions equivalent to those obtained in other gapless situations (e.g., superconductors containing magnetic impurities). In fact, it is possible to establish such an equivalence relation for all the response functions[5] except the magnetic susceptibility. (The reason for that difference is the same as in the case of a superconducting thin film in the presence of a parallel magnetic field.) This result is true insofar as one neglects the effect of fluctuations of the order parameter, which is then given—for $H\simeq H_{c2}$ (i.e., in the perpendicular-field case)—by the Abrikosov solution.

The purpose of the present paper is to study the time variations of the order parameter, their coupling to external perturbations, and their effect on various transport properties of dirty type-II superconductors.

In the next section we develop a formalism which enables us to deal with the problem of fluctuations of the Abrikosov structure in a quite general way. The method is very similar to the one used by Kadanoff and Falko[6] in their study of the effects of Coulomb screening.

In Sec. III we concentrate on the solution of the equation describing the time evolution of these fluctuations for dirty type-II superconductors in the high-field region ($H\sim H_{c2}$). We find that, among many modes, two are of special physical interest: The first one is longitudinal; it couples to the density oscillations. It is of diffusion type at all temperatures. The second is transverse and couples to the current oscillations. At low temperatures ($T\ll T_{c0}$) it is essentially a damped oscillation; when $T\to T_{c0}$ its damping decreases and it becomes also of diffusion type.

These results hold provided the scales of time and space variations, ω^{-1} and q^{-1}, satisfy the "dirty limit" conditions: $\omega^{-1}\gg\tau$ and $q^{-1}\gg l$. (τ and l are the electron collision time and mean free path, and ξ_0 is the BCS coherence length of the pure-host superconductor.) This is a generalization of the result previously obtained by Abrahams and Tsuneto[7] in the region $T\sim T_{c0}$. The contribution of these modes to the thermal conductivity, the ultrasonic attenuation coefficients, and the spin susceptibility turns out to be negligible. Therefore, these responses are still given by expressions equivalent to the corresponding ones calculated for other gapless situations.[2]

However, these modes play an important role in the transverse electromagnetic conductivity (calculated in Sec. IV) which becomes strongly anisotropic. This new contribution is specific to the detailed form of the order parameter at equilibrium. Thus it cannot be generalized by an equivalence relation[5] to the other gapless situations.

Therefore, the observation of the electromagnetic

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1 K. Maki, Physics 1, 21 (1964); P. G. de Gennes, Physik Kondensierten Materie 3, 79 (1964); C. Caroli, M. Cyrot, and P. G. de Gennes, Solid State Commun. 4, 17 (1966).
2 K. Maki, in *Superconductivity*, edited by R. Parks (Marcel Dekker Publishing Company, Inc., to be published).
3 C. Caroli and M. Cyrot, Phys. Kondensierten Materie 4, 285 (1965).
4 K. Maki, Phys. Rev. 141, 331 (1966).
5 P. Fulde and K. Maki, Solid State Commun. 5, 21 (1967).
6 L. P. Kadanoff and I. I. Falko, Phys. Rev. 136, A1170 (1964).
7 E. Abrahams and T. Tsuneto, Phys. Rev. 152, 416 (1966).

response (e.g., by surface-impedance measurements) should demonstrate the existence of the transverse oscillations in dirty type-II superconductors.

More generally, the following two criteria allow us to tell which response functions contain a nonnegligible contribution from these collective modes: (1) The perturbation has to break time reversal symmetry; (2) that perturbation has to be spin-independent, since the collective oscillations all belong to the spin-singlet state.

II. FORMULATION

We shall consider here the change of the superconducting order parameter Δ induced by an external perturbation δU (which can be a vector potential, scalar potential, etc.). Extending a linear response theory, we can express this change in Δ in terms of correlation functions (i.e., retarded products of operators). First of all, let us write down the total perturbing Hamiltonian which is given by

$$H_{\mathrm{I}}=\int(V\delta U+\Psi^{\dagger}\delta\Delta+\delta\Delta^{\dagger}\Psi)\,d^3r, \tag{1}$$

where V is written as a bilinear form in the electron-field operators

$$\psi_\sigma(\mathbf{r}) \quad\text{and}\quad \psi_\sigma^{\dagger}(\mathbf{r}),$$

and

$$\Psi^{\dagger}(\mathbf{r},t)=\psi_{\uparrow}^{\dagger}(\mathbf{r},t)\psi_{\downarrow}^{\dagger}(\mathbf{r},t).$$

Here the first term in Eq. (1) is the external perturbation. The second and the third terms are due to the variation of the interaction Hamiltonian;

$$H_g=\int[\Psi^{\dagger}(\mathbf{r},t)\Delta(\mathbf{r},t)+\Delta^{\dagger}(\mathbf{r},t)\Psi(\mathbf{r},t)]d^3r \tag{2}$$

in the presence of the perturbation δU. $\Delta(r, t)$ in Eq. (2) has to be determined self-consistently by

$$\Delta(\mathbf{r},t)=-|g|\langle\Psi(\mathbf{r},t)\rangle. \tag{3}$$

In the spirit of the linear response theory[6] the change of the order parameter $\delta\Delta^{\dagger}$ induced by the perturbation H_{I} is written as

$$\delta\Delta^{\dagger}=-|g|\langle[\Psi^{\dagger},V]\rangle\delta U+|g|\langle[\Psi^{\dagger},\Psi]\rangle\delta\Delta^{\dagger}$$

or

$$\delta\Delta_{\mathbf{q}\omega}{}^{\dagger}=-\big((1-|g|\langle[\Psi^{\dagger},\Psi]\rangle)^{-1}|g|\langle[\Psi^{\dagger},V]\rangle\delta U\big)_{\mathbf{q}\omega}. \tag{4}$$

Here ω is the frequency, $\mathbf{q}$ the wave vector, and $\langle[\ ]\rangle$ means the average value of the retarded product taken on the Gibbs ensemble of the unperturbed state of the system. $\{\ \}_{\omega\mathbf{q}}$ is a symbolic form of the expression

$$\delta\Delta_{\mathbf{q}\omega}{}^{\dagger}=-|g|\times\int\frac{d^3q'}{(2\pi)^3}\int\frac{d^3q''}{(2\pi)^3}\{1-|g|\langle[\Psi^{\dagger},\Psi]\rangle\}^{-1}{}_{\mathbf{q},\mathbf{q}''\omega}\times\langle[\Psi^{\dagger},V]\rangle_{\mathbf{q}'',\mathbf{q}'\omega}\delta U_{\mathbf{q}'\omega}. \tag{5}$$

As one can easily see from the above expression, the zeros of the denominator

$$1-|g|\langle[\Psi^{\dagger},\Psi]\rangle_{\mathbf{q}\omega}=0 \tag{6}$$

give rise to collective modes associated with fluctuations of the order parameter.

[In Eq. (6) we take $\langle[\Psi^{\dagger},\Psi]\rangle_\omega$ to be diagonal in momentum, since we will see later that in the high-field region it is sufficient to calculate this retarded product to zeroth order in the order parameter; i.e., the equilibrium state is then the normal state, which is translationally invariant.]

