environmental regimes. That is, the theory may be expressed in the limit of weak coupling by assuming that nearly all particles are in the single-particle zero-momentum state or, alternatively, the theory may be expressed in such a way as to furnish a good representation for a strongly-coupled system in which the elementary excitations exist in a "collisionless" regime. For the latter case, the theory gives a valid description of the low-lying excited states only.

Other systems which can be considered by using this formalism are those with $He^3$ ions in neutral $He^4$, or $\alpha$ particles in neutral $He^4$, or neutral $He^4$–$He^6$ systems. For example, Severene[20] has shown that even for minute concentrations of $He^3$ ions in $He^4$, a fundamentally new phenomenon appears in the presence of an external magnetic field, namely the scattering mechanism of magnetoresistance.

The intent of this work has been to develop techniques which may point the way toward a more sophisticated formalism, one which is capable of giving satisfactory predictions about the behavior of complex aggregates.

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[20] G. Severene, Physica **27**, 465 (1961).

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# Theory of Transport Processes in Nearly Ferromagnetic Fermi Liquids

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A theory of the low-temperature transport processes of thermal conduction and spin diffusion in Fermi liquids near the ferromagnetic state is developed on the assumption that the predominant particle scattering at low temperatures occurs from persistent spin fluctuations, or "paramagnons," of the type discussed recently by Doniach and Engelsberg and by Berk and Schrieffer. A Boltzmann equation for this type of scattering is set up and solved by the usual variational procedure to obtain expressions for the coefficients of thermal conductivity K and spin-diffusion $D$. The latter are evaluated by using a simple model approximation for the paramagnon spectral density function $A_q(\omega)$, based on the correct RPA result at long wavelengths. It is found that in the limit $T\rightarrow 0$, $K$ varies as $T^{-1}$ and $D$ as $T^{-2}$, in accordance with the predictions of the Landau theory. As $T$ increases from absolute zero, however, $K$ and $D$ fall off increasingly less rapidly than $T^{-1}$ and $T^{-2}$, respectively, and actually increase with increasing $T$ after passing through minima. The theory can therefore account for the observed temperature dependence of the thermal conductivity and spin diffuison of liquid $He^3$. In particular, the theoretical predictions are in close agreement with the low-pressure thermal-conductivity and diffusion data obtained by the Illinois group.

## 1. SYNOPSIS

THE concept of particle scattering from persistent spin fluctuations, or "paramagnons," in nearly ferromagnetic Fermi liquids has been applied recently by Doniach and Engelsberg[1] to interpret the low-temperature specific heat of liquid $He^3$. In this paper the equilibrium theory[2] of Doniach and Engelsberg is extended to nonequilibrium[3] in order to discuss the processes of thermal conduction and spin diffusion. This is carried through by including fermion-paramagnon scattering processes into the Boltzmann equation for the single particle distribution function $f_{\mathbf{k}s}$. As a first approximation to the description of transport processes we formulate the Boltzmann equation for *bare* fermion states, i.e., the fermions scattered from paramagnons are considered to be bare fermions. The treatment is therefore in contrast to the approach of the Landau theory[4,5] where the relaxation of quasiparticles is considered. Following standard variational procedures[6] the Boltzmann equation is used to obtain formal expressions for the coefficients of thermal conductivity $K$ and spin diffusion $D$. In doing this we assume that the paramagnetic excitations can be treated as at equilibrium. The expressions for $K$ and $D$ are evaluated by using a simple model approximation for the paramagnon spectral density function $A_q(\omega)$, based on the correct RPA result[7,1] at long wavelengths. The

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[1] S. Doniach and S. Engelsberg, Phys. Rev. Letters **17**, 577 (1966); S. Doniach, S. Engelsberg, and M. J. Rice, in Proceedings of the 10th International Conference on Low Temperature Physics, Moscow, 1966 (to be published); N. F. Berk and J. R. Schrieffer, Phys. Rev. Letters **17**, 433 (1966); Proceedings of the 10th International Conference on Low Temperature Physics, Moscow, 1966 (to be published).

[2] Similar calculations have also been made in the latter reference at the $T=0$ limit.

[3] An analogous theory for the electrical resistivity of paramagnetic transition metals has been discussed by D. L. Mills and P. Lederer, J. Phys. Chem. Solids **27**, 1805 (1966).

[4] A. A. Abrikosov and I. M. Khalatnikov, in *Reports on Progress in Physics* (The Physical Society, London, 1959), Vol. 22, p. 329.

[5] D. Hone, Phys. Rev. **121**, 669 (1961).

[6] J. M. Ziman, *Electrons and Phonons* (Clarendon Press, Oxford, England, 1960), Chap. 7.

[7] T. Izuyama, D. J. Kim, and R. Kubo, J. Phys. Soc. Japan **18**, 1025 (1963).

results, valid for temperatures $T \ll T_f$, where $T_f$ denotes the fermi temperature for the noninteracting fermi gas, show that in the limit $T \to 0$, $K$ varies as $T^{-1}$ and $D$ as $T^{-2}$ in agreement with the predictions of the Landau theory. As $T$ increases from zero, however, it is found that $K$ and $D$ fall off increasingly less rapidly than $T^{-1}$ and $T^{-2}$, respectively, and actually increase with increasing $T$ after passing through minima. The theory can therefore account qualitatively for the observed temperature variation of the thermal conductivity and spin-diffusion of liquid He$^3$.[8,9] In particular the theoretical temperature dependence is found to fit closely the low-pressure thermal-conductivity data of Anderson *et al.*[10] and the low-pressure diffusion data of Hart and Wheatley.[8]

## 2. BOLTZMANN EQUATION AND TRANSPORT COEFFICIENTS

In this section a Boltzmann equation is set up to describe transport processes in a fermi liquid. This is formulated in terms of an effective Hamiltonian for the short-range repulsion between fermions which is taken to have the form

$$H_{\text{int}} = I \iint d\mathbf{x} d\mathbf{x}' \psi_\uparrow{}^\dagger(\mathbf{x}) \psi_\uparrow(\mathbf{x}) \delta(\mathbf{x}-\mathbf{x}') \psi_\downarrow{}^\dagger(\mathbf{x}') \psi_\downarrow(\mathbf{x}'), \tag{2.1}$$

$$H_{\text{int}} = I \sum_{\mathbf{k},\mathbf{k}'} \sum_{\mathbf{q}} a_{\mathbf{k}+\mathbf{q}\,\uparrow}{}^\dagger a_{\mathbf{k}\uparrow} a_{\mathbf{k}'-\mathbf{q}\,\downarrow}{}^\dagger a_{\mathbf{k}'\,\downarrow}$$

on transforming to momentum space. Here $I$ is a disposable parameter which will be fitted from the measured paramagnetic susceptibility in the application of the theory to liquid He$^3$. The fermion creation operators $\psi_s{}^\dagger(\mathbf{x})$ are related to the momentum-space creation operators $a_{\mathbf{k}s}{}^\dagger$ by

$$\psi_s{}^\dagger(\mathbf{x}) = \Omega^{-1/2} \sum_{\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{x}) a_{\mathbf{k}s}{}^\dagger, \tag{2.2}$$

