Study of Branch Points in the Angular-Momentum Plane*†

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A study is made of the Amati-Fubini-Stanghellini (AFS) type of approximation to the amplitudes associated with the exchange of a single Regge pole and an elementary spinless particle, and with the exchange of two Regge poles. The location, motion, and nature of the singularities in the complex-angular-momentum plane of the s reaction which appear in these approximations, and their cancellation in the full diagram, are considered in detail; the singularities are found to be of two general types: branch points whose positions are independent of, and dependent on, particle masses. Only the former singularities determine the asymptotic behavior of the AFS amplitudes in the physical scattering region, while the latter appear only on the physical sheet via the mass-independent branch points at unphysical momentum transfers. The same method used in the study of the AFS approximation to the diagrams which do not have the AFS-type singularities is applied to the analysis of the Mandelstam diagrams for which the above-mentioned cancellation of the cuts does not occur; the analysis, although less rigorous, suggests that the location and nature of the singularities in the j plane are the same as those found for the AFS type of approximations to the simpler versions of these diagrams. With a number of approximations which, although plausible, are hard to justify rigorously, an estimate is made of the contribution to the amplitude coming from the angular-momentum cut.

I. INTRODUCTION

It was originally noticed by Amati, Fubini, and Stanghellini (AFS) that if one combines two Regge poles according to two-body unitarity in the t channel, and then disperses the resultant absorptive part in t, one arrives at an amplitude which exhibits moving branch points in the angular-momentum plane of the s reaction.[1] Although the cuts suggested by AFS were later found by Mandelstam to be absent in the diagram considered by them,[2] these cuts are nevertheless believed to be present in more complicated diagrams such as the ones shown in Figs. 10 and 11 (see Refs. 2–5); their crucial feature is the appearance of the crossed lines. The presence of the Mandelstam cuts is the result of inelastic contributions to the unitarity relation, and is particular to the relativistic problem (for potential scattering the crossed graphs do not occur). If such singularities indeed exist then they cannot be ignored, since it was shown by the above authors that their contribution to the amplitude at large t is similar to that of a Regge pole (except for logarithmic factors), where the trajectory function $\alpha(s)$ is replaced by $\lambda(s)$:

$$\lambda(s)=2\alpha(s/4)-1.$$

[Actually, AFS did not write it in this form; we shall see, however, that the above expression for $\lambda(s)$ is rigorously true.] Thus if $\alpha(s)$ is the Pomeranchuk trajectory, for example, then the branch point will coincide with the position of the Pomeranchuk pole at $s=0$ (i.e., in the forward direction), while for $s\lesssim 0$ and large t, the cut will dominate over the pole. If on the other hand $\alpha(0)=1-\epsilon$, then there exists a region of small momentum transfers where the pole will dominate over the cut. For s sufficiently negative, however, the situation might very well get turned around, with the cut giving the dominant contribution. In addition it was indicated by Mandelstam[2] and shown by Gribov *et al.*[4] that the generalization of $\lambda(s)=2\alpha(s/4)-1$ to the case where we exchange n identical Regge poles is

$$\lambda_n(s)=n\alpha(s/n^2)-n+1,$$

which shows that the trajectories $\lambda_n(s)$ become flatter as we increase n. Thus, if $\alpha(s)$ is the Pomeranchuk trajectory, for example, then for sufficiently large energies the above singularities would dominate even more strongly than the singularity at $\lambda=2\alpha(s/4)-1$ the contribution from the Pomeranchuk pole. The above discussion was concerned with angular-momentum branch points that arise from the multiple exchange of identical trajectories. In general one will, of course, have to consider the contribution to the amplitude coming from the exchange of different trajectories; the location of the associated angular-momentum branch points, however, can no longer be given by a simple formula such as the one discussed above. In view of what has been said, it is desirable to get as clear an understanding as possible regarding the existence or nonexistence of these cuts in various types of diagrams, the location and nature of the various branch points one is dealing with, and, if possible, the strength of the discontinuities involved.

Let us now review in more detail the history of branch points in the angular-momentum plane. Following the suggestion of Amati, Fubini, and Stanghellini that the

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1 D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters **1**, 29 (1962); Nuovo Cimento **26**, 896 (1962).
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3 C. Wilkin, Nuovo Cimento **31**, 377 (1964).
4 V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martorosyan, Phys. Rev. **139**, B184 (1965); see also Ya. I. Azimov, A. A. Ansel'm, V. N. Gribov, G. S. Danilov, and I. T. Dyatlov, Zh. Eksperim. i Teor. Fiz. **48**, 1776 (1965) [English transl.: Soviet Phys.—JETP **21**, 1189 (1965)].
5 J. C. Polkinghorne, J. Math. Phys. **4**, 1396 (1963).

continued partial-wave amplitude is not a meromorphic function of the angular momentum, Mandelstam analyzed a modified version of the AFS diagram (see Fig. 1), and shown that the cuts suggested by the above authors were merely the result of a poor approximation to the unitarity relation. At the same time he was able to show that in a certain approximation (to be discussed below) the diagram of Fig. 10 does give rise to a branch point in the angular-momentum plane whose location is identical to that obtained from an AFS type of approximation to the corresponding diagram of Fig. 1.[2] The essential features of Fig. 10 are its right- and left-hand portions (i.e., the "crosses") which when considered by themselves exhibit a third double spectral function with respect to the s reaction. The proof of the above result is rather involved. It seems worthwhile, however, to give a brief summary of the general method used, which leaves little to offer where ingenuity is concerned. Rather than make an elastic unitarity approximation with respect to the t reaction in the diagram of Fig. 1 (which would be the analog of the AFS procedure) Mandelstam applies three-body unitarity in the s channel. By a clever choice of variables for the three-body intermediate state, and equipped with the knowledge of the singularity structure of each half of the diagram, Mandelstam is able to show from the large-t behavior of the amplitude that the AFS singularity is absent from the diagram, at least in the three-body unitarity approximation. The method used in the proof depends strongly on the fact that the left- and right-hand portions of the diagram do not possess a third double spectral function in the above-mentioned sense; the method therefore cannot be extended to the diagram of Fig. 10. In order to establish the existence of the singularity in the latter diagram, Mandelstam makes use of the fact that if there exists a bound state or resonance of spin σ lying on the Regge trajectory, then the diagram will have a Gribov-Pomeranchuk singularity at $j=\sigma-1$, where j is the angular momentum in the s reaction (the elementary exchange is taken to have zero spin, for simplicity). He is then able to show, by a number of ingenious tricks, that the singularity can be made to disappear by moving the AFS cut past the point $j=\sigma-1$; such a phenomenon of course requires that the angular-momentum plane exhibit a sheet structure.[6]