Strictly speaking, the reduction of the original pair interaction gives rise not only to H_g [Eq. (2)] but also to a density–density coupling of the form

$$g\int d^3r\, n(\mathbf{r},t)\langle n(\mathbf{r},t)\rangle, \tag{7}$$

so that the complete equation for the collective modes should be, instead of (6),

$$\{[1-|g|\langle[\Psi^{\dagger},\Psi]\rangle+2|g|\langle[n,\Psi]\rangle\times(1-|g|\langle[\Psi^{\dagger},\Psi]\rangle)^{-1}\langle[\Psi^{\dagger},n]\rangle]\delta\Delta\}_{\omega\mathbf{q}}=0. \tag{8}$$

This gives rise to the Anderson–Bogoliubov modes of the neutral superconductor.[8,9]

In the high-field region ($H\sim H_{c2}$) the first term in Eq. (8) is of zeroth order in Δ (the order parameter), while the last one can be easily shown to be of order $|\Delta|^2$. Since in this region Δ is vanishingly small, this last term is negligible and Eq. (8) is reduced to Eq. (6).

When we choose the equilibrium state in which the average is taken in Eq. (6) to be the normal state, the solution with $\omega=0$ determines the transition temperature or the critical field[10] (H_{c2} or H_{c3} depending on the geometry) and the corresponding eigensolution for $\Delta(\mathbf{r})$ (i.e., the Abrikosov solution[11] or the de Gennes–Saint James solution[12]). The linearized time-dependent Ginzburg–Landau equation[13] for the order parameter can be derived simply from Eq. (6) by expanding $\langle[\Psi^{\dagger},\Psi]\rangle_{\omega\mathbf{q}}$ in powers of ω as well as $\mathbf{q}$.

We can describe the total variation in the average value of an operator W [also bilinear in $\psi_\sigma(\mathbf{r})$ and

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[8] J. R. Schrieffer, *Theory of Superconductivity* (W. A. Benjamin, Inc., New York, 1964), p. 174.

[9] P. W. Anderson, Phys. Rev. **112,** 1900 (1958); N. N. Bogoliubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau Enterprises, Inc., New York, 1959).

[10] One should notice that this is true only because the transition at the upper critical field is of second order. When the transition is of first order, the $\omega=0$ solution of Eq. (8) determines the supercooling field, see for instance L. P. Gor'kov, Zh. Eksperim i Teor. Fiz. **37,** 833 (1959) [English transl.: Soviet Phys.—JETP **10,** 593 (1960)].

[11] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32,** 1442 (1957) [English transl.: Soviet Phys.—JETP **5,** 1174 (1957)].

[12] P. G. de Gennes and D. Saint James, Phys. Letters **4,** 151 (1963).

[13] See for instance, Ref. 7.

$\psi_\sigma^\dagger(\mathbf{r})]$ induced by the external perturbation δU as

$$\delta\langle W\rangle_{\omega\mathbf{q}}=\{\langle[W,V]\rangle\delta U+\langle[W,\Psi^\dagger]\rangle\delta\Delta+\langle[W,\Psi]\rangle\delta\Delta^\dagger\}_{\omega\mathbf{q}}$$
$$=\{[\langle[W,V]\rangle+|g|\langle[W,\Psi^\dagger]\rangle(1-|g|\langle[\Psi^\dagger,\Psi]\rangle)^{-1}\langle[\Psi,V]\rangle$$
$$+|g|\langle[W,\Psi]\rangle(1-|g|\langle[\Psi^\dagger,\Psi]\rangle)^{-1}\langle[\Psi^\dagger,V]\rangle]\delta U\}_{\omega\mathbf{q}}, \quad (9)$$

where the first term in the bracket gives the ordinary response discussed previously, while the second and third terms are due to the fluctuations in the order parameter, as described above. Therefore the present formalism enables us to analyze these fluctuations and their effects on various responses, simply in terms of retarded products which can be calculated easily by means of the usual thermal Green's-function techniques. In the charged system we have to consider the electromagnetic interaction terms, which are treated in the following. We here use the convention that the vector potential is purely transverse and that the scalar potential describes the total longitudinal component.

A. Longitudinal Response

Since a scalar potential couples to the density operator $n=\sum_\sigma\psi_\sigma^\dagger(\mathbf{r})\psi_\sigma(\mathbf{r})$, it induces a variation of the charge density, which according to Eq. (9), is given by

$$\delta\rho_{\mathbf{q}\omega}=e\{e\langle[n,n]\rangle\delta\phi+\langle[n,\Psi^\dagger]\rangle\delta\Delta+\langle[n,\Psi]\rangle\delta\Delta^\dagger\}_{\mathbf{q}\omega}, \quad (10)$$

where

$$\delta\Delta_{\mathbf{q}\omega}^\dagger=e|g|\{(1-|g|\langle[\Psi^\dagger,\Psi]\rangle^L)^{-1}\langle[\Psi^\dagger,n]\rangle\delta\phi\}_{\mathbf{q}\omega}, \quad (11)$$

$$\delta\Delta_{\mathbf{q}\omega}=e|g|\{(1-|g|\langle[\Psi,\Psi^\dagger]\rangle^L)^{-1}\langle[\Psi,n]\rangle\delta\phi\}_{\mathbf{q}\omega}, \quad (12)$$

$\delta\phi$ is the total scalar potential, and the symbol $\langle\ \rangle^L$ means that we have selected the longitudinal part of $\langle[\Psi,\Psi^\dagger]\rangle$.

Combining Eqs. (10) to (12) with Maxwell equations, and eliminating $\delta\Delta$ and $\delta\Delta^\dagger$, we have

$$(4\pi e^2)^{-1}[q^2-\omega^2]\delta\phi_{\mathbf{q}\omega}{}^i=\{[\langle[n,n]\rangle+|g|\langle[n,\Psi^\dagger]\rangle(1-|g|\langle[\Psi,\Psi^\dagger]\rangle^L)^{-1}\langle[\Psi,n]\rangle$$
$$+|g|\langle[n,\Psi]\rangle(1-|g|\langle[\Psi^\dagger,\Psi]\rangle^L)^{-1}\langle[\Psi^\dagger,n]\rangle]\delta\phi\}_{\omega\mathbf{q}}, \quad (13)$$

where $\delta\phi_{\mathbf{q}\omega}{}^i$ is the induced scalar potential. Here we can neglect the term $\omega^2\delta\phi_{\omega\mathbf{q}}{}^i$ which is much smaller than the first term on the left-hand side of Eq. (13).

As in the case of Eq. (10), in the high-field region which we are interested in, the last two terms in Eq. (13) are of order $|\Delta|^2$, so they can be neglected in the study of plasma oscillations, which then satisfy the usual equations for the normal state (i.e., we set $\delta\phi^i=\delta\phi$ in the absence of any driving potential).

$$1-(4\pi e^2/q^2)\langle[n,n]\rangle_{\mathbf{q}\omega}=0. \quad (14a)$$

Proceeding in exactly the same way, we can eliminate $\delta\Delta$ and $\delta\phi$ from Eqs. (10) to (12), and we obtain the dispersion equation for the longitudinal fluctuations of the order parameter to lowest order in Δ

$$1-|g|\langle[\Psi^\dagger,\Psi]\rangle_{\mathbf{q}\omega}{}^L=0. \quad (14b)$$

Generally speaking, in the charged system, because of the long-range nature of the Coulomb interaction, the coupling between these two types of modes is by no means negligible. In fact, in the BCS case the fluctuations of the order parameter couple to the density fluctuations so strongly that the corresponding modes are removed into the region of plasma frequency.[9] It is only because of the smallness of Δ in the vicinity of a second-order transition that we can here neglect this coupling (e.g., this is also true for the BCS case in the vicinity of T_{c0}).