where $\Omega$ denotes the volume and the subscripts the spin orientation ("↑" or "↓"). As in Ref. 1 (hereafter referred to as DE) we make the assumption that the dominant scattering processes in the fermi liquid close to the ferromagnetic state at low temperatures is the scattering of fermions from paramagnons, or persistent spin fluctuations. In the present treatment this process is assumed to be defined in terms of a transition rate between *bare* fermion states calculated in terms of a Born approximation to the scattering of fermions from spin fluctuations. That is, in calculating the transition probability for particle-particle scattering we take account within RPA of the final-state interaction of a particle and a hole as illustrated in Fig. 1. This approximation is formulated, following Kadanoff and Baym,[11] in terms of the analytically continued retarded and advanced fermion self-energy functions $\Sigma_s{}^{\gtrless}(\mathbf{k}, \omega)$ for emission and reabsorption of paramagnons taken on the bare particle energy shell. This leads to the following Boltzmann equation for the one-particle distribution function $f_{\mathbf{k}s}$:

$$(\partial f_{\mathbf{k}s}/\partial t) + (\hbar\mathbf{k}/m)\cdot\nabla_{\mathbf{r}} f_{\mathbf{k}s} = \partial f_{\mathbf{k}s}/\partial t\,|_{\text{coll}}, \tag{2.3}$$

where

$$\partial f_{\mathbf{k}s}/\partial t\,|_{\text{coll}} = -f_{\mathbf{k}s}\Sigma_s{}^{>}(\mathbf{k}, \epsilon_{\mathbf{k}s}) + (1-f_{\mathbf{k}s})\Sigma_s{}^{<}(\mathbf{k}, \epsilon_{\mathbf{k}s}). \tag{2.4}$$

Here $\epsilon_{\mathbf{k}\uparrow} = \epsilon_{\mathbf{k}\downarrow} = \hbar^2k^2/2m$ denotes the bare fermion energy and $m$ its mass. The fermion-paramagnon scattering processes of Fig. 1 are now taken into account in Eq. (2.4) by using the self-energy formula given in DE in which the one particle Green's function $G_s{}^0(\mathbf{k}, \omega)$ is replaced by a corresponding nonequilibrium Green's function $g_s{}^0(\mathbf{k}, \omega)$ defined in terms of the one-particle distribution function $f_{\mathbf{k}s}$ by[11]

$$\begin{aligned} g_s{}^{>}(\mathbf{k}, \omega) &= 2\pi\delta(\epsilon_{\mathbf{k}s}-\omega)(1-f_{\mathbf{k}s}), \\ g_s{}^{<}(\mathbf{k}, \omega) &= 2\pi\delta(\epsilon_{\mathbf{k}s}-\omega) f_{\mathbf{k}s}, \end{aligned} \tag{2.5}$$

where $g_s{}^{\gtrless}(\mathbf{k}, \omega)$ are defined by the relations

$$g_s{}^{\gtrless}(\mathbf{k}, \omega) = (\pm i) \int_{-\infty}^{\infty} e^{i\omega t} dt\, g_s{}^{\gtrless}(\mathbf{k}, t),$$

$$\begin{aligned} g_s{}^0(\mathbf{k}, t) &= g_s{}^{>}(\mathbf{k}, t) \qquad (t>0), \\ &= g_s{}^{<}(\mathbf{k}, t) \qquad (t<0), \end{aligned}$$

with

$$g_s{}^0(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} e^{i\omega t} dt\, g_s{}^0(\mathbf{k}, t).$$

The paramagnon propagator[12] $S_{\mathbf{q}}(t)$ which enters the self-energy expression of DE is assumed to be defined

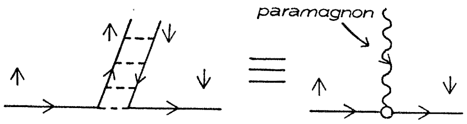

FIG. 1. Diagrams denoting the scattering of a fermion from a paramagnon calculated within the random-phase approximation.

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[8] H. R. Hart, Jr., and J. C. Wheatley, Phys. Rev. Letters **4**, 1 (1960).

[9] J. C. Wheatley, in *Quantum Fluids*, edited by D. F. Brewer (North-Holland Publishing Company, Amsterdam, 1966).

[10] A. C. Anderson, J. I. Connolly, O. E. Vilches, and J. C. Wheatley, Phys. Rev. **147**, 86 (1966); J. I. Connolly, Ph.D. thesis, University of Illinois, 1965 (unpublished).

[11] G. Baym and L. Kadanoff, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962), Chap. 9.

[12] Explicitly $S_{\mathbf{q}}(t) = i\langle T\{\sigma_{\mathbf{q}}{}^-(^-t)\sigma_{-\mathbf{q}}{}^+(0)\}\rangle$, where $\sigma_{\mathbf{q}}{}^\pm(s)$ denote the spin-density operators $\sigma_{\mathbf{q}}{}^\pm = \sum_{\mathbf{k}} a_{\mathbf{k}+\mathbf{q}\uparrow,\downarrow}{}^\dagger a_{\mathbf{k}\downarrow,\uparrow}$, taken at time $s$ in the Heisenberg representation; $T$ denotes the time-ordering operator and $\langle A \rangle$ the equilibrium average of $A$.

for thermodynamic equilibrium as in Ref. 1. We then have

$$\Sigma_{\uparrow}^{\gtrless}(\mathbf{k},\omega)=I^2\sum_{\mathbf{q}}\int_{-\infty}^{\infty}\frac{d\omega_1}{(2\pi)}\,g_{\downarrow}^{\gtrless}(\mathbf{k}+\mathbf{q},\,\omega+\omega_1)\,S_{\mathbf{q}}^{\gtrless}(\omega_1),\tag{2.6}$$

$$\Sigma_{\downarrow}^{\gtrless}(\mathbf{k},\omega)=I^2\sum_{\mathbf{q}}\int_{-\infty}^{\infty}\frac{d\omega_1}{(2\pi)}\,g_{\uparrow}^{\gtrless}(\mathbf{k}-\mathbf{q},\,\omega-\omega_1)\,S_{\mathbf{q}}^{\gtrless}(\omega_1),$$

where

$$S_{\mathbf{q}}^{\gtrless}(\omega)=(-i)\int_{-\infty}^{\infty}e^{i\omega t}S_{\mathbf{q}}^{\gtrless}(t)\,dt,$$

with

$$\begin{aligned}S_{\mathbf{q}}(t)&=S_{\mathbf{q}}^{<}(t)\qquad(t<0),\\&=S_{\mathbf{q}}^{>}(t)\qquad(t>0).\end{aligned}$$