This method, however, cannot be used to either prove or disprove the existence of the angular-momentum cut for diagrams whose right- or left-hand portions do not have the above-mentioned double spectral functions, since they do not possess the Gribov-Pomeranchuk singularity. It was shown subsequently by Wilkin that if the cut is to exist, both the right- and left-hand portions of the diagram must possess a third double spectral function in the sense that we have mentioned previously.[3] Wilkin's method consisted in treating the various diagrams as Feynman graphs, thus avoiding the complications introduced by multiparticle unitarity. He finds that unless both the right- and left-hand portions of the diagram possess a third double spectral function, one may distort the integration contours in such a manner that the Regge pole never assumes its characteristic asymptotic form anywhere along the path of integration; with the amplitude vanishing like $1/t^2$ for $t\rightarrow\infty$, he then concludes that the AFS singularity must be absent in such diagrams. Although this method is quite general, it nevertheless does not provide us with a deeper understanding of just how the AFS cut is generated, and of the mechanism responsible for its cancellation.

Several other authors have investigated the moving branch points in the angular-momentum plane. Thus Gribov *et al.*[4] considered the possibility of establishing these branch points directly from the structure of the multiparticle unitarity condition for the partial-wave amplitude continued to complex angular momenta j. On the basis of a definite assumption regarding the form of this analytic continuation, they are able to obtain, among other results, the above singularity at $j=2\alpha(s/4)-1$ for the double Regge pole exchange case, and its generalization to the exchange of n Regge poles: $j_n=n\alpha(s/n^2)-n+1$. In addition they obtain a formula for the discontinuity across the above-mentioned branch point which has the general form of a unitarity relation involving the amplitudes for the production of particles with complex spin (that is, Regge poles).[7] The singularities associated with the exchange of one or two Regge poles have been further considered by Simonov[8] using the form of the many-particle unitarity relation for complex j proposed by Gribov *et al.* An alternative approach has been proposed by Polkinghorne,[5] who has analyzed the diagram of Fig. 10 using the Feynman representation of the amplitude; in this approach Regge cuts result from pinches in the interior of the hypercontour of integration where the coefficient of the asymptotic variable t vanishes.[9] The absence of the AFS-type singularities in the diagrams of Figs. 1 and 7, and their presence in the diagrams of Figs. 10 and 11, can, in all of the above approaches, be ultimately stated in terms of the absence or presence of the already

[6] Mandelstam's analysis of the diagram of Fig. 2 is still only approximate, since he considered only the contribution to the s-channel unitarity relation coming from the three-body intermediate state in which one pair of particles interact to form a Regge pole.

[7] The general form of the Regge-pole unitarity condition proposed in Ref. 4 has been confirmed by Polkinghorne, using single Regge-pole insertions in the Froissart-Gribov continuation, and with the help of some results from perturbation-theory models; see J. C. Polkinghorne, J. Math. Phys. **6**, 1960 (1965).

[8] Yu. A. Simonov, Zh. Eksperim. i Teor. Fiz. **48**, 242 (1965) [English transl.: Soviet Phys.—JETP **21**, 160 (1965)].

[9] The methods of Refs. 7 and 5 have further been applied by P. Osborne and J. C. Polkinghorne to the analysis of more general type of Regge pole in insertions [Cambridge, 1966 (to be published)].

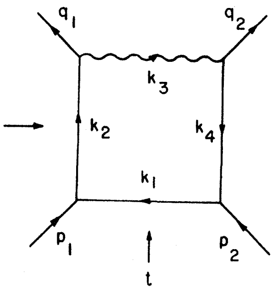

FIG. 1. Box diagram in which one of the elementary lines has been replaced by a Regge pole (denoted by the wiggly line).

mentioned third double spectral function, a fact which had originally been suggested by Mandelstam.[2]

In this paper we shall mainly concentrate on the detailed study of the branch points in the angular-momentum plane which occur for an AFS-type of approximation to the diagrams of Figs. 1 and 7. The philosophy behind this approach is that we expect the location of the j-plane singularities, as well as their general nature (that is, square-root type, logarithmic type, etc.) to be the same for the corresponding diagrams shown in Figs. 10 and 11. The organization of the paper will be as follows: in Sec. II we extract the leading contribution at large t to the Feynman amplitude associated with the diagram of Fig. 1, and show that the AFS approximation corresponds to ignoring certain singularities of the integrand. We then proceed to write the amplitude as a contour integral in the energy plane of the exchanged Regge pole and investigate the analytic structure of the integrand in detail. The nature of the branch points is established, and the discontinuities across the various cuts evaluated; we then obtain the correct form for the asymptotic behavior in t of the AFS amplitude, which in turn tells us the nature and location of the leading branch point in the angular momentum plane of the s reaction; we conclude the section by exhibiting the mechanism which is responsible for the cancellation of the cuts, and with some general remarks.

In Sec. III we make a similar analysis of the diagram involving the exchange of two Regge poles.

Finally, in Sec. IV, we consider the more complicated diagrams of Figs. 10 and 11, which, as originally suggested by Mandelstam, actually have the AFS-type singularities. Their analysis is of course substantially more complicated and we have to make a number of approximations (which do not seem unreasonable) in order to arrive at a numerical estimate of the large-t contribution to the amplitude coming from the leading angular-momentum branch point.