B. Transverse Response

We now apply the general expression (9) to the response to a transverse vector potential described by the Hamiltonian:

$$\mathcal{H}_T=\int j_\mu\delta A_\mu d^3r,$$

where j_μ is the μ component of the current operator.

$$j_\mu=\sum_\sigma(2mi)^{-1}\{(\nabla_\mu\psi_\sigma^\dagger(\mathbf{r})\psi_\sigma(\mathbf{r})-\psi_\sigma^\dagger(\mathbf{r})\nabla_\mu\psi_\sigma(\mathbf{r}))\}.$$

We then obtain for the change in the current density

$$\delta\langle j_\mu\rangle_{\mathbf{q}\omega}=\{[\langle[j_\mu,j_\nu]\rangle+|g|\langle[j_\mu,\Psi^\dagger]\rangle(1-|g|\langle[\Psi^\dagger,\Psi]\rangle^T)^{-1}\langle[\Psi,j_\nu]\rangle$$
$$+|g|\langle[j_\mu,\Psi]\rangle(1-|g|\langle[\Psi^\dagger,\Psi]\rangle^T)^{-1}\langle[\Psi^\dagger,j_\nu]\rangle]\delta A\}_{\mathbf{q}\omega}. \quad (15)$$

$\langle[\Psi^\dagger, \Psi]\rangle^T$ is the transverse part of $\langle[\Psi^\dagger, \Psi]\rangle$. Combining (15) with Maxwell's equations, we obtain an equation which determines the transverse collective modes. For the same reasons as in the longitudinal case, provided we concentrate on the high-field region, the coupling between current- and order-parameter fluctuations can be neglected. Thus there are two distinct classes of transverse modes: The mode associated with current fluctuations is, in the pure case, the helicon mode; the second one is a transverse fluctuation of the order parameter, and has a dispersion relation analogous to (14b).

III. COLLECTIVE MODES IN THE HIGH-FIELD REGION ($H \sim H_{c2}$)

The formalism that we have developed in the preceding section enables us to analyze the collective modes in terms of retarded products.

In this section we restrict ourselves to type-II superconductors in the high-field domain corresponding to a perpendicular-field goemetry; i.e., $H\sim H_{c2}$. In fact, the same kind of treatment could be applied to the surface superconductivity regime (parallel-field geometry; i.e., $H\sim H_{c3}$). The only difference is that the equilibrium form of the order parameter is in the former case the Abrikosov solution, and in the latter one the de Gennes–Saint James solution.

The calculations of the various retarded products which come into play in the dispersion relations and response functions are completely analogous to the ones which have been carried out previously for dirty type-II superconductors. [2–4] So, here we will only sketch the procedure and give the results of these calculations, leaving out all details (the main ones are given in the Appendix).

The products of operators like n, J_μ, $\Psi^\dagger$, Ψ are expanded in terms of the Gor'kov G and F functions.[14]

Furthermore, in the high-field region which we consider here we can expand these Green's functions in powers of $\Delta(\mathbf{r})$.[15] The developments have the well-known form

$$G_\omega(\mathbf{r},\mathbf{r}') = \langle G_\omega^0(\mathbf{r},\mathbf{r}')\rangle_i - \int \langle G_\omega^0(\mathbf{r},\mathbf{l})\Delta(\mathbf{l})G_{-\omega}^0(\mathbf{m},\mathbf{l})\Delta^\dagger(m)G_\omega^0(\mathbf{m},\mathbf{r}')\rangle_i d^3m d^3l,$$

$$F_\omega(\mathbf{r},\mathbf{r}') = \int \langle G_\omega^0(\mathbf{r},\mathbf{l})\Delta(\mathbf{l})G_{-\omega}^0(\mathbf{r}',\mathbf{l})\rangle_i d^3l. \quad (16)$$

G_ω^0 is the Green's function in the normal state in the presence of field and $\langle\ \rangle_i$ mean that the average has to be taken on impurity configurations (the rules for doing such averaging on all the relevant diagrams are given in the Appendix).

Then from Eq. (16) it is obvious that the products which contain the same number of creation and annihilation operators (i.e., $\langle[n, n]\rangle$, $\langle[\Psi^\dagger, \Psi]\rangle\cdots$) have finite values in the normal state, whereas the leading term in "cross products" like $\langle[n, \Psi]\rangle$ is proportional to Δ.

One can then easily calculate all the products provided that the external frequency and momentum satisfy the "dirty-limit" condition; i.e., $\omega\tau\ll1$, $ql\ll1$ (τ and l are the electron-collision time and mean free path).

Since the fluctuations of the order parameter are given by Eq. (6), let us first consider $\langle[\Psi^\dagger, \Psi]\rangle_{q\omega}$;

$$\langle[\Psi^\dagger, \Psi]\rangle_{q\omega} = |g|^{-1} + N(0) \times \left\{ \ln\frac{T_{c0}}{T} + \psi(\tfrac{1}{2}) - \psi\left(\tfrac{1}{2} - \frac{i\omega - D\mathbf{Q}^2}{4\pi T}\right)\right\}, \quad (17)$$

where $\mathbf{Q}$ is the differential operator which is the gauge-invariant generalization of $\mathbf{q}$:

$$\mathbf{Q} = \mathbf{q} + 2e\mathbf{A}.$$

$D = \frac{1}{3}v_F^2\tau$ is the diffusion coefficient,[16] v_F the Fermi velocity of the alloy, $N(0) = mp_F/2\pi^2$ is the density of states at the Fermi level, and $\psi(z)$ is the digamma function.

The ground state ($\omega=0$ solution) of Eq. (6) is simply the Abrikosov solution. In order to find the excited states, which correspond to the various types of fluctuations, it is convenient to define the functions

$$\phi_{n,k,p_z} = (\Pi^+)^n \exp\{-eH_{c2}(x-(k/2eH_{c2}))^2\} \times \exp[i(ky+p_z z)],$$

$$\Pi^+ = Q_x + iQ_y.$$

(The Abrikosov solution corresponds to a superposition of various $\phi_{0,k,0}$.) The ϕ_{n,k,p_z} satisfy the relation[17]

$$DQ^2\phi_{n,k,p_z} = D\{p_z^2 + (2n+1)2eH_{c2}\}\phi_{n,k,p_z} \equiv \epsilon_n(p_z)\phi_{n,k,p_z}, \quad (18)$$

so that they form a complete set of eigensolutions for Eq. (6). The general dispersion relation is given by

$$\ln\frac{T_{c0}}{T} + \psi(\tfrac{1}{2}) - \psi\left(\tfrac{1}{2} - \frac{i\omega - \epsilon_n(p_z)}{4\pi T}\right) = 0. \quad (19)$$

It is easy to see that the $n=0$ mode transforms like a scalar, the $n=1$ one like a vector; more generally, the nth mode transforms like an irreducible tensor of the nth order. So, the nth mode appears only in the response

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[14] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).

[15] This expansion is valid for all values of ω in the dirty case since one can show that all the retarded products $\langle[A,B]\rangle_{qq'\omega}$ are analytic at all temperatures in the neighborhood of $\omega=0$.

[16] In all the calculations we consider the impurity scattering to be isotropic, so that $\tau_{\text{transport}}=\tau$.