If we introduce the spectral density function

$$A_{\mathbf{q}}(\omega)=S_{\mathbf{q}}^{>}(\omega)-S_{\mathbf{q}}^{<}(\omega),$$

we may write[3]

$$\begin{aligned}S_{\mathbf{q}}^{<}(\omega)&=[n(\omega)+1]A_{\mathbf{q}}(\omega),\\S_{\mathbf{q}}^{>}(\omega)&=n(\omega)A_{\mathbf{q}}(\omega),\end{aligned}\tag{2.7}$$

where $n(\omega)=[\exp(\beta\hbar\omega)-1]^{-1}$ is the Planck function involving Boltzmann's constant $k_B=1/\beta T$. On substituting (2.6) into (2.4) and making use of (2.5) and (2.7) we obtain

$$\begin{aligned}\left.\frac{\partial f_{\mathbf{k}\uparrow}}{\partial t}\right|_{\mathrm{coll}}=\frac{I^2}{\hbar}\sum_{\mathbf{q}}\int_{-\infty}^{\infty}d\omega A_{\mathbf{q}}(\omega)\,\delta(\epsilon_{\mathbf{k}+\mathbf{q}\,\downarrow}-\epsilon_{\mathbf{k}\uparrow}-\hbar\omega)\\\times\{(1-f_{\mathbf{k}\uparrow})f_{\mathbf{k}+\mathbf{q}\,\downarrow}[n(\omega)+1]-f_{\mathbf{k}\uparrow}(1-f_{\mathbf{k}+\mathbf{q}\,\downarrow})n(\omega)\},\end{aligned}\tag{2.8a}$$

$$\begin{aligned}\left.\frac{\partial f_{\mathbf{k}\downarrow}}{\partial t}\right|_{\mathrm{coll}}=\frac{I^2}{\hbar}\sum_{\mathbf{q}}\int_{-\infty}^{\infty}d\omega A_{\mathbf{q}}(\omega)\,\delta(\epsilon_{\mathbf{k}-\mathbf{q}\,\uparrow}-\epsilon_{\mathbf{k}\uparrow}+\hbar\omega)\\\times\{(1-f_{\mathbf{k}\downarrow})f_{\mathbf{k}-\mathbf{q}\,\uparrow}n(\omega)-f_{\mathbf{k}\downarrow}(1-f_{\mathbf{k}-\mathbf{q}\,\uparrow})[n(\omega)+1]\}.\end{aligned}\tag{2.8b}$$

As in DE we use the RPA result[7] to estimate the paramagnon propagator leading to a spectral density for paramagnons given in terms of the susceptibility response function $\chi_{\mathbf{q}}^{0}(\omega)$ for the noninteracting fermi gas by[7,3]

$$A_{\mathbf{q}}(\omega)=[2\hbar\,\mathrm{Im}\chi_{\mathbf{q}}^{0}(\omega)/|\,(1-I\chi_{\mathbf{q}}^{0}(\omega))\,|^2],\tag{2.9}$$

with

$$\chi_{\mathbf{q}}^{0}(\omega)=\sum_{\mathbf{k}}\frac{f_{\mathbf{k}+\mathbf{q}\,\downarrow}^{0}-f_{\mathbf{k}\uparrow}^{0}}{\epsilon_{\mathbf{k}\uparrow}-\epsilon_{\mathbf{k}+\mathbf{q}\,\downarrow}-\hbar\omega-i\delta}\qquad(\delta\rightarrow0+),$$

where $f_{\mathbf{k}s}^{0}$ denotes the equilibrium Fermi function. The Boltzmann equation (2.3), together with the approximations (2.8) and (2.9), form the basis of the calculations of this paper of the transport coefficients for the interacting fermi liquid. It should be emphasized that in this treatment we are considering the approach to equilibrium of the one-particle distribution defined for bare particle states. This is to be contrasted with the Landau phenomenological treatment in which the relaxation of quasiparticles is considered.

Expressions for the transport coefficients may now be formulated from (2.3) and (2.8) by following standard variational methods.[6] Specifying the deviation of $f_{\mathbf{k}s}$ from its equilibrium value $f_{\mathbf{k}s}^{0}$ by the function $\phi_{\mathbf{k}s}$, where

$$f_{\mathbf{k}s}=f_{\mathbf{k}s}^{0}-(\partial f_{\mathbf{k}s}^{0}/\partial\epsilon_{\mathbf{k}s})\phi_{\mathbf{k}s},\tag{2.10}$$

the collision integral (2.8a) may be linearized in $\phi_{\mathbf{k}s}$ to give

$$\left.\frac{\partial f_{\mathbf{k}\uparrow}}{\partial t}\right|_{\mathrm{coll}}=(k_BT)^{-1}\int\frac{d^3k'}{(2\pi)^3}\,V(\mathbf{k}'\downarrow;\mathbf{k}\uparrow)\{\phi_{\mathbf{k}'\downarrow}-\phi_{\mathbf{k}\uparrow}\},\tag{2.11}$$

where we have defined the transition rate

$$\begin{aligned}V(\mathbf{k}'\downarrow;\mathbf{k}\uparrow)=\frac{(2\pi)^3I^2}{\hbar}\sum_{\mathbf{q}}\int_{-\infty}^{\infty}d\omega A_{\mathbf{q}}(\omega)n(\omega)f_{\mathbf{k}\uparrow}^{0}(1-f_{\mathbf{k}'\downarrow}^{0})\\\times\delta(\epsilon_{\mathbf{k}'\downarrow}-\epsilon_{\mathbf{k}\uparrow}-\hbar\omega)\,\delta(\mathbf{k}'-\mathbf{k}-\mathbf{q}).\end{aligned}\tag{2.12}$$

In deriving (2.11) and (2.12) we have made use of the detailed balancing condition

$$(1-f_{\mathbf{k}\uparrow}^{0})f_{\mathbf{k}+\mathbf{q}\,\downarrow}^{0}[n(\omega)+1]=f_{\mathbf{k}\uparrow}^{0}(1-f_{\mathbf{k}+\mathbf{q}\,\downarrow}^{0})n(\omega),\tag{2.13}$$

which results from the vanishing of the left hand sides of (2.8) at equilibrium. The equilibrium distribution $f_{\mathbf{k}s}^{0}$ is

$$f_{\mathbf{k}s}^{0}=[\exp(\beta(\epsilon_{\mathbf{k}s}-\mu_s))+1]^{-1},$$

with the chemical potential $\mu_s$ determined by the requirement

$$\int\frac{d^3k}{(2\pi)^3}f_{\mathbf{k}s}^{0}=n,$$

where $n$ denotes the equilibrium concentration of fermions of specified spin. Equation (2.8b) may also be linearized to give

$$\left.\frac{\partial f_{\mathbf{k}\downarrow}}{\partial t}\right|_{\mathrm{coll}}=(k_BT)^{-1}\int\frac{d^3k'}{(2\pi)^3}\,V(\mathbf{k}'\uparrow;\mathbf{k}\downarrow)\{\phi_{\mathbf{k}'\uparrow}-\phi_{\mathbf{k}\downarrow}\}.\tag{2.14}$$

We note that the transition rate (2.12) is symmetrical in the sense that

$$V(\mathbf{k}'s';\mathbf{k}s)=V(\mathbf{k}s;\mathbf{k}'s')\qquad(s\neq s').\tag{2.15}$$