II. THE SINGLE REGGE POLE EXCHANGE DIAGRAM

A. The AFS Approximation

In this section we analyze the diagram of Fig. 1 which in the elastic unitarity approximation gives rise to cuts in the angular-momentum plane. Rather than start from the unitarity relation, as was done by Amati, Fubini, and Stanghellini,[1] and also by Mandelstam,[2] we shall follow Wilkin[3] and treat the diagram as a Feynman graph. Our methods will, however, be adapted to the specific purpose of exhibiting in as clear a way as possible the moving singularities in the angular-momentum plane, and the mechanism which is responsible for their cancellation.

Consider then the Feynman amplitude corresponding to the diagram of Fig. 1:

$$A(s,t)=C\int dk_1^2dk_2^2dk_3^2dk_4^2J(k_n^2;s,t)\frac{1}{k_1^2-m^2+i\epsilon}\times\frac{1}{k_2^2-m^2+i\epsilon}\,\frac{1}{k_4^2-m^2+i\epsilon}R(k_3^2,t;k_2^2,k_4^2)\,,\quad(2.1)$$

where C is an over-all constant, $J(k_n^2;s,t)$ is the Jacobian for the transformation

$$d^4k_1\to\prod_{n=1}^{4}dk_n^2\,,$$

and $R(k_3^2,t;k_2^2,k_4^2)$ is the off-the-mass-shell amplitude associated with the exchange of a Regge pole with trajectory $\alpha(k_3^2)$; the invariants s and t are defined by $s=(q_1-p_1)^2$, and $t=(p_1+p_2)^2$. As we shall see later, we do not require an explicit expression for the Regge pole in order to prove the cancellation of the AFS cut; only its general properties are needed.

Now the Jacobian, $J(k_n^2;s,t)$, is given (we suppress the arguments) by the following expression

$$J=\theta(D)/(D)^{1/2}\,,\quad(2.2\text{a})$$

where

$$D=-16\det|2k_i\cdot k_j|\,.\quad(2.2\text{b})$$

Evaluation of the determinant yields, for $s/t\ll1$, and $m^2/t\ll1$,

$$D=16t^2\bigg\{4sk_3^2-(k_1^2-s-k_3^2)^2+4\left(\frac{s}{t}\right)\bigg[(k_3^2+m^2-k_2^2)(k_3^2+m^2-k_4^2)+(k_1^2-s-k_3^2)(k_3^2+m^2-\tfrac{1}{2}k_2^2-\tfrac{1}{2}k_4^2)+\frac{s-8m^2}{4t}(k_2^2-k_4^2)^2\bigg]\bigg\}\,.\quad(2.3)$$

Now we are interested only in the leading contribution to (2.1) for $t\to\infty$; we therefore may approximate the right-hand side of (2.3) by the first two terms, since the remainder becomes comparable in magnitude only when k_2^2 or k_4^2 (or both) become of the order of t, in which case the contribution to the integral is already strongly suppressed due to the presence of the Feynman propagators. Hence for $t\to+\infty$, $s<0$, we expect the leading contribution to (2.1) to be given by (notice that the Jacobian is to be taken positive):

$$A(s,t)\underset{t\to+\infty}{\approx}\frac{C}{4t}\int dk_1^2dk_3^2\tau(k_1^2,k_3^2,s)\frac{1}{k_1^2-m^2+i\epsilon}$$
$$\times\int dk_2^2dk_4^2\frac{1}{[k_2^2-m^2+i\epsilon][k_4^2-m^2+i\epsilon]}$$
$$\times R(k_3^2,t;k_2^2,k_4^2),\quad(2.4)$$

where $\tau(k_1^2,k_3^2,s)$ is the usual triangle function defined by

$$\tau(x,x',x'')$$
$$=\frac{\theta(-x^2-x'^2-x''^2+2xx'+2xx''+2x'x'')}{(-x^2-x'^2-x''^2+2xx'+2xx''+2x'x'')^{1/2}}.\quad(2.5)$$

Consider the amplitude (2.4); we notice that the integrand associated with the k_2^2 and k_4^2 integrations has poles at $k_2^2=k_4^2=m^2$, as well as singularities in these variables arising from the Regge pole amplitude; since for k_4^2 fixed we expect that $R\approx1/k_2^2$ for $k_2^2\to\infty$ (and viceversa),[10] the integrals certainly converge and we may evaluate them using Cauchy's theorem; let us separate the contributions to (2.4) coming from the above-mentioned poles and the singularities of R; we have

$$A(s,t)\underset{t\to+\infty}{\longrightarrow}[A(s,t)]_{\mathrm{AFS}}+B(s,t),$$

where $[A(s,t)]_{\mathrm{AFS}}$ is obtained by ignoring the singularities of R in k_2^2 and k_4^2,

$$[A(s,t)]_{\mathrm{AFS}}=-\pi^2\frac{C}{t}\int dk_3^2R(\alpha(k_3^2),t)$$
$$\times\int dk_1^2\tau(k_1^2,k_3^2,s)\frac{1}{k_1^2-m^2},\quad(2.6)$$