[17] Note that Eq. (18) is strictly valid at $H=H_{c2}$ only. This is consistent with the fact that we calculate $\langle[\Psi^\dagger,\Psi]\rangle_{ql\omega}$ to zeroth order in Δ and that we neglect $\{\langle[\Psi^+, n]\rangle\}^2$ in the dispersion equation.

function corresponding to an operator having the same transformation properties.

Consequently, the $n=0$ mode contributes only to the density–density correlations, the $n=1$ one to the thermal (or transverse electromagnetic) current–current correlations, $\cdots$. The modes with $n\geq 2$ can be discarded for two reasons:

(1) The only possible contribution from these modes to a linear response function would appear in the collision-drag term of the ultrasonic-attenuation coefficient ($n=2$ mode)[18]; but, since ultrasonic attenuation corresponds to a time-reversal conserving perturbation, this effect can be shown to be negligible.

(2) The damping coefficient of the fluctuation modes [which is proportional to $\epsilon_n(q)$] increases linearly with n.

Therefore, let us only consider the $n=0$ and $n=1$ cases, to which we will refer as longitudinal and transverse, respectively.

A. Longitudinal Modes

In this case ($n=0$), using Eq. (19) and the equation which defines H_{c2},[1,2]

$$\ln(T/T_{c0})=\psi(\tfrac{1}{2})-\psi(\tfrac{1}{2}+(\epsilon_0/4\pi T)), \qquad (20)$$

where $\epsilon_0=2DeH_{c2}$, we obtain for the dispersion relation

$$i\omega=Dp_z^2. \qquad (21)$$

This means that the time evolution of the longitudinal modes is of diffusion type. It is interesting to note that the corresponding diffusion coefficient, which is the same as the one ruling density correlations, is therefore independent of temperature. This feature holds only in the dirty limit; it is not maintained in the opposite limit of a pure superconductor,[19] for example.

The cross product relevant to longitudinal perturbations is

$$\langle[\Psi^\dagger, n]\rangle_{\mathbf{qq}'\omega} =\frac{iN(0)}{4\pi T}\bigg\{(-i\omega+Dq'^2)^{-1}\bigg[\psi\left(\tfrac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)-\psi(\tfrac{1}{2}+\rho)\bigg]+(-i\omega+DQ^2)^{-1}\bigg[\psi\left(\tfrac{1}{2}-\frac{i\omega}{4\pi T}+\rho_1\right)-\psi(\tfrac{1}{2}+\rho)\bigg]$$
$$+(-i\omega-DQ^2)^{-1}\bigg[\psi\left(\tfrac{1}{2}-\frac{i\omega}{4\pi T}+\rho_1\right)-\psi\left(\tfrac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)\bigg]\bigg\}\,\Delta^\dagger(\mathbf{q}-\mathbf{q}'), \qquad (22)$$

where $\Delta^\dagger(q)$ is the Fourier transform of the order parameter at equilibrium—which is here described by the Abrikosov solution—and

$$\rho=\epsilon_0/4\pi T, \qquad \rho_1=\rho+Dp_z^2/4\pi T, \qquad \mathbf{Q}=\mathbf{q}+2eA.$$

In the limit of small ω (i.e., $\omega\ll T_{c0}$, T_{c0} being the transition temperature in zero field), Eq. (22) can be expanded in powers of ω, and

$$\langle[\Psi^\dagger, n]\rangle_{\mathbf{qq}'\omega} =\frac{iN(0)}{4\pi T}\frac{(-i\omega)}{DQ^2+i\omega}\left\{\frac{Dq^2+Dq'^2}{2\pi T(-i\omega+Dq'^2)}\psi'(\tfrac{1}{2}+\rho)-\frac{2}{DQ^2-i\omega}[\psi(\tfrac{1}{2}+\rho_1)-\psi(\tfrac{1}{2}+\rho)]\right\}\Delta^\dagger(\mathbf{q}-\mathbf{q}')+O(\omega^2). \quad (23)$$

The above equation implies that $\langle[\Psi^\dagger, n]\rangle_{\mathbf{qq}'\omega}$ vanishes like ω for small frequency, if $Dq^2\gg\omega$. This is in fact the case for ultrasonic measurements. Therefore the contribution from the longitudinal modes to the attenuation coefficient gives rise to a small correction (higher order in ω^2), which can be neglected in usual experimental situations.

The change induced in the order parameter by a scalar potential $\phi_{\mathbf{q}\omega}$ is given by

$$\delta\Delta_{\mathbf{q}\omega}=\left\{\frac{|g|}{1-|g|\langle[\Psi,\Psi^\dagger]\rangle^L}\langle[\Psi,n]\rangle\delta\phi\right\}_{\mathbf{q}\omega}. \qquad (24)$$

Substituting Eqs. (17) and (22) in the above expression, we see that the change in the order parameter also vanishes like ω for low frequency and $\omega\ll Dq^2$.

In order to compute the effect of the superconducting transition on the plasma mode, one has to calculate all retarded products involved in Eq. (14) up to second order in $\Delta(\mathbf{r})$. However, the correction due to these terms is likely to be of the order of $(\Delta/\omega_{\mathrm{pl}})^2$, where ω_{pl} is the plasma frequency.

B. Transverse Modes

These modes correspond to the $n=1$ excited states of Eq. (18). Proceeding as in the case of longitudinal

[18] T. Tsuneto, Phys. Rev. **121**, 402 (1961).

[19] C. Caroli and K. Maki, following paper, Phys. Rev. 159, 316 (1967).

modes, we find the dispersion relation

$$i\omega = 2\epsilon_0 + Dp_z^2. \tag{25}$$

This means that the transverse mode is simply damped at low temperature (where the damping coefficient $\epsilon_0 \cong \Delta_{00}$, Δ_{00} being the BCS gap at $T=0$ and in zero field) while at higher temperature ($T \sim T_c$) it becomes essentially of diffusion type.

These modes couple to the electrical and thermal current.

$$\langle[\Psi, J_\mu]\rangle_{qq'\omega} = \frac{e\tau N}{2m}(2q_\mu - q'_\mu + 4eA_\mu)\left\{(2\epsilon_0 + Dq_z^2 - i\omega)^{-1}\left[\psi\left(\tfrac{1}{2} - \frac{i\omega}{4\pi T} + \rho_2\right) - \psi(\tfrac{1}{2}+\rho)\right]\right.$$
$$\left. + (2\epsilon_0 + Dq_z^2 + i\omega)^{-1}\left[\psi\left(\tfrac{1}{2} - \frac{i\omega}{4\pi T} + \rho_2\right) - \psi\left(\tfrac{1}{2} + \rho - \frac{i\omega}{2\pi T}\right)\right]\right\}\Delta^\dagger(\mathbf{q}-\mathbf{q}'), \tag{26}$$

where

$$\rho_2 = 3\rho + Dq_z^2/4\pi T.$$

It is clear from Eq. (26) that $\langle[\Psi, J_\mu]\rangle_{qq'\omega}$ does not vanish at zero frequency. This implies that the correction to the transverse electromagnetic response arising from the fluctuations of Δ is not negligible and that even a static transverse vector potential can induce nonnegligible variations of Δ.