This result follows from (2.12) with the use of (2.13) and the observation from (2.9) that $A_{\mathbf{q}}(\omega)=-A-\mathbf{q}(-\omega)$. With the symmetry property (2.15) the collision integrals (2.11) and (2.14) are of the standard form of conventional transport theory.[6] It then follows that the quantity

$$\dot{S}=(2k_BT^2)^{-1}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\sum_{s\neq s'}V(\mathbf{k}'s';\,\mathbf{k}s)\times\{\phi_{\mathbf{k}'s'}-\phi_{\mathbf{k}s}\}^2 \qquad (2.16)$$

can be interpreted as the rate of irreversible entropy production in terms of which the coefficients of thermal conductivity $K$ and spin diffusion[13] $D$ may be expressed as[14,6]

$$1/K=(T^2\dot{S}/J_H^2), \qquad (2.17)$$

$$\frac{1}{D}=\frac{T\dot{S}}{2J_{N\uparrow}^2(\partial\mu_\uparrow/\partial n)}=\frac{T\dot{S}}{2J_{N\downarrow}^2(\partial\mu_\downarrow/\partial n)}, \qquad (2.18)$$

where $\mathbf{J}_{Ns}$ and $\mathbf{J}_H$ are, respectively, the macroscopic particle flux (for a given spin orientation) and energy flux, given in terms of $\phi_{\mathbf{k}s}$ by

$$\mathbf{J}_{Ns}=-\int\frac{d^3k}{(2\pi)^3}\frac{\hbar\mathbf{k}}{m}\phi_{\mathbf{k}s}\frac{\partial f_{\mathbf{k}s}{}^0}{\partial\epsilon_{\mathbf{k}s}}, \qquad (2.19)$$

$$\mathbf{J}_H=-\int\frac{d^3k}{(2\pi)^3}\frac{\hbar\mathbf{k}}{m}(\epsilon_{\mathbf{k}s}-\mu_s)\phi_{\mathbf{k}s}\frac{\partial f_{\mathbf{k}s}{}^0}{\partial\epsilon_{\mathbf{k}s}}. \qquad (2.20)$$

In obtaining the above expression for the diffusion coefficient we have assumed zero net magnetization. This will imply[5] that $\phi_{\mathbf{k}\uparrow}=-\phi_{\mathbf{k}\downarrow}$. This is to be contrasted with the case of thermal conduction where by symmetry we have $\phi_{\mathbf{k}\uparrow}=\phi_{\mathbf{k}\downarrow}$. Bearing in mind these considerations, Eqs. (2.16)–(2.20) can be used to express $K$ and $D$ in terms of $\phi_{\mathbf{k}\uparrow}$ (say) only:

$$K^{-1}=\left(k_B^{-1}\iint d^3kd^3k'V(\mathbf{k}'\downarrow;\,\mathbf{k}\uparrow)\{\phi_{\mathbf{k}'\uparrow}-\phi_{\mathbf{k}\uparrow}\}^2\Big/\left|\frac{2\hbar}{m}\int d^3k\mathbf{k}(\epsilon_{\mathbf{k}\uparrow}-\mu_\uparrow)\phi_{\mathbf{k}\uparrow}\frac{\partial f_{\mathbf{k}\uparrow}{}^0}{\partial\epsilon_{\mathbf{k}\uparrow}}\right|^2\right), \qquad (2.21)$$

$$D^{-1}=\left((2k_BT)^{-1}\iint d^3kd^3k'V(\mathbf{k}'\downarrow;\,\mathbf{k}\uparrow)\{\phi_{\mathbf{k}'\uparrow}+\phi_{\mathbf{k}\uparrow}\}^2\Big/\left|\frac{\hbar}{m}\int d^3k\mathbf{k}\phi_{\mathbf{k}\uparrow}\frac{\partial f_{\mathbf{k}\uparrow}{}^0}{\partial\epsilon_{\mathbf{k}\uparrow}}\right|^2\frac{\partial\mu_\uparrow}{\partial n_\uparrow}\right). \qquad (2.22)$$

The standard method of evaluating these expressions consists of introducing an explicit approximation, or trial function, for $\phi_{\mathbf{k}s}$. For the calculation of the thermal conductivity $K$ we shall take the usual trial function[6,4]

$$\phi_{\mathbf{k}s}=(\epsilon_{\mathbf{k}s}-\mu_s)\,\mathbf{k}\cdot\mathbf{u}\times\text{constant}, \qquad (2.23)$$

which is close to the true solution for $T\ll T_f$. Here $\mathbf{u}$ denotes a unit vector in the direction of the temperature gradient. The appropriate trial function for the calculation of $D$ is[5]

$$\begin{aligned}\phi_{\mathbf{k}\uparrow}&=\mathbf{k}\cdot\mathbf{u}_\uparrow\times\text{constant}\\&=-\phi_{\mathbf{k}\downarrow},\end{aligned} \qquad (2.24)$$

where $\mathbf{u}_\uparrow$ denotes a unit vector in the direction of the concentration gradient of ↑-spin fermions. Equations (2.21)–(2.24) together with a simple model approximation for the spectral density $A_\mathbf{q}(\omega)$ will be used to estimate $K$ and $D$ in the following section.

## 3. MODEL CALCULATIONS

In order to obtain tractable results for the transport coefficients we propose the following model approximation for the spectral density $A_\mathbf{q}(\omega)$:

$$\begin{aligned}A_\mathbf{q}(\omega)&=\alpha\omega/q &&(0\leq|\omega|\leq\omega_q),\\&=0 &&(|\omega|>\omega_q),\end{aligned} \qquad (3.1)$$

where

$$\begin{aligned}\hbar\omega_q&=(4\kappa_0{}^2\epsilon_f/\pi\bar{I})(q/k_f) &&(0\leq q\leq Q),\\&=0 &&(q>Q),\end{aligned} \qquad (3.2)$$

and

$$\alpha=\pi N(\epsilon_f)\bar{I}\hbar^2k_f/2\kappa_0{}^4\epsilon_f. \qquad (3.3)$$

In the above equations $\epsilon_f$ denotes the fermi energy $\hbar^2k_f{}^2/2m$, $N(\epsilon_f)$ the single particle density of states, $\sum_\mathbf{k}\delta(\epsilon_{\mathbf{k}s}-\epsilon_f)$, $\bar{I}=N(\epsilon_f)I$, and $1/\kappa_0{}^2=1/(1-\bar{I})$ the RPA exchange enhancement factor [7,1] for the interacting fermi-liquid. $Q$, a cutoff wave vector, is to be regarded as a disposable parameter and will be fitted from the observed specific heat in the application of the model to liquid He$^3$. The model represents an approximation to the actual form of the RPA formula (2.9) evaluated at $T=0$ for small $q/2k_f$,[3,15] and serves as a reasonable approximation for long-wavelength paramagnons. Explicitly the model neglects the contribution to $A_\mathbf{q}(\omega)$, which is strongly peaked about $\omega_q$ for $\kappa_0{}^2\ll1$, from the higher-frequency region $|\omega|>\omega_q$ while it extrapolates the correct low-frequency form of $A_\mathbf{q}(\omega)$ up to a maximum wave vector $Q$. We note here that it is the scattering of fermions from long-wavelength paramagnons (small-angle scattering) which will give the predominant contribution to the thermal conductivity $K$ and spin diffusion $D$. This may be seen generally by considering the distortion of $f_{\mathbf{k}s}$ about its equilibrium value $f_{\mathbf{k}s}{}^0$ given by the solutions (2.23) and (2.24) corresponding, respectively, to ther-

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[13] Spin diffusion is sometimes referred to as "self-diffusion."