$R(\alpha(k_3^2),t)$ being the on-the-mass-shell Regge pole amplitude associated with the exchange of a Regge trajectory $\alpha(k_3^2)$ [i.e., $R(\alpha(k_3^2),t)\equiv R(k_3^2,t;m^2,m^2)$], and $B(s,t)$ is a remainder which we shall assume makes a negligible contribution for large t (we will see later that this is actually not the case). If $A(s,t)$ were in fact dominated by (2.6) for $t\to\infty$, then it would have angular-momentum branch points of the Amati-Fubini-Stanghellini type; to emphasize this fact (which we shall prove shortly) we have labeled the quantity (2.6) with the subscript AFS, and shall refer to such amplitudes obtained via the above-described pole approximations as "AFS-type" amplitudes. [It should be noticed that (2.6) is not identical with the high-energy approximation obtained according to the AFS prescription; see Ref. 1.] The advantage of our present systematic approach in extracting a specific part of the high-energy contribution to (2.1) is that it will allow us to demonstrate explicitly how the "remainder" $B(s,t)$ is able to account for the absence of the AFS singularities in the full amplitude (2.1). It is also interesting to note that for $t\to+\infty$ formula (2.6) is precisely the discontinuity across the two-particle, t-channel, normal threshold cut associated with a physical intermediate state formed by the particles with four momenta k_2 and k_4; the prescription for computing this discontinuity has been given by Cutkosky,[11] and is seen to be equivalent to ours [except that in the present approach (2.6) is an approximation to the *amplitude* itself]. One further remark seems appropriate here since it will be relevant later on: The above discussion has dealt with the limit $t\to+\infty$, ($s<0$) of the amplitude (2.1), corresponding to large energies in the physical region of the t reaction; to arrive at a corresponding formula for $t\to-\infty$ we can proceed in two ways: we either continue formula (2.1) to the domain $t\to-\infty$, $s<0$ (which would certainly be a very difficult task due to the presence of the θ function in the definition of the Jacobian), or we simply recognize that $t\to-\infty$, $s<0$, corresponds to the high-energy limit in the u reaction; now one may easily convince oneself that in the physical region of the u reaction the Feynman amplitude associated with the diagram of Fig. 1 is still given by exactly the same formula (2.1), except that t is negative. Since the Jacobian, $J(k_n^2;s,t)$, is, however, a positive quantity, it follows that we must take the positive determination of the square root of D [see Eq. (2.3)], and hence the positive square root of the over-all factor $16t^2$ in (2.3). We hence conclude that in the limit $t\to-\infty$, $A(s,t)$ approaches the negative of (2.4).

B. Representation of the AFS Amplitude as a Contour Integral in the Energy Plane of the Exchanged Regge Pole

We now wish to write (2.6) as a contour integral in the energy plane of the exchanged Regge pole. To this effect we change the integration variables in (2.6) from k_1^2 and k_3^2, to x and k_z, where[12]

$$x=k_3^2-(1/4s)(k_1^2-s-k_3^2)^2,$$
$$k_z=(k_1^2-s-k_3^2)/2(-s)^{1/2},$$

and substitute for $R(\alpha(k_3^2),t)$ the expression[13]

$$R(\alpha(k_3^2),t)=\gamma(k_3^2)C(\alpha)\xi_\pm(\alpha)t^{\alpha(k_3^2)}/\sin\pi\alpha(k_3^2),\quad(2.7)$$

[10] See the Appendix of Ref. 2, p. 1141.

[11] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960); see also M. Fowler, *ibid.* 3, 936 (1962).

[12] For small scattering angles the quantity k_z is the z component of $\mathbf{k}_3$ in the cm system of the t reaction, with the z axis taken perpendicular to the direction of the incident momentum $\mathbf{p}_1$ in the plane formed by the vectors $\mathbf{p}_1$ and $\mathbf{q}_1$.

[13] Whenever there is no confusion possible, we suppress the argument of the trajectory function $\alpha(u)$; we also omit all ($\pm$) signature labels if they are not pertinent to the discussion.

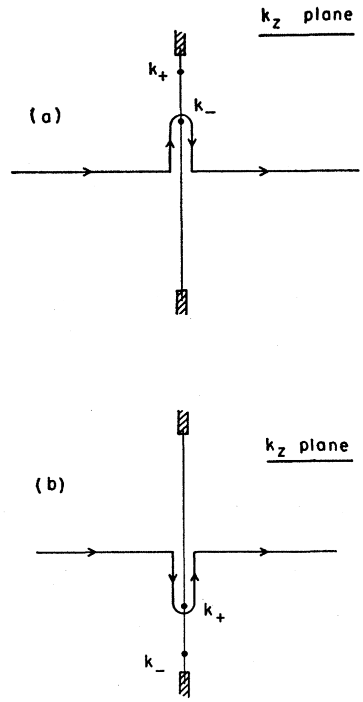

FIG. 2. (a) The new integration contour of (2.8) in the k_z plane as it appears when $s>m^2-x$, for the case where $(-s)^{1/2}$ has been continued to $s>0$ according to $(-s)^{1/2}=is^{1/2}$; only the singularities at $k_\pm\equiv is^{1/2}\pm i(m^2-x)^{1/2}$ are shown; the shaded portions on the imaginary axis are cuts associated with the branch points that arise from the normal threshold singularities in $k_3{}^2$ of the Regge-pole amplitude (they extend to $\pm i\infty$). (b) Same as in Fig. 2a, except that $(-s)^{1/2}$ has been continued to $s>0$ according to $(-s)^{1/2}=-is^{1/2}$.

where $C(\alpha)$ is the coefficient of z^α in the asymptotic expansion of the Legendre function $P_\alpha(z)$, and where $\gamma(k_3{}^2)$ is a reduced residue function which is related to the full residue $\beta(k_3{}^2)$ of the Regge pole by

$$\gamma(k_3{}^2)=-\tfrac{1}{2}\pi(2\alpha+1)\beta(k_3{}^2)/(2q^2)^\alpha,\quad q^2=-m^2+k_3{}^2/4.$$

Finally, $\xi_\pm(\alpha)$ is the usual signature factor

$$\xi_\pm(\alpha)=\exp(-i\pi\alpha)\pm1.$$

We then obtain for the AFS amplitude, valid for $s<0$,

$$[A(s,t)]_{\rm AFS}=-\pi^2C\int_{-\infty}^{0}\frac{dx}{(-x)^{1/2}}\int_{-\infty}^{+\infty}dk_z\frac{\gamma(x-k_z{}^2)}{x-[k_z-\sqrt{(-s)}]^2-m^2}\times C(\alpha)\xi(\alpha)t^{\alpha(x-k_z{}^2)-1}/\sin[\pi\alpha(x-k_z{}^2)].\quad(2.8)$$