The explicit form of the fluctuation of the order parameter is given by

$$\delta\Delta_{q\omega}^\dagger = |g|\{(1 - |g|\langle[\Psi^\dagger, \Psi]\rangle)^{-1}\langle[\Psi^\dagger, j_\mu]\rangle\delta A_\mu\}_{q\omega}. \tag{27}$$

Substituting Eq. (26), we see that the fluctuation of the Abrikosov structure is in a direction parallel to the microwave current A_μ. This might seem, at first glance, to be in contradiction with the flux flow experiments,[20] where the vortex lines move perpendicularly to the current (or external electrical field). We would like to stress that in the present situation there exists no net total current [i.e., $\langle \mathbf{j}(\mathbf{r})\rangle_{\text{Av}} = 0$], and hence the situation is completely different from those in the flux flow experiments. At this point it may be useful to emphasize that the modes defined by Eq. (25) are completely different from the helicon modes, which in the dirty limit do not exist, since they are damped with a characteristic time τ (characteristic time for the loss of velocity correlations). On the contrary, the lifetime of the present modes is the characteristic time for the loss of pair correlations.

Finally, one can readily be convinced that the coupling between these fluctuations and the energy current

$$\mathbf{j}_E = -(2m)^{-1}\sum_\sigma \psi_\sigma^\dagger \nabla\psi_\sigma + (\nabla\psi_\sigma^\dagger)\psi_\sigma$$

is proportional to $\omega/(-i\omega + Dq^2 + 2\epsilon_0)$. Since the thermal conductivity is related to the $\omega=0$ limit of $\langle[\Psi^\dagger, \mathbf{j}_E]\rangle_{qq'\omega}$, it contains no contribution from these modes. Therefore the previous result[3] is valid for the case of thermal conductivity.

IV. ANISOTROPIC ELECTROMAGNETIC CONDUCTIVITY

Since we have seen that the effect of fluctuations of Δ on ultrasonic attenuation and thermal conductivity is negligible, we will now concentrate on the electromagnetic response, which seems to provide the only way of detecting these modes.

We can write the complex conductivity as

$$\sigma_{\mu\nu}(\mathbf{q}, \mathbf{q}', \omega) = (i\omega)^{-1}Q_{\mu\nu}(\mathbf{q}, \mathbf{q}', \omega),$$

where

$$Q_{\mu\nu}(\mathbf{q}=\mathbf{q}'=o, \omega) = \langle[j_\mu, j_\nu]\rangle_{00\omega} + R_{\mu\nu}(o, \omega) + R_{\nu\mu}(o, \omega)$$

$$R_{\mu\nu}(0, \omega) = -|g|\{\langle[\Psi^\dagger, j_\mu]\rangle \times (1 - |g|\langle[\Psi^\dagger, \Psi]\rangle^T)^{-1}\langle[j_\nu, \Psi]\rangle\}_{q=q'=0,\omega}. \tag{28}$$

We only need to know $Q_{\mu\nu}(\omega, \mathbf{q}=\mathbf{q}'=0)$ since $\mathbf{q}$ and $\mathbf{q}'$ are essentially of the order of the inverse penetration depth λ^{-1}. Since in the dirty limit the electrodynamics of the superconductor is local ($\lambda \gg l$) we can put $q=q'=0$. In that case $\langle[j_\mu, j_\nu]\rangle_{00,\omega}$ has the simple expression

$$\langle[j_\mu, j_\nu]\rangle_{00\omega} = \sigma\delta_{\mu\nu}\left\{-i\omega + \frac{\langle|\Delta|^2\rangle}{2\pi T}\left[\psi'\left(\tfrac{1}{2} - \frac{i\omega}{2\pi T} + \rho\right) + \left(\frac{2\pi T}{-i\omega} + \frac{2\pi T}{-i\omega + \epsilon_0}\right)\left\{\psi\left(\tfrac{1}{2} - \frac{i\omega}{2\pi T} + \rho\right) - \psi(\tfrac{1}{2}+\rho)\right\}\right]\right\}, \tag{29}$$

$\sigma = (Ne^2\tau/m)$ is the normal-state static conductivity and $\psi'(z) = [d\psi(z)/dz]$. The correction to $Q_{\mu\nu}$ arising from the collective modes is, using Eq. (26) and the expression of $\langle[\Psi^\dagger, \Psi]\rangle^T$,

$$1 - |g|\langle[\Psi^\dagger, \Psi]\rangle_{q,\omega}{}^T = |g|N(0)\left\{\psi\left(\tfrac{1}{2} - \frac{i\omega}{4\pi T} + \rho_2\right) - \psi(\tfrac{1}{2}+\rho)\right\}, \tag{30}$$

$$[R_{\mu\nu}(\omega) + R_{\nu\mu}(\omega)] = -2\sigma\epsilon_0(\delta_{\mu x}\delta_{\nu x} + \delta_{\mu y}\delta_{\nu y})\left\{(2\epsilon_0 - i\omega)^{-1}\left[\psi\left(\tfrac{1}{2} - \frac{i\omega}{4\pi T} + 3\rho\right) - \psi(\tfrac{1}{2}+\rho)\right]\right.$$
$$\left. + (2\epsilon_0 + i\omega)^{-1}\left[\psi\left(\tfrac{1}{2} - \frac{i\omega}{4\pi T} + 3\rho\right) - \psi\left(\tfrac{1}{2} - \frac{i\omega}{2\pi T} + \rho\right)\right]\right\}^2\left[\psi\left(\tfrac{1}{2} - \frac{i\omega}{4\pi T} + 3\rho\right) - \psi(\tfrac{1}{2}+\rho)\right]^{-1}. \tag{31}$$

[20] See for instance Y. B. Kim, C. F. Hempstead, and A. R. Strand, Phys. Rev. **139**, A1163 (1965).

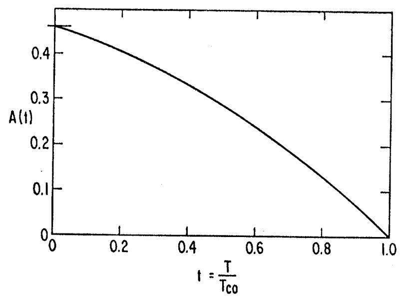

FIG. 1. The temperature dependence of $A(t)$, which appears in the expression of $Q_\perp(\omega)$, is plotted against the reduced temperature $t=T/T_{c0}$.

In the derivation of the above equation we made use of the relation[21]

$$\int \Delta^*(2)\Pi_\mu(2)\Pi_\nu(1)\Delta(1)\,|_{1=2}\,d\mathbf{1}=\tfrac{1}{2}(\delta_{\mu+}\delta_{\nu-})$$

$$\times\int(\Pi_+\Delta(1))^*(\Pi_+\Delta(1))d\mathbf{1}=\delta_{\mu+}\delta_{\nu-}eH\langle|\Delta|^2\rangle\int d\mathbf{1}, \quad (32)$$

since $\Pi_-\Delta=0$, where

$$\Pi_\mu=(i\nabla_\mu+2eA_\mu), \qquad \Pi_\pm=\Pi_x\pm i\Pi_y. \quad (33)$$

We see from Eq. (31) that this correction is highly anisotropic, since it does not contain any (zz) component. Thus, the transverse fluctuations only couple with vector potentials having their polarization vector in the plane perpendicular to the static field. We note that $Q_{xx}=Q_{yy}$ which follows from the high symmetry of the Abrikosov structure in the high-field region. This fact, added with the dirty-limit condition, eliminates the possibility of any "gyrotropic" effect[22] (i.e., we have $Q_{xy}=Q_{yx}=0$). At low frequency ($\omega\ll\pi T_{c0}$, which is usually satisfied by microwaves) the expression (31) reduces to