[14] S. R. DeGroot, *Thermodynamics of Irreversible Processes* (North-Holland Publishing Company, Amsterdam, 1951), pp. 101–106.

[15] S. Doniach, Proc. Phys. Soc. (London) (to be published).

mal conduction and spin diffusion. As discussed by Ziman,[16] the discontinuity in sign of (2.23) as $\epsilon_{qs}$ passes through the fermi surface means that at low temperatures, equilibrium is restored predominantly by the relaxation of fermion-hole pairs that are narrowly separated in **k** space at the Fermi surface. In the present discussion such processes are animated by the emission or absorption of long-wavelength (small $q$) paramagnons. The same conclusion also results for Eq. (2.24) where a discontinuity in sign occurs when the direction of the spin is reversed at the fermi surface. This seems a plausible explanation of the subsequent success of the above approximations in fitting $K$ and $D$ [17] to experiment (to be discussed in Sec. 4).

The expressions for the transport coefficients given in Sec. 2 may now be evaluated by following essentially the same procedure as used in electron-phonon theory.[18] We first consider the thermal conductivity. Since we have $\epsilon_{\mathbf{k}\uparrow}=\epsilon_{\mathbf{k}\downarrow}=\hbar^2k^2/2m$ we shall from now on drop the spin subscript from $\epsilon_{\mathbf{k}s}$ and $f_{\mathbf{k}s}{}^0$. Also since we are concerned with the low-temperature region $k_BT\ll\mu_s$, we put $\mu_\uparrow=\mu_\downarrow=\epsilon_f$, where

$$k_f=(6\pi^2n)^{1/3}.$$

Then with the trial function (2.23) and the transition rate (2.12), the expression (2.21) for $K$ becomes

$$K^{-1}=\frac{I^2\Omega}{\hbar k_B}\int d^3q\int_{-\infty}^{\infty}d\omega A_{\mathbf{q}}(\omega)B_{\mathbf{q}}(\omega)n(\omega)/A, \tag{3.4}$$

where

$$B_{\mathbf{q}}(\omega)=\int d^3kf_{\mathbf{k}}{}^0(1-f_{\mathbf{k+q}}{}^0)\delta(\epsilon_{\mathbf{k+q}}-\epsilon_{\mathbf{k}}-\hbar\omega)$$
$$\times\{\mathbf{u}\cdot[(\mathbf{k}+\mathbf{q})(\epsilon_{\mathbf{k+q}}-\epsilon_f)-\mathbf{k}(\epsilon_{\mathbf{k}}-\epsilon_f)]\}^2, \tag{3.5}$$

and

$$A=\left|\frac{2\hbar}{m}\int d^3k\mathbf{k}(\epsilon_{\mathbf{k}}-\epsilon_f)^2\mathbf{u}\cdot\mathbf{k}\,\frac{\partial f_k{}^0}{\partial\epsilon_{\mathbf{k}}}\right|^2. \tag{3.6}$$

The volume $\Omega$ appears as a factor in (3.4) as the result of replacing the summation over **q** by an integral. At this point it is convenient to define the characteristic temperatures $k_BT_f=\epsilon_f$ and

$$k_B\Theta=\hbar\omega_Q, \tag{3.7}$$

where $\omega_q$ is defined by (3.2). In calculating (3.4) we will consistently neglect terms of order $(T/T_f)$ relative to the others. The result is then the first term in a power series in $(T/T_f)$, the coefficients of which are functions of $(T/\Theta)$. On eliminating the delta function in the integrand of (3.5), $B_{\mathbf{q}}(\omega)$ reduces to the integral

$$B_{\mathbf{q}}(\omega)=\left(\frac{2m}{\hbar^2}\right)^2\frac{\pi}{6q}\int_{\epsilon_0}^{\infty}d\epsilon\,\frac{[2m\omega^2(\epsilon+\hbar\omega)+(\epsilon-\epsilon_f)q^2(\epsilon-\epsilon_f+\hbar\omega)]}{\{\exp[\beta(\epsilon-\epsilon_f)]+1\}\{1+\exp[-\beta(\epsilon+\hbar\omega-\epsilon_f)]\}}. \tag{3.8}$$

The lower limit $\epsilon_0$, which guarantees the vanishing of the argument of the delta function, is given by $\epsilon_0=\hbar^2k_0{}^2/2m$, where

$$k_0=|\,q/2-m\omega/\hbar q\,|. \tag{3.9}$$

In view of (3.1) the wave vector $p=m|\omega|/\hbar q$ will never be greater than $m\omega_q/\hbar q$, which by (3.2) is $\sim\kappa_0{}^2k_f$, showing that $p\ll k_f$ for $\kappa_0{}^2\rightarrow0$. Then for the larger part of the subsequent integration over $\omega$ and **q** we have $\epsilon_0$ below the Fermi surface in the nearly ferromagnetic fermi liquid, so that the lower limit in (3.8) may be effectively set equal to zero. Following Wilson[18] we can then expand (3.8) to obtain

$$B_{\mathbf{q}}(\omega)=\left(\frac{2m}{\hbar^2}\right)^2\frac{\pi}{6q}(k_BT)^3\frac{\bar{\omega}}{[1-\exp(-\bar{\omega})]}$$
$$\times\{\bar{\omega}^2k_f{}^2+\tfrac{1}{3}\pi^2q^2-\tfrac{1}{6}\bar{\omega}^2q^2\}, \tag{3.10}$$

where $\bar{\omega}=\hbar\omega/k_BT$. The denominator of (3.4) can be similarly calculated to yield

$$A=\{(8\pi^3/9\hbar)(k_BT)^2k_f{}^3\}^2. \tag{3.11}$$

On inserting (3.10) into (3.4) and introducing the approximation (3.1) and (3.2) for $A_{\mathbf{q}}(\omega)$, the integrals over $\omega$ and **q** are easily formulated. Using (3.11) and (3.3), formula (3.4) for $1/K$ can then be expressed in the form

$$1/TK=\alpha_Kw_K(T/\Theta), \tag{3.12}$$

where $\alpha_K$ is the constant

$$\alpha_K=\frac{27W_K(0)}{8\pi}\cdot\frac{\bar{I}^3\bar{Q}\hbar}{\epsilon_f{}^2\kappa_0{}^4k_f}\qquad(\bar{Q}\equiv Q/k_f), \tag{3.13}$$

and

$$w_K(T/\Theta)=W_K(T/\Theta)/W_K(0), \tag{3.14}$$

with

$$W_K(t)=[J_{(4)}(t^{-1})-tJ_{(5)}(t^{-1})]$$
$$+\tfrac{1}{9}(\bar{Q}^2\pi^2)[J_{(2)}(t^{-1})-t^3J_{(5)}(t^{-1})]$$
$$-\tfrac{1}{18}\bar{Q}^2[J_{(4)}(t^{-1})-t^3J_{(7)}(t^{-1})]. \tag{3.15}$$

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[16] See Ref. 6, Chap. 9, p. 386.