Consider the integrand of the k_z integration; it is singular at

$$k_z=(-s)^{1/2}\pm i(m^2-x)^{1/2}\quad(2.9\text{a})$$

and also at

$$k_z=\pm i(u_n-x)^{1/2},\quad(2.9\text{b})$$

where the latter singularities arise from the normal threshold branch points of the Regge trajectory, $\alpha(k_3{}^2)$, and reduced residue, $\gamma(k_3{}^2)$, and from the vanishing of $\sin\pi\alpha(k_3{}^2)$ at the bound states and resonances which lie on the trajectory (the resonance poles are reached by going through the normal threshold cuts); u_n gives the position of these singularities in the $k_3{}^2$ plane. So far the integral (2.8) is valid for $s<0$. As we increase s through negative values, the complex singularities (2.9a) move towards the imaginary axis, which they reach for $s=0$. For $s>0$ the singularities remain on the imaginary axis, both moving either up or down depending on the continuation chosen for the function $(-s)^{1/2}$; as s becomes larger than m^2-x, one of the singularities will cross the real k_z axis and drag the integration contour along the imaginary axis, as is shown in Figs. 2a and 2b. We now make a final change of variables from k_z and x, to $u\equiv x-k_z{}^2$ and x. The above discussion in the k_z plane was only intended to serve as a crutch for a better understanding of the analysis that follows, as well as of the similarity existing between the single and double Regge pole exchange diagrams. With the above change of variables, (2.8) becomes[14]

$$[A(s,t)]_{\rm AFS}=\tfrac{1}{2}i\int_{C_u}du\,c(u,s)\times\left[\frac{\exp(-i\pi\alpha)\pm1}{\sin\pi\alpha(u)}\right]t^{\alpha(u)-1},\quad(2.10\text{a})$$

where

$$c(u,s)=-i\pi^2C\gamma(u)C(\alpha)I(u,s)\quad(2.10\text{b})$$

and

$$I(u,s)=\int_{-\infty}^{0}\frac{dx}{(-x)^{1/2}(x-u)^{1/2}}\times\frac{1}{u+s-m^2+2(-s)^{1/2}(x-u)^{1/2}}.\quad(2.10\text{c})$$

The contour C_u is shown in Fig. 3. It is clear from (2.10c) that $I(u,s)$ will have a branch point at $u=0$ which

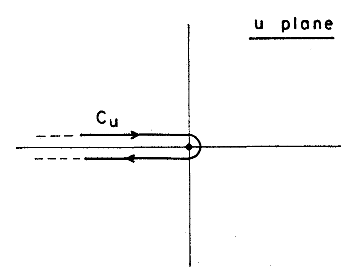

FIG. 3. The contour C_u of the integral (2.10a); only the singularity of $c(u,s)$ at $u=0$ is shown.

[14] To obtain the form (2.10a) we have replaced the original contour in the u plane, which extends along the negative u axis and encircles the branch point at $u=x$, by a fixed contour that encloses the point $u=0$. This is always possible, since nowhere in the integration region are we forced to distort this contour from its fixed position. Subsequent interchange of the x and u integrations results in formula (2.10a).

arises from the collision of the square-root singularity at $x=u$ with the upper limit of integration; if we cut the u plane from $u=0$ along the negative u axis, then the contour C_u is seen to extend around this cut; the branch of the square-root function, $(x-u)^{1/2}$, to be taken is evidently given by

$$(x-u)^{1/2}=+i|(x-u)^{1/2}|, \quad \text{for} \quad u>x,$$

and corresponds to displacing the k_z-integration contour in (2.8) slightly into the upper half of the complex plane.

Next we wish to examine the singularity structure of the function $c(u,s)$ appearing in (2.10a), and to continue the integral to the positive s region. The reason for making such a detailed study is that we believe that the motion, and nature of the angular-momentum branch points to be inferred from the present study are the same for the more complicated diagrams which do exhibit angular-momentum cuts; the latter diagrams will be considered in a later section. Now $c(u,s)$ is defined by (2.10b), where $I(u,s)$ is given by (2.10c); the integral may be readily evaluated; one finds

$$I(u,s)=\frac{i}{[K(u,s,m^2)]^{1/2}} \times\ln\left(\frac{[K(u,s,m^2)]^{1/2}-(u+s-m^2)}{[K(u,s,m^2)]^{1/2}+(u+s-m^2)}\right)\exp(i\pi), \quad (2.11a)$$

where

$$K(u,s,m^2)=(u+s-m^2)^2-4su, \quad (2.11b)$$

and where, for $s<0$ and $u>0$, the phase of the quantity appearing within brackets in the argument of the log is to be taken zero. Throughout this paper we adopt the conventions that: (a) all square roots are to be taken positive if their discriminant is positive, and (b) $\ln z$ is taken to be real for $z>0$; all phases will therefore be explicitly exhibited. We now examine (2.11a) for three real domains of the variable s.

1. $s<0$

From (2.11a) we see that the possible singular points of $I(u,s)$ are located at

$$(1) \quad u=[m\pm i\surd(-s)]^2,$$

and

$$(2) \quad u=0,$$

where, for $s<0$, the latter singularity arises from the vanishing of the denominator in the argument of the logarithm. If we cut the u plane from $u=0$ along the negative real axis, then the contour C_u of the integral (2.10a) extends around this cut, and the value of $I(u,s)$ on that contour is obtained by continuing (2.11) in u to the points $u_0\pm i\epsilon$, where $u_0<0$; from here on we shall refer to that sheet of the logarithmic branch point on which C_u appears as the "leading sheet." We now verify that $I(u,s)$ is singular at $u=[m-i\surd(-s)]^2$ and regular

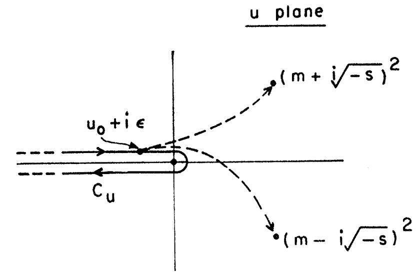

FIG. 4. Paths of continuation leading from the point $u_0+i\epsilon$, located on the contour C_u, to the possible singular points of $I(u,s)$ at $u=[m\pm i\surd(-s)]^2$.