$$Q_{zz}(\omega)=Q_{\parallel}=\sigma\{-i\omega(1+[\langle|\Delta|^2\rangle/2(2\pi T)^2]$$
$$\times[\rho^{-1}\psi'(\tfrac{1}{2}+\rho)+3\psi''(\tfrac{1}{2}+\rho)])$$
$$+(\langle|\Delta|^2\rangle/\pi T)\psi'(\tfrac{1}{2}+\rho)\}, \quad (34)$$

$$Q_{xx}(\omega)=Q_{yy}(\omega)=Q_\perp$$
$$=Q_{zz}-\sigma\{(2\langle|\Delta|^2\rangle/\epsilon_0)(\psi(\tfrac{1}{2}+3\rho)-\psi(\tfrac{1}{2}+\rho)$$
$$-(i\omega/4\pi T)[\psi'(\tfrac{1}{2}+3\rho)-2\psi'(\tfrac{1}{2}+\rho)])\}, \quad (35)$$

where, as usual

$$\langle|\Delta|^2\rangle=\frac{H_{c2}-H}{\beta_A[2\kappa_2^2(T)-1]}\frac{eT}{\sigma}[\psi'(\tfrac{1}{2}+\rho)]^{-1}. \quad (36)$$

[21] P. G. de Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, Inc., New York, 1966), p. 204.

[22] In a pure isotropic material one expects a finite "gyrotropic" effect, though it should be extremely small [i.e., $\sigma_{xy}=-\sigma_{yx}\sim\sigma_{xx}(B/H_{c2})(\Delta/E_F)$, where B is the magnetic induction and E_f the Fermi energy].

Making use of Eq. (36) we can rewrite Eqs. (34) and (35) in the form

$$Q_\parallel(\omega)=-\sigma i\omega+\frac{e}{\pi}\frac{H_{c2}-H}{\beta_A[2\kappa_2^2(t)-1]}\left[1-\frac{i\omega}{8\pi T_{c0}}C_\parallel(t)\right], \quad (34')$$

$$C_\parallel(t)=t^{-1}\left[\rho^{-1}+\frac{3\psi''(\tfrac{1}{2}+\rho)}{\psi'(\tfrac{1}{2}+\rho)}\right], \qquad t=T/T_{c0}, \quad (34'')$$

$$Q_\perp(\omega)=-\sigma i\omega+\frac{e}{\pi}\frac{H_{c2}-H}{\beta_A[2\kappa_2^2(t)-1]}$$
$$\times\left[A(t)-\frac{i\omega}{8\pi T_{c0}}C_\perp(t)\right], \quad (35')$$

$$A(t)=1-\frac{\psi(\tfrac{1}{2}+3\rho)-\psi(\tfrac{1}{2}+\rho)}{2\rho\psi'(\tfrac{1}{2}+\rho)}, \quad (35'')$$

$$C_\perp(t)=C_\parallel(t)-t^{-1}\rho^{-1}\left[\frac{\psi'(\tfrac{1}{2}+3\rho)}{\psi'(\tfrac{1}{2}+\rho)}-2\right]. \quad (35''')$$

The universal functions of the reduced temperature $C_\parallel$, $C_\perp$, and A are plotted on Figs. 1 and 2. Equation (35) implies that

$$\mathrm{Re}Q_\perp<\mathrm{Re}Q_\parallel, \qquad \mathrm{Im}Q_\perp<\mathrm{Im}Q_\parallel,$$

namely, the transverse fluctuations increase both the penetration depth and the absorption of the electromagnetic wave. Since the current fluctuations perpendicular to the static external field couple to these transverse fluctuations, they have effectively a larger penetration depth than the current fluctuations parallel to the external field.

The surface impedance is expressed in terms of $Q_{\mu\nu}(\omega)$ as

$$Z_{\parallel,\perp}=i\omega[4\pi/cQ_{\parallel,\perp}(\omega)]^{1/2}, \quad (37)$$

depending on the direction of the microwave current,

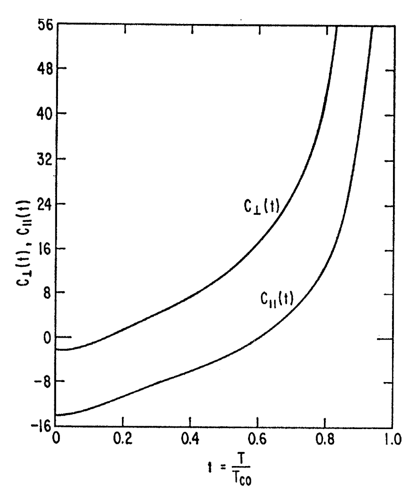

FIG. 2. The coefficients appearing in the absorptive part of $Q_\parallel(\omega)$ and $Q_\perp(\omega)$ are plotted against the reduced temperature t.

where $Q_{||}$ and $Q_\perp$ are given in Eq. (34). Substituting Eq. (35) we have in the immediate vicinity of H_{c2} (i.e., $\langle|\Delta|^2\rangle \ll \omega T_{c0}$),

$$Z_{||}=R_n(1-i)\Big(1-[\langle|\Delta|^2\rangle/4(2\pi T)^2]\times\{\rho^{-1}\psi'(\tfrac{1}{2}+\rho)+3\psi''(\tfrac{1}{2}+\rho)\}-(i\langle|\Delta|^2\rangle/2\omega\pi T)\psi'(\tfrac{1}{2}+\rho)\Big) \quad (38)$$

and

$$Z_\perp=Z_{||}+R_n(1-i)(\langle|\Delta|^2\rangle/\epsilon_0)\times\{(4\pi T)^{-1}[\psi'(\tfrac{1}{2}+3\rho)-2\psi'(\tfrac{1}{2}+\rho)]+(i/\omega)[\psi(\tfrac{1}{2}+3\rho)-\psi(\tfrac{1}{2}+\rho)]\}, \quad (39)$$

where $R_n=(2\pi\omega/\sigma)^{1/2}$ is the surface resistance of the normal metal. From the above expression, we see that the surface resistance for the microwave current in the x (or y direction) is larger than for a current in the z direction. Let us define the surface resistance for the currents in the (x, y) plane and in the z direction as $R_\perp$ and $R_{||}$, respectively. Since $R_\perp$ and $R_{||}$ decrease linearly in field in the vicinity of H_{c2} as the external (static) field decreases, it is convenient to introduce the new quantities

$$r_{||,\perp}=\partial R_{||,\perp}/\partial H\,|_{H=H_{c2}}. \quad (40)$$

Then the ratio of these two quantities is given by a universal function[23] which characterizes the anisotropy in the surface resistance

$$\frac{r_\perp}{r_{||}}=1-\frac{\psi(\frac{1}{2}+3\rho)-\psi(\frac{1}{2}+\rho)}{2\rho\psi'(\frac{1}{2}+\rho)}=A(t), \quad (41)$$

the asymptotic behaviors of which are given by

$$r_\perp/r_{||}=1-[1-(56/\pi^4)\zeta(3)(1-t)]=0.70(1-t) \quad \text{for } t\cong1, \quad (42)$$

$$r_\perp/r_{||}=1-\tfrac{1}{2}[\ln3+(\tfrac{1}{12}\ln3-\tfrac{1}{27})(4\gamma t)^2]=0.451-0.406t^2 \quad \text{for } t\ll1, \quad (43)$$

with $\gamma=1.78$.