[17] We expect that a calculation of the viscosity coefficient on the same model for $A_{\mathbf{q}}(a)$ would break down quantitatively since it can be shown that the viscosity would receive its dominant contribution from the short-wavelength paramagnons.

[18] A. Wilson, *The Theory of Metals* (Cambridge University Press, New York, 1953), Chap. 9.

The function $J_{(v)}(y)$ ($v$ integer) is the definite integral

$$J_{(v)}(y)=\int_0^y \frac{x^v dx}{(e^x-1)(1-e^{-x})}\,. \tag{3.16}$$

Formula (2.22) for $1/D$ may be similarly calculated for our model (3.1)–(3.3). Using the appropriate trial function (2.24) together with the transition rate (2.12) we find

$$1/DT^2=\alpha_D w_D(T/\Theta), \tag{3.17}$$

where

$$\alpha_D=3\pi W_D(0)\bar{I}^3\bar{Q}k_f{}^2k_B{}^2\hbar/\epsilon_f{}^3\kappa_0{}^4, \tag{3.18}$$

and

$$w_D(T/\Theta)=W_D(T/\Theta)/W_D(0), \tag{3.19}$$

with

$$W_D(t)=[J_{(2)}(t^{-1})-tJ_{(3)}(t^{-1})] -\tfrac{1}{12}\bar{Q}^2[J_{(2)}(t^{-1})-t^3J_{(5)}(t^{-1})]. \tag{3.20}$$

As in the previous calculation of $K$, we have neglected terms of order $(T/T_f)$ relative to others.

For a given value of the cutoff parameter $\bar{Q}=Q/k_f$ we note that $w_K(T/\Theta)$ and $w_D(T/\Theta)$ are functions of the variable $(T/\Theta)$ only. In Appendix B we have computed these functions in the range $0\leq(T/\Theta)\leq1$ for two values of $\bar{Q}$ which result from a subsequent fit of the theory to liquid He$^3$.

## 4. APPLICATION TO LIQUID He$^3$

In this section we apply the theory of the two previous sections to a discussion of liquid He$^3$. Firstly we examine the temperature dependences of $K$ and $D$ that are predicted by the results of Sec. 3 at low temperatures.

At low temperatures $T$ for which $T/\Theta\ll1$ the integrals $J_{(v)}(\Theta/T)$ defined by (3.16) may be replaced by constants $J_{(v)}(\infty)$ which may be expressed in terms of Riemann zeta functions $\zeta(v)$ by[18]

$$J_{(v)}(\infty)=v!\zeta(v)\equiv J_{(v)}{}^0. \tag{4.1}$$

For the particular values of $v$ encountered in Sec. 3, $2\leq v\leq7$, this will be legitimate for $T\lesssim\Theta/10$. It then follows from (3.12), (3.15), (3.17), and (3.20) that $K$ and $D$ have temperature dependences of the form

$$1/TK=\alpha_K-\beta_K(T/\Theta)+O[(T/\Theta)^3], \tag{4.2}$$

$$1/DT^2=\alpha_D-\beta_D(T/\Theta)+O[(T/\Theta)^3], \tag{4.3}$$

where the coefficients of $(T/\Theta)$ on the left-hand sides of (4.2) and (4.3) are constants given by

$$\alpha_K/\beta_K=\{J_{(4)}{}^0(1-\bar{Q}^2/18)+(\pi^2\bar{Q}^2/9)J_{(2)}{}^0\}/J_{(5)}{}^0, \tag{4.4}$$

$$\alpha_D/\beta_D=J_{(2)}{}^0(1-\bar{Q}^2/12)/J_{(3)}{}^0, \tag{4.5}$$

with $\alpha_K$ and $\alpha_D$ defined as in Sec. 3. By (3.2) and (3.3) the explicit formula for $\Theta$ is

$$\Theta=(4/\pi)(\kappa_0{}^2/\bar{I})\bar{Q}T_f. \tag{4.6}$$

In the limit $T\rightarrow0$ it is seen from (4.2) and (4.3) that $1/TK$ and $1/DT^2$ are constants independent of $T$ in agreement with the predictions of the Landau theory.[4,5] As $T$ increases from zero, however, $K$ and $D$ fall off increasingly less rapidly than $T^{-1}$ and $T^{-2}$, respectively, reach minima, and then increase with increasing $T$. This qualitative prediction is in agreement with the experimental work on liquid He$^3$.[8,10]

We note that in the nearly ferromagnetic fermi liquid $\Theta\lll T_f$. Thus there exists a special fermi-liquid region, for example, $\Theta\lesssim T\ll T_f$, in which Landau theory will fail. This point is most directly illustrated by the calculation of the single-fermion self-energy given in *DE*.

In applying our theory to liquid He$^3$ we shall fit the parameter $\bar{I}$ and hence $\kappa_0{}^2$ from the observed static paramagnetic susceptibility $\chi$ by using the result

$$\chi=\chi_0/(1-\bar{I}), \tag{4.7}$$

obtained in DE for the interacting fermi liquid, where $\chi_0=2N(\epsilon_f)$ is the Pauli susceptibility for the noninteracting fermi gas. As discussed by the latter, data on the low-temperature susceptibility of He$^3$ at high and low pressures given by Wheatley[9] lead to the values of $\kappa_0{}^2$ shown in Table I. The corresponding values of $T_f$ are also shown. In order to fit the values of the cutoff parameter $\bar{Q}$ we use the model approximation of Sec. 3 for $A_q(\omega)$ to calculate the renormalized fermion mass $m^*$ which results from the virtual fermion-paramagnon scattering processes discussed in DE. In Appendix A we find that this is

$$m^*/m=1+(\bar{Q}^2/2\pi)(\bar{I}^2/\kappa_0{}^2). \tag{4.8}$$

This result is to be contrasted with the result [(A5) of Appendix A] established in DE, where a logarithmic dependence on $\bar{Q}^2/\kappa_0{}^2$ was found. The difference arises because the latter authors used a more complete approximation for $A_q(\omega)$, allowing for the higher frequency region $\omega\geq\omega_q$ which we neglected. The result (4.8) together with the appropriate finite temperature corrections, is nevertheless sufficiently good to account for the observed low temperature variation of the specific heat in liquid He$^3$. This is demonstrated in

TABLE I. Values of parameters for He$^3$.

| Pressure (atm) | 0.28 | 27 |
|---|---|---|
| $\kappa_0{}^2$ | 1/9 | 1/21 |
| $\bar{Q}$ | 1.4 | 1.3 |
| $\theta$ | 1.1°K | 0.5°K |
| $T_f$ | 5.0°K | 6.2°K |

Appendix A. The low-temperature measurements of Abel *et al.*[19] give $m^*/m$ equals 3.1 and 5.8 at 0.28 and 27.0 atm, respectively. Using these values together with the previously determined values of $\kappa_0^2$, (4.8) leads to values of $\bar{Q}$ of 1.4 and 1.3 at the lower and higher pressures, respectively. By (4.6) we find $\Theta=1.1°\mathrm{K}$ at 0.28 atm and $\Theta=0.5°\mathrm{K}$ at 27 atm. In using (4.8) to fit $\bar{Q}$ we have made the extreme assumption that all of the observed mass enhancement is due to the mechanism considered here. The fitted values of $\bar{Q}$ do not vary significantly with pressure and correspond to minimum wavelengths of the order of twice the mean interparticle separation $\{3/(4\pi 2n)\}^{1/3}$.