at $u=[m+i\surd(-s)]^2$ where these points are reached via the paths shown in Fig. 4. Let

$$z=\frac{[K(u,s,m^2)]^{1/2}-(u+s-m^2)}{[K(u,s,m^2)]^{1/2}+(u+s-m^2)}. \quad (2.12)$$

Recalling that the phase of z is zero for $s<0$ and $u>0$, one may readily verify that

$$z\to\exp(\mp i\pi), \quad \text{as} \quad u\to[m\pm i\surd(-s)]^2.$$

It is then an easy matter to show that $I(u,s)$ is singular at $u=[m-i\surd(-s)]^2$ and regular at $u=[m+i\surd(-s)]^2$. The continuation of (2.11) to the remaining sheets of the logarithmic branch point at $u=0$ may also be readily effected; one finds that $I(u,s)$ is singular at $u=[m+i\surd(-s)]^2$ on all sheets of the logarithm with the exception of the leading one, and singular at $u=[m-i\surd(-s)]^2$ on every sheet but the one which is reached by a counterclockwise continuation around the branch point at $u=0$. Finally one obtains the discontinuity of $I(u,s)$ across the logarithmic branch point by continuing (2.11) to the points $u_0\pm i\epsilon$, where $u_0<0$; the quantity z, defined by (2.12), then acquires a corresponding phase $\mp i\pi$, and the discontinuity becomes

$$\{\mathrm{disc}_u I(u,s)\}_{u=0}=2\pi/[K(u,s,m^2)]^{1/2}. \quad (2.13)$$

2. $0<s<m^2$

As s becomes positive, the complex singularities of $I(u,s)$ located at $u=[m+i\surd(-s)]^2$ and $u=[m-i\surd(-s)]^2$ move onto the real axis; if $(-s)^{1/2}$ is continued to $s>0$ according to $(-s)^{1/2}=is^{1/2}$, then their positions will be given by $u=(s^{1/2}-m)^2\equiv u_-$ and $u=(s^{1/2}+m)^2\equiv u_+$, respectively. If we had chosen the other branch of the square root, then the above order of u_- and u_+ would be interchanged. For the remainder of this section we shall restrict ourself to the case where $(-s)^{1/2}$ has been continued to $s>0$ according to the above given prescription; the other possibility may be discussed just as easily and leads, of course, to the same conclusions with regard to the singularities of the s channel partial-wave amplitude in the angular-momentum plane. [That this must be so becomes evident when one follows the motion of the singularities of the

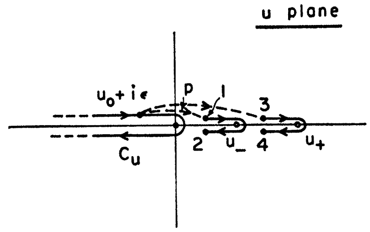

FIG. 5. The complex u plane, showing the paths along which the discontinuities of $I(u,s)$ across $u=u_+$ and $u=u_-$ are evaluated.

integrand of (2.8) in the k_z plane for the two possible continuations of $(-s)^{1/2}$.] With the above convention, $u_- \equiv (s^{1/2}-m)^2$ and $u_+ \equiv (s^{1/2}+m)^2$ are the respective continuations of the complex locations of the singularities of $I(u,s)$ at $u=[m+i\sqrt{(-s)}]^2$ and $u=[m-i\sqrt{(-s)}]^2$; since for $s<m^2$ neither one of the above singularities could have left their respective logarithmic sheet, it is clear from our previous discussion of the case $s<0$ that $I(u,s)$ will be regular at $u=u_-$ and singular at $u=u_+$ on the leading sheet, and that it will be regular at the latter point on the sheet reached by a counterclockwise continuation around the logarithmic branch point at $u=0$. The above expectations may be readily verified by starting from an expression for $I(u,s)$ valid for $s>0$ and $u=u_0+i\epsilon$, where the latter point is located just above the left hand cut on the leading sheet; one finds (recall that all phases are exhibited explicitly)

$$I(u_0+i\epsilon,\, s) = \frac{i}{[K(u_0,s,m^2)]^{1/2}} \ln\left(\frac{[K(u_0,s,m^2)]^{1/2}-(u_0+s-m^2)}{[K(u_0,s,m^2)]^{1/2}+(u_0+s-m^2)}\right) \times \exp(-i\pi),\quad u_0<0,\quad s>0. \quad (2.14)$$

As one continues this expression in u along the path P shown in Fig. 5 one finds that at $u=u_1$, $z_1=|z_1|\exp(+i\pi)$, while at $u=u_2$, $z_2=(1/|z_1|)\exp(+i\pi)$, where z_i is the value of (2.12) at $u=u_i$ [in the figure we have denoted the points u_i by their subscripts i]; the discontinuity of $I(u,s)$ across $u=u_-$ therefore vanishes; to compute the discontinuity of $I(u,s)$ across $u=u_+$ we notice that if $u_-<u<u_+$,

$$z=\left(\frac{a-ib}{a+ib}\right)\exp(+i\pi)=\exp(i\varphi)\exp(+i\pi);$$

thus one finds that at $u=u_3$ and $u=u_4$, $z_3=\exp(i\varphi_3)\times\exp(+i\pi)$ and $z_4=\exp(-i\varphi_3-4i\pi)\exp(+i\pi)$, respectively; hence the discontinuity becomes

$$\{\mathrm{disc}_u I(u_3,s)\}_{u=u_+}=4\pi/[K(u_3,s,m^2)]^{1/2}. \quad (2.15)$$

3. $s>m^2$

The case $s>m^2$ can be analyzed in a similar manner as above; one finds that $z_1=|z_1|\exp(-i\pi)$, $z_2=(1/|z_1|)\times\exp(-i\pi)$, $z_3=\exp(i\varphi_3)\exp(-i\pi)$, and $z_4=\exp(-i\varphi_3)\times\exp(-i\pi)$; we are therefore led to the conclusion that

$$\{\mathrm{disc}_u I(u_1,s)\}_{u=u_-}=4\pi/[K(u_1,s,m^2)]^{1/2}. \quad (2.16)$$

while the discontinuity of $I(u,s)$ across $u=u_+$ is still given by (2.15). Thus, for $s>m^2$, $I(u,s)$ is found to be singular at u_+ and u_- on the leading sheet of the logarithmic branch point; in fact, one may verify that $I(u,s)$ is singular at these points on all sheets of the log, except for the one reached by a counterclockwise continuation. Finally one may readily check that the discontinuity of $I(u,s)$ across the branch point at $u=0$, is still given by (2.13).