The anisotropy in r increases with the temperature from about 55% at $T=0$ to 100% close to T_{c0}. This means that in the vicinity of T_{c0}, the reactive part of the complex conductivity for a microwave current flowing in the plane perpendicular to the static field vanishes completely (i.e., the term coming from $\langle[j_\mu, j_\nu]\rangle$ cancels exactly the one coming from the $R_{\mu\nu}$'s and hence the penetration depth increases much faster than $(H_{c2}-H)^{-1/2}$.

Such an anisotropy in r has already been noticed[24] and is in rough qualitative agreement with our results. However, more experiments would be desirable in order to make a quantitative comparison, since the existing ones were concerned only with the parallel geometry (i.e., the static field H parallel to the surface of the sample).

We might point out that the same kind of calculations could be done for the surface sheath regime ($H\sim H_{c3}$), provided we knew the wave functions for the excited states with the appropriate boundary condition [i.e., $(i\nabla+2e\mathbf{A})_n\phi=0$ at the boundary]. We expect, in this situation, an anisotropy in the surface resistance similar to the one discussed above.

V. CONCLUSION

It appears from our results that, besides the usual (plasma and helicon) collective modes, there exists also in a type-II superconductor a class of modes associated with fluctuations of the order parameter. These modes have no correspondent in the homogeneous superconductor, since they are intimately connected with the space variations of the order parameter at equilibrium. For example, in the BCS state in the absence of a magnetic field $\langle[\Psi^\dagger, j_\mu]\rangle$ vanishes identically so that there is no contribution to the electromagnetic response from the collective oscillation. It is clear from our calculations that they exist, not only in the Abrikosov structure and in the surface sheath regions, but also, for instance, in the intermediate state, close to the superconducting interface. However, in the latter case the equation ruling their time evolution is strongly coupled with the one governing the electromagnetic fluctuations.

Although our calculation applies only to the high-field region, these fluctuations of Δ exist all over the mixed state or surface sheath range. However, contrary to the case where Δ is small, in lower field they mix with the density or current oscillations (which are the collective modes of vortex lines discussed by several authors),[25] and the various modes are no longer the solutions of independent dispersion equations.

We have seen also that, among many response functions, the electromagnetic response function is most suitable for the study of these collective modes, since it contains a significant contribution coming from the transverse collective modes of the order parameter. Hence, measurements of the surface impedance would be of particular interest.

APPENDIX

We shall present here the calculation of various retarded products, which have appeared in the main text.

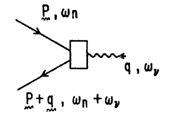

FIG. 3. P-wave vertex with a finite external momentum.

[23] Precisely speaking Eq. (41) is not valid in the immediate vicinity of $T_{c0}[1-(T/T_{c0})\ll1]$, since it assumes that $\epsilon_0(T)>\omega$.

[24] M. Cardona, G. Fischer, and B. Rosenblum, Phys. Rev. Letters 12, 101 (1964).

[25] P. G. de Gennes and J. Matricon, Rev. Mod. Phys. 36, 45 (1964); A. L. Fetter, P. C. Hohenberg, and P. Pincus, Phys. Rev. 147, 140 (1966).

A. Calculation of $[\langle\Psi^\dagger, \Psi]\rangle_{q\omega}$

The retarded product is obtained from the thermal product by analytical continuation.[26] The term of lowest order in $\Delta(\mathbf{r})$ of the thermal product is given by

$$\langle[\Psi^\dagger, \Psi]\rangle_{\nu,\mathbf{q}} = \frac{T}{V}\sum_n \int \langle G_{\omega_n}(\mathbf{r},\mathbf{l}) G_{-\omega'_n}(\mathbf{r},\mathbf{l})\rangle_i \exp[i\mathbf{q}\cdot(\mathbf{r}-\mathbf{l})]d^3r,$$

$$\omega_n=(2n+1)\pi T, \qquad \omega_\nu = 2\nu\pi T, \qquad \omega'_n=\omega_n-\omega_\nu, \tag{A1}$$

where n and ν are integers and $\langle\ \rangle_i$ indicates that the average has to be taken over the random configurations of impurity atoms. Using the standard techniques of the impurity-scattering problem[27] (A1) is reduced to

$$T\sum_{n=-\infty}^{\infty}\int \frac{d^3p}{(2\pi)^3}(i\tilde{\omega}_n-\xi_p)^{-1}(i\tilde{\omega}_n'+\xi_p-\mathbf{v}\cdot\mathbf{q})^{-1}\eta_{|\omega_n+\omega_n'|/2,\mathbf{q}}$$

$$=i\frac{mp_F}{(2\pi)^2}T\sum_{n=-\infty}^{\infty}\int d\Omega_v\,\theta(\omega_n\omega_n')[i(\tilde{\omega}_n+\tilde{\omega}_n')-\mathbf{v}\cdot\mathbf{q}]^{-1}\eta_{|\omega_n+\omega_n'|/2,\mathbf{q}}$$

with $|v|=v_F$

$$\langle[\Psi^\dagger, \Psi]\rangle_{\nu,\mathbf{q}} = N(0)\left\{\ln\frac{\gamma\omega_D}{\pi T}-\psi\left(\tfrac{1}{2}+\frac{\omega_\nu}{4\pi T}+\frac{Dq^2}{4\pi T}\right)\right\}, \tag{A2}$$

where

$$\tilde{\omega}_n=\omega_n[1+(2\tau|\omega_n|)^{-1}]$$

$$\eta_{|\omega+\omega'|/2}=\{1-(\tau|\tilde{\omega}+\tilde{\omega}'|)^{-1}(1-\tau Dq^2)\}^{-1}, \quad \text{for} \quad \omega\omega'>0$$

$$=1, \qquad \text{for} \quad \omega\omega'<0 \tag{A3}$$

$$\theta(x)=1, \quad \text{for} \quad x>0$$

$$=0, \quad \text{for} \quad x<0 \tag{A4}$$

and $\psi(z)$ is a digamma function. After analytical continuation we have

$$1-|g|\langle[\Psi^\dagger, \Psi]\rangle_{\omega\mathbf{q}} = |g|N(0)\left\{\psi\left(\tfrac{1}{2}-\frac{i\omega}{4\pi T}+\frac{Dq^2}{4\pi T}\right)-\psi(\tfrac{1}{2})\right\}. \tag{A5}$$

Finally, the effect of the magnetic field is introduced by the simple transformation[1]

$$\mathbf{q}\to\mathbf{q}+2e\mathbf{A}.$$

B. Calculation of $\langle[\Psi^\dagger, n]\rangle$ and $\langle[\Psi^\dagger, j_\mu]\rangle$

First let us consider $\langle[\Psi^\dagger, n]\rangle$. It is important to note that the above retarded product depends on two momenta,

$$\langle[\Psi^\dagger, n]\rangle_{\mathbf{q},\mathbf{q}',\omega} = \int_{-\infty}^{0} dt\, e^{i\omega t}\int d^3r d^3r' \exp[i(\mathbf{q}\cdot\mathbf{r}-\mathbf{q}'\cdot\mathbf{r}')]\langle[\Psi^\dagger(\mathbf{r}'t), n(\mathbf{r}0)]\rangle. \tag{A6}$$