We note that on the basis of (4.2), valid for $T \ll \Theta$, a plot of $1/TK$ against $T$ gives a straight line which intercepts the $1/TK$ axis at $\alpha_K$ and the $T$ axis at $(\alpha_K/\beta_K)\Theta$. By (4.4) the latter intercept can yield a value for $\Theta$ if $\bar{Q}$ is known. These considerations are applied to fit formula (3.12) for $1/TK$ to the low-pressure data of Anderson *et al.*[10] Plotting $1/TK$ against $T$ and using the previously determined value of 1.4 for $\bar{Q}$, a straight line drawn through the low-temperature points gives $\Theta=1.04°\mathrm{K}$ and $\alpha_K=2.5\times10^{-2}$ c.g.s. units. These values of $\alpha_K$ and $\Theta$ were used in (3.12) to fit the higher-temperature points. The resulting fit is shown in Fig. 2. Agreement is remarkable. The value $\Theta=1.04°\mathrm{K}$ found here is close to the value $\Theta=1.1°\mathrm{K}$ which we calculated from (4.6) using the fitted values of $\bar{Q}$ and $\kappa_0^2$. The magnitude of $\alpha_K$ which results from using the low-pressure parameters of Table I in the theoretical formula (3.13) is $6.2\times10^{-2}$ c.g.s. units, in reasonable agreement with the value just obtained from drawing the straight line through the low-temperature points. We point out here that use of the single-trial function (2.23) for the calculation of the thermal conductivity actually serves only to place an upper bound on $\alpha_K$ [6] and likewise on $\alpha_D$ in the calculation of $1/DT^2$ with (2.24).

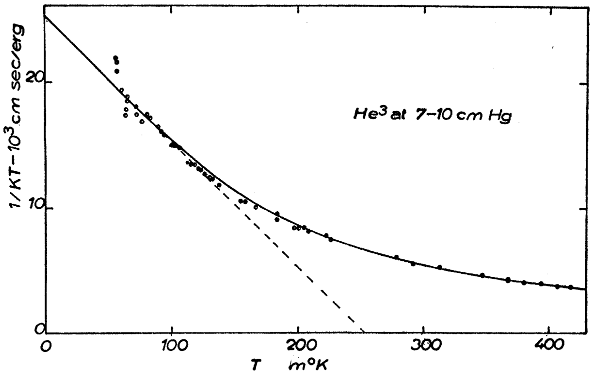

FIG. 2. $1/TK$ versus $T$ for liquid He at low pressure. The broken line is the continuation of the straight line which was drawn through the lowest-temperature points in order to fit $\alpha_K$ and $\Theta$ on the basis of (4.2). The points were taken from the published data of Anderson *et al.* (Ref. 10). The full line is the theoretical curve computed from (3.12).

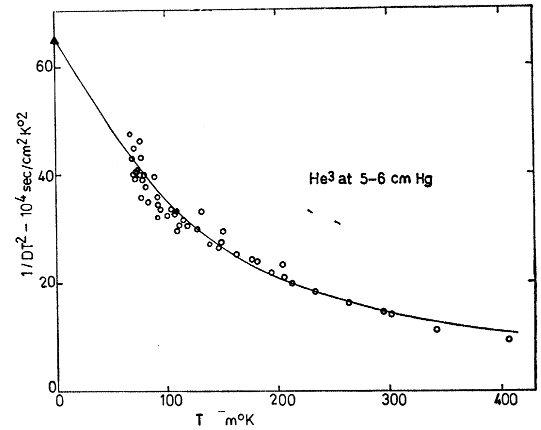

FIG. 3. $1/DT^2$ versus $T$ for liquid He at low pressure. The open circles denote the data of Hart and Wheatley (Ref. 8). The triangular point drawn on the $1/DT^2$ axis represents the value of $1/DT^2$ at $T=0$ as determined from the ultra-low-temperature measurements of Abel *et al.* (Ref. 21). The full line is the theoretical curve computed from (3.17).

We have also fitted formula (3.17) for $1/DT^2$ to the low-pressure diffusion data of Hart and Wheatley.[8] Here we used a value for $\alpha_D$ equal to $6.5\times10^5$ c.g.s. units, obtained independently from the ultra-low-temperature measurements[20] of Abel *et al.*,[21,22] while $\bar{Q}$ was fixed as before at 1.4.[22] It was then found necessary to use a value for $\Theta$ equal to 0.5°K, this being somewhat lower than the value used for the thermal conductivity data. The fit obtained in this way is shown in Fig. 3. The value $\alpha_D=6.5\times10^5$ c.g.s. units is close to the theoretical value which may be calculated from (3.18) and Table I to be $\alpha_D=8.3\times10^5$ c.g.s. units.

We conclude, therefore, that it is the paramagnetic scattering of fermions which account for the low-temperature processes of thermal conduction and spin diffusion in liquid He³.

## ACKNOWLEDGMENTS

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[19] W. R. Abel, A. C. Anderson, W. C. Black, and J. C. Wheatley, Phys. Rev. **147**, 111 (1966).

[20] In this temperature range, which extended down to a few millidegrees, it was found that the temperature dependence of the diffusion coefficient could be represented by a $T^2$ law, that is $D=1/(\alpha_D T^2)$.

[21] W. R. Abel, A. C. Anderson, W. C. Black, and J. C. Wheatley, Phys. Rev. Letters **14**, 129 (1965); Physics **1**, 337 (1965).

[22] Our calculation of $C_v/C_v^0$ includes a contribution of order $(T/\Theta)^4$ [see Eq. (A3)] which was neglected in the corresponding calculation performed in DE. The neglect of this term probably accounts for the breakdown of the latter authors' fit to the high-pressure specific-heat data above 20 m°K.

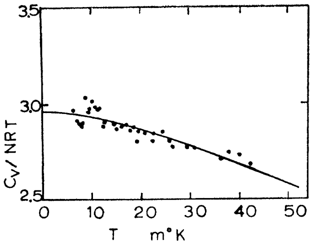

FIG 4. Heat capacity of liquid He$^3$ at low pressure after Abel *et al.* (Ref. 19). The theoretical curve was computed from (A.3) with $m^*/m=3.1$ and $\Theta=1.4$°K.