C. Asymptotic Behavior of the AFS Amplitude, and Location and Nature of the Singularities in the j Plane

From the above discussion of the singularities of $I(u,s)$, and therefore also of $c(u,s)$, we immediately obtain the proper continuation of the integral (2.10a) to $s>0$: for $s<m^2$ the contour of C_u of (2.10a) extends around the logarithmic branch point of $c(u,s)$ at $u=0$, and the asymptotic behavior in t of the integral is determined by this singularity. The discontinuity of $c(u,s)$ across the left-hand cut associated with the logarithmic branch point is given by $-i\pi^2 C\gamma(u)C(\alpha)\{\mathrm{disc}I(u,s)\}$, where we must substitute (2.13) for $\mathrm{disc}I(u,s)$. [This expression remains valid also for $s>m^2$.] As s becomes larger than m^2, the singularity of $I(u,s)$ [or $c(u,s)$] at $u=(s^{1/2}-m)^2$, which for $s<m^2$ was absent from the leading logarithmic sheet, now appears on the leading sheet via the branching at $u=0$ and drags the contour C_u to the right as we keep increasing s; thus for $s>m^2$ the new asymptotic behavior of the amplitude (2.10a) will be determined by the singularity at $u=(s^{1/2}-m)^2$; the discontinuity of $c(u,s)$ across this branch point, for $0<u<(s^{1/2}-m)^2$, is given by $-i\pi^2 C\gamma(u)C(\alpha)\times\{\mathrm{disc}I(u,s)\}_{u_-}$, where (2.16) is to be substituted for the discontinuity of $I(u,s)$. The integral (2.10) may therefore be cast into the following form, valid for all real s:

$$[A(s,t)]_{\mathrm{AFS}}=\pi^3 C\int_{-\infty}^{0}\frac{du}{[(u+s-m^2)^2-4su]^{1/2}}\frac{\gamma(u)}{\sin\pi\alpha(u)}C(\alpha)\xi(\alpha)t^{\alpha(u)-1}+2\pi^3 C\theta(s-m^2)\times\int_0^{(s^{1/2}-m)^2}\frac{du}{[(u+s-m^2)^2-4su]^{1/2}}\frac{\gamma(u)}{\sin\pi\alpha(u)}c(\alpha)\xi(\alpha)t^{\alpha(u)-1}. \quad (2.17)$$

The asymptotic behavior of the AFS amplitude may be obtained immediately from the expression (2.17); thus for s strictly less than m^2 (only the first integral then contributes) the integrand is seen to approach a constant as $u \to 0$, while for $s > m^2$, the integrand of the second integral diverges as $u \to (s^{1/2}-m)^2$; now for large t the leading contribution to (2.17) comes from the upper integration limit of the first or second integral, depending on whether $s < m^2$ or $s > m^2$, respectively. The leading term in (2.17) is then readily obtained by expanding the trajectory function $\alpha(u)$ around the appropriate upper limit of integration and neglecting the variations of any other slowly varying factors in the integrand; one finds

$$[A(s,t)]_{\mathrm{AFS}} \underset{t\to\infty}{\longrightarrow} B(s) t^{\alpha(0)-1}/\ln t, \quad \text{for} \quad s<m^2, \tag{2.18a}$$

where

$$B(s) = \pi^3 C \frac{\gamma(0)}{\sin\pi\alpha} \frac{C(\alpha)\xi(\alpha)}{\alpha'(0)(m^2-s)}, \tag{2.18b}$$

and $\alpha \equiv \alpha(0)$. Similarly one finds that for $s > m^2$

$$[A(s,t)]_{\mathrm{AFS}} \underset{t\to\infty}{\longrightarrow} G(s) t^{\alpha[(s^{1/2}-m)^2]-1}/[\ln t]^{1/2}, \tag{2.19a}$$

where

$$G(s) = \pi^{7/2} C \frac{\gamma(u_-)}{\sin\pi\alpha(u_-)} \frac{C(\alpha)\xi(\alpha)}{[m\alpha'(u_-)s^{1/2}]^{1/2}} \tag{2.19b}$$

and $\alpha = \alpha(u_-)$, $u_- = (s^{1/2}-m)^2$.

If the AFS amplitude has a Sommerfeld-Watson representation (in which case the large-t behavior of the amplitude is determined by the leading singularities in the j plane of the s reaction), then we conclude from the asymptotic expressions (2.18) and (2.19) that the leading branch points in the angular-momentum plane associated with the s reaction are located at $j = \alpha(0)-1$ and $j = \alpha((s^{1/2}-m)^2)-1$, for $s < m^2$ and $s > m^2$, respectively, and, furthermore, that these singularities are of the logarithmic type, and inverse square-root type. Since for $s < m^2$ the singularity at $j = \alpha((s^{1/2}-m)^2)-1$ no longer determines the asymptotic behavior of the AFS amplitude, it must have moved onto an "unphysical" sheet via the logarithmic branch point at $j = \alpha(0)-1$; notice that the latter singularity does not depend on any mass parameters, while the former one depends on m, the mass of the exchanged elementary particle. The location of the singularities as well as their logarithmic and square-root nature agrees with the results obtained by Mandelstam,[2] Wilkin,[3] Gribov *et al.*,[4] and Simonov,[8] in connection with the single Regge-pole exchange diagram for which the cancellation of the cuts does not occur (see Fig. 10, for example).

D. Concluding Remarks

Before closing our discussion of the AFS amplitude associated with the single Regge-pole exchange diagram we wish to make two further remarks concerning (a) the signature of the partial-wave amplitude in which the leading branch points appear, and (b) the generation of the normal threshold branch points in s of $[A(s,t)]_{\mathrm{AFS}}$. We begin with the first-mentioned point. Assuming that (2.1) is dominated at large t by the expression (2.10a), we have

$$A(s,t) \underset{t\to\pm\infty}{\longrightarrow} \frac{i}{2|t|} \int_{C_u} du\, c(u,s) \times[(-t)^{\alpha(u)} \pm t^{\alpha(u)}]/\sin\pi\alpha(u), \tag{2.20}$$

where we have substituted $[(-t)^{\alpha(u)} \pm t^{\alpha(u)}]$ for $[\exp(-i\pi\alpha(u)) \pm 1]t^{\alpha(u)}$, and have written $|t|$ instead of t to include the limit $t \to -\infty$ [see the discussion at the end of Sec. (II.A)]. From (2.20) it follows that for large t the amplitude is even or odd under the transformation $t \to -t$ depending on whether we exchange a Regge pole of even or odd signature, respectively. Now any amplitude that satisfies the usual one-dimensional dispersion relation can be written in the form

$$A(s,z_s) = \tfrac{1}{2}[A^+(s,z_s) + A^+(s,-z_s) + A^-(s,z_s) - A^-(s,-z_s)], \tag{2.21}$$

where z_s is the cosine of the center-of-mass (cm) scattering angle for the s reaction (i.e., $z_s = 1 + t/2q_s^2$, q_s being the corresponding c.m. momentum), and where

$$A^\pm(s,z_s) = \frac{1}{\pi}\int_{t_0}^{\infty} dt' \frac{A_t(s,t')}{t'-t(s,z_s)} \pm \frac{1}{\pi}\int_{u_0}^{\infty} du' \frac{A_u(s,u')}{u'-u(s,-z_s)};$$

here $A_t(s,t)$ and $A_u(s,u)$ are the t-channel and u-channel absorptive parts of $A(s,t)$ respectively, and t_0 and u_0 are the squares of the lowest-normal thresholds in the t and u reactions; since for large t, $z \propto t$, it follows from (2.20) and (2.21) that in the limit $t \to \infty$ only the positive (negative) signature amplitude will contribute to $[A(s,t)]_{\mathrm{AFS}}$ if we exchange a Regge pole of positive (negative) signature. Thus the leading branch points in the angular-momentum plane appear in the analytically continued partial-wave amplitude of the same signature as that of the exchanged Regge pole.

We now turn to the second point and show how the normal threshold branch points of $[A(s,t)]_{\mathrm{AFS}}$ in s are generated; they are expected to result from the coincidence of the poles and normal threshold branch points of the Regge-pole amplitude with the pole of the propagator associated with the elementary particle exchange; the latter manifests itself in the singularity of $c(u,s)$ at $u = (s^{1/2}-m)^2$. We notice first of all that the integrand of (2.20) has poles at $u = M_i^2$, where M_i are the masses of the *physical* bound states or resonances lying on the Regge trajectory [the latter singularities are reached by going through the normal threshold cuts of the trajectory function $\alpha(u)$]; they are therefore a solution to

$$\alpha^+(M_i^2) = 0, 2, 4, \cdots,$$

and

$$\alpha^-(M_j^2)=1, 3, 5, \cdots,$$

[we have suppressed the signature labels on the trajectory function in (2.20)]. The residues of the poles at the remaining integers of $\alpha^\pm(u)$ vanish on account of the signature. Now for $s<m^2$, the contour C_u extends along the negative u axis and encircles the branch point of $c(u,s)$ at $u=0$; as we increase s above m^2, a new singularity at $u=(s^{1/2}-m)^2$ appears on the leading sheet, and the contour C_u will be pinched between this singularity and the above-mentioned poles when

$$s_i=(m+M_i)^2.$$

This gives the position of the two-body normal threshold branch points. The higher-normal threshold branch points are generated in exactly the same way by the pinching of the contour C_u between the moving singularity at $u=(s^{1/2}-m)^2$, and the normal threshold singularities at $u=u_N$ of the Regge-pole amplitude; their locations are given by $s_N=(m+u_N^{1/2})^2$. In addition to the above normal threshold branch points the continued partial-wave amplitude of definite signature, $b^\pm(j,s)$, will have singularities in s arising from the moving branch point at $j=\alpha((s^{1/2}-m)^2)-1$, as was originally pointed out by Mandelstam.[2] Their locations are given by

$$s(j)=\{m+[\lambda(j+1)]^{1/2}\}^2, \qquad (2.22)$$

where λ is the inverse function corresponding to $\alpha(u)$. For definiteness sake let us consider the case in which we exchange a trajectory of even signature (the Pomeranchuk, for example). The singularity at $j=\alpha[(s^{1/2}-m)^2]-1$ will then appear in the even signature partial-wave amplitude; from (2.22) we see that if j is an even integer [$b^+(j,s)$ then coincides with the physical partial wave amplitude], then the singularity $s(j)$ would coincide with an unphysical threshold branch point at $s_{j+1}=(m+M_{j+1})^2$, where M_{j+1} is the mass of an unphysical bound state (or resonance) with odd spin $\sigma=j+1$, lying on an even signature trajectory. Unless the discontinuity across this singularity vanishes for j an even integer, it would manifest itself in the scattering amplitude as well; now we have seen that $[A(s,t)]_{\mathrm{AFS}}$ has only physical cuts in s; we hence conclude that for even j, $b^+(j,s)$ is nonsingular at $s=s_{j+1}$; this agrees with the conclusion reached by Mandelstam.[2]

E. Cancellation of the Cuts

So far we have assumed that (2.6) gives the dominant contribution to (2.1) at large energies; this has led us to conclude that the amplitude has angular-momentum branch points of the AFS type. As was originally suggested by Mandelstam,[2] these singularities are absent in the true amplitude; the purpose of this section is to exhibit the simplicity of the cancellation mechanism. To this effect we return to formula (2.4), which one expects