The term of lowest order in $\Delta(r)$ is given by

$$\langle[\Psi^\dagger, n]\rangle_{\mathbf{q},\mathbf{q}',\nu} = T\sum_n\int d^3\mathbf{r}d^3\mathbf{r}'d^3\mathbf{l}\exp[i(\mathbf{q}\cdot\mathbf{r}-\mathbf{q}'\cdot\mathbf{r}')]\langle G_{\omega_n}(\mathbf{r},\mathbf{l})\Delta(\mathbf{l})G_{-\omega_n}(\mathbf{r}',\mathbf{l})G_{\omega_n'}(\mathbf{r}',\mathbf{r})\rangle_i,$$

$$\langle[\Psi^\dagger, n]\rangle_{\mathbf{q},\mathbf{q}',\nu} = T\sum_n\int\frac{d^3p}{(2\pi)^3}(i\tilde{\omega}_n-\xi_p)^{-1}(i\tilde{\omega}_n+\xi_p+\mathbf{v}\cdot\mathbf{q}_1)^{-1}$$

$$\times[i\tilde{\omega}_n'-\xi_p-\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1)]^{-1}\eta_{\omega_n,\mathbf{q}_1}\Delta(\mathbf{q}_1)\eta_{(\omega_n+\omega_n')/2,\mathbf{q}}\eta_{(\omega_n-\omega_n')/2,\mathbf{q}'}\big|_{\mathbf{q}_1=\mathbf{q}'-\mathbf{q}}$$

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[26] See for instance: A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzialoshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963), Sec. 37-2.
[27] See Ref. 26, Chap. 7.

$$=iTN(0)\sum_{n=-\infty}^{\infty}\int d\Omega_v\left\{\frac{\omega_n}{|\omega_n|}\theta(\omega_n\omega_n')(2i\tilde{\omega}_n+\mathbf{v}\cdot\mathbf{q}_1)^{-1}[i(\tilde{\omega}_n+\tilde{\omega}_n')-\mathbf{v}\cdot\mathbf{q}]^{-1}\eta_{\omega_n,q_1}\eta_{(\omega_n+\omega_n')/2,q}\right.$$

$$\left.+\frac{\omega_n}{|\omega_n|}\theta(-\omega_n\omega_n')(2i\tilde{\omega}_n-\mathbf{v}\cdot\mathbf{q}_1)^{-1}[i(\tilde{\omega}_n-\tilde{\omega}_n')-\mathbf{v}\cdot\mathbf{q}']^{-1}\eta_{\omega_n q_1}\eta_{(\omega_n-\omega_n'/2),q'}\right\}\Delta(q_1)\,|_{q_1=q'-q}$$

$$=\frac{iN(0)}{4\pi T}\left\{(\omega_\nu+Dq^2)^{-1}\left[\psi\left(\tfrac{1}{2}+\frac{\omega_\nu}{4\pi T}+\frac{Dq_1^2}{4\pi T}\right)-\psi\left(\tfrac{1}{2}+\frac{Dq_1^2}{4\pi T}\right)\right]+[D(q^2-q_1^2)+\omega_\nu]^{-1}\right.$$

$$\times\left[\psi\left(\tfrac{1}{2}+\frac{\omega_\nu}{4\pi T}+\frac{Dq^2}{4\pi T}\right)-\psi\left(\tfrac{1}{2}+\frac{Dq_1^2}{4\pi T}\right)\right]-[D(q^2-q_1^2)-\omega_\nu]^{-1}$$

$$\left.\times\left[\psi\left(\tfrac{1}{2}+\frac{\omega_\nu}{4\pi T}+\frac{Dq^2}{4\pi T}\right)-\psi\left(\tfrac{1}{2}+\frac{\omega_\nu}{2\pi T}+\frac{Dq_1^2}{4\pi T}\right)\right]\right\}\Delta(\mathbf{q}_1)_{q_1=q'-q}. \tag{A7}$$

From the above equation we have

$$\langle[\Psi^\dagger,n]\rangle_{q,q',\omega}=\frac{iN(0)}{4\pi T}\left\{(-i\omega+Dq'^2)^{-1}\left[\psi\left(\tfrac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)-\psi(\tfrac{1}{2}+\rho)\right]+(Dq^2-\epsilon_0-i\omega)^{-1}\right.$$

$$\times\left[\psi\left(\tfrac{1}{2}-\frac{i\omega}{4\pi T}+\frac{Dq^2}{4\pi T}\right)-\psi(\tfrac{1}{2}+\rho)\right]-(Dq^2-\epsilon_0+i\omega)^{-1}$$

$$\left.\times\left[\psi\left(\tfrac{1}{2}-\frac{i\omega}{4\pi T}+\frac{Dq^2}{4\pi T}\right)-\psi\left(\tfrac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)\right]\right\}\Delta(\mathbf{q}'-\mathbf{q}), \tag{A8}$$

where in the above expressions $\mathbf{q}$ is understood to be $\mathbf{q}-2e\mathbf{A}$ because it operates on Δ^1. Here we have made use of the relation

$$D(\mathbf{q}-2e\mathbf{A})^2\Delta(\mathbf{r})=\epsilon_0\Delta(\mathbf{r}). \tag{A9}$$

Similarly, we have

$$\langle[\Psi^\dagger,j_\mu]\rangle_{q_1q_1'\omega}=\frac{e\tau_{tr}N}{2m}\left\{(2\epsilon_0-i\omega)^{-1}\left[\psi\left(\tfrac{1}{2}-\frac{i\omega}{4\pi T}+3\rho\right)-\psi(\tfrac{1}{2}+\rho)\right]+(2\epsilon_0+i\omega)^{-1}\right.$$

$$\left.\times\left[\psi\left(\tfrac{1}{2}-\frac{i\omega}{4\pi T}+3\rho\right)-\psi\left(\tfrac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)\right]\right\}(q_\mu'-2q_\mu+4eA_\mu)\Delta(\mathbf{q}-\mathbf{q}'),\quad\text{for}\quad\mu=x\text{ or }y, \tag{A10}$$

$$\langle[\Psi^\dagger,j_\mu]\rangle_{qq'\omega}=\frac{e\tau_{tr}N}{2m}\left\{(Dq_z'^2-i\omega)^{-1}\left[\psi\left(\tfrac{1}{2}-\frac{i\omega-Dq_z'^2}{4\pi T}+\rho\right)-\psi(\tfrac{1}{2}+\rho)\right]+(Dq_z'^2+i\omega)^{-1}\right.$$

$$\left.\times\left[\psi\left(\tfrac{1}{2}-\frac{i\omega-Dq_z'^2}{4\pi T}+\rho\right)-\psi\left(\tfrac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)\right]\right\}(q_\mu'-2q_\mu+4eA_\mu)\Delta(\mathbf{q}-\mathbf{q}'),\quad\text{for}\quad\mu=z. \tag{A11}$$

Here we have made use of the fact

$$D(q-2eA)^2(q_x-2eA_x)\Delta(\mathbf{r})=3\epsilon_0(q_x-2eA_x)\Delta(\mathbf{r}).$$

Note that the rule for renormalizing the p-wave vertex of Fig. 3 amounts to the replacement of $(2\mathbf{p}+\mathbf{q})$ by $(2\mathbf{p}+\alpha\mathbf{q})$, where

$$\alpha=\frac{2p_Fv_F}{3i}\frac{\omega_n}{|\omega_n|}(|\omega_\nu|+Dq^2)^{-1},\quad\text{if}\quad\omega_n(\omega_n+\omega_\nu)>0$$

$$=0,\qquad\qquad\text{if}\quad\omega_n(\omega_n+\omega_\nu)<0.$$