## APPENDIX A

Here we apply our model approximation (3.1)–(3.3) for $A_{\mathbf{q}}(\omega)$ to calculate the corrections to the bare fermion mass, and hence specific heat, that arise from fermion-paramagnon interaction. To do this it is convenient to use the formula derived in DE for the entropy shift $\Delta S(T\neq 0)$ caused by this interaction:

$$\Delta S=-2\sum_{\mathbf{k}}(\partial f_{\mathbf{k}}^0/\partial T)\Delta\epsilon(\mathbf{k},\epsilon_{\mathbf{k}}),\qquad(\text{A.1})$$

where $f_k^0$ denotes the equilibrium fermi function $\epsilon_{\mathbf{k}}=k^2/2m-\epsilon_f(\hbar\equiv 1)$ and $\Delta\epsilon$, the shift in the single fermion energy which within the model (2.1), is given in terms of $A_{\mathbf{q}}(\omega)$ by

$$\Delta\epsilon(\mathbf{k},\omega)=\frac{I^2m}{(2\pi)^3k}\int_0^Q dq\,q\int_0^\infty d\omega_1 A_{\mathbf{q}}(\omega_1)\int_{-\infty}^\infty d\epsilon_{\mathbf{k}'}\frac{\partial f_{\mathbf{k}'}^0}{\partial\epsilon_{\mathbf{k}'}}\times\ln\left|\frac{\omega-\epsilon_{\mathbf{k}'}+\omega_1}{\omega-\epsilon_{\mathbf{k}'}-\omega_1}\right|.\qquad(\text{A.2})$$

Using (3.1)–(3.3) for $A_{\mathbf{q}}(\omega)$ and the well-known expansion

$$-\int_{-\infty}^\infty\phi(\epsilon_{\mathbf{k}})d\epsilon_{\mathbf{k}}\frac{\partial f_{\mathbf{k}}^0}{\partial\epsilon_{\mathbf{k}}}=\phi(0)+\tfrac{1}{6}\pi^2(k_BT)^2\ddot{\phi}(0)+\cdots,$$

valid for $T\ll T_f$, (A.1) and (A.2) may be evaluated in powers of $(T/\Theta)$ to give a specific heat $C_v$ at low temperatures of the form

$$\frac{C_v}{C_v^0}=\frac{m^*}{m}-\tfrac{32}{5}\pi^2\left(\frac{m^*}{m}-1\right)\left(\frac{T}{\Theta}\right)^2\times\left[1+\tfrac{3}{4}\ln\left(\frac{\Theta}{T}\right)-\tfrac{75}{14}\pi^2\left(\frac{T}{\Theta}\right)^2\right]+O\left[\left(\frac{T}{\Theta}\right)^6\right],\qquad(\text{A.3})$$

where the effective fermion mass $m^*$ at $T=0$ is

$$m^*/m=1+(\bar{Q}^2/2\pi)(\bar{I}^2/\kappa_0^2),\qquad(\text{A.4})$$

and

$$C_v^0=\tfrac{2}{3}\pi^2N(\epsilon_f)k_B^2T.$$

As mentioned in Sec. 4, (A.4) does not contain the logarithmic dependence obtained in the corresponding result of DE,

$$m^*/m=1+3\bar{I}\ln[1+(\bar{Q}^2\bar{I}/12\kappa_0^2)],\qquad(\text{A.5})$$

which was derived from (A.1) on the basis of a more refined approximation for $A_{\mathbf{q}}(\omega)$. We stress, however, that (A.3) exhibits the same temperature dependence for $C_v$ as found in DE. Thus the effective mass $C_v/C_v^0$ falls off as $(T/\Theta)^2\ln(\Theta/T)$ in the vicinity of $T=0$. In order to check the plausibility of the result (A.3) we have applied it to fit the low-temperature data on the specific heat of liquid He$^3$, obtained by Abel *et al.*[19] at 0.28 atm. Taking $m^*/m=3.1$ [19] formula (A.3) gives the fit shown in Fig. 4 on using a value for $\Theta$ equal to 1.4°K. This value of $\Theta$ is in fair agreement with the theoretical value $\Theta=1.1$°K obtained in Sec. 4 (Table I) from the fitted values of $\bar{Q}$ and $\kappa_0^2$. We conclude, therefore, that the approximations of the present paper are also sufficiently good to account for the observed low-temperature variation of the specific heat in liquid He$^3$.[22]

TABLE II. Values of the function $w_K$ for different values of $T/\Theta$ and $\bar{Q}$.

| $T/\Theta$ | $\bar{Q}=1.4$ $w_K(T/\Theta)$ | $\bar{Q}=1.4$ $w_D(T/\Theta)$ | $\bar{Q}=1.3$ $w_K(T/\Theta)$ | $\bar{Q}=1.3$ $w_D(T/\Theta)$ |
|---|---|---|---|---|
| 0.00 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.02 | 0.9177 | 0.9476 | 0.9160 | 0.9490 |
| 0.04 | 0.8359 | 0.8956 | 0.8325 | 0.8983 |
| 0.06 | 0.7550 | 0.8444 | 0.7499 | 0.8482 |
| 0.08 | 0.6757 | 0.7942 | 0.6686 | 0.7990 |
| 0.10 | 0.5998 | 0.7454 | 0.5908 | 0.7511 |
| 0.20 | 0.3282 | 0.5397 | 0.3132 | 0.5465 |
| 0.30 | 0.2064 | 0.4050 | 0.1917 | 0.4109 |
| 0.40 | 0.1477 | 0.3190 | 0.1349 | 0.3241 |
| 0.50 | 0.1147 | 0.2616 | 0.1036 | 0.2659 |
| 0.60 | 0.0938 | 0.2211 | 0.0841 | 0.2284 |
| 0.70 | 0.0893 | 0.1912 | 0.0708 | 0.1944 |
| 0.80 | 0.0688 | 0.1680 | 0.0612 | 0.1710 |
| 0.90 | 0.0608 | 0.1499 | 0.0540 | 0.1527 |
| 1.00 | 0.0545 | 0.1355 | 0.0483 | 0.1378 |

## APPENDIX B

In this Appendix we tabulate values of the functions $w_K(T/\Theta)$ and $w_D(T/\Theta)$ in the range $0\leq T/\Theta\leq 1$ for two values of $\bar{Q}$, $\bar{Q}=1.3$ and $\bar{Q}=1.4$, these values of $\bar{Q}$ being representative of liquid He$^3$ at high and low pressures, respectively (see Sec. 4). By (3.14), (3.15), (3.19), and (3.20) $w_K(T/\Theta)$ and $w_D(T/\Theta)$ involve the definite integrals $J_{(v)}(x)$ defined by (3.16). In the range $0\leq T/\Theta\leq 1$, we have $x\geq 1$ so that we may employ the series expansion

$$J_{(v)}(x)=v!\sum_{s=1}^\infty\frac{1}{s^v}-\frac{x^v}{e^x-1}-v!\sum_{s=1}^\infty\frac{e^{-sv}}{s^v}\left(1+sx+\cdots+\frac{(v-1)x^{v-1}}{(v-1)!}\right),\qquad(\text{B.1})$$

due to Sondheimer.[23] This expansion has been used to evaluate $w_K(T/\Theta)$ and $w_D(T/\Theta)$ correct to four places of decimals. The results for the two values of $\bar{Q}$ are shown in Table II.

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[23] E. H. Sondheimer, Proc. Roy. Soc. (London) **A203**, 75 (1950); see also Ref. 18, p. 336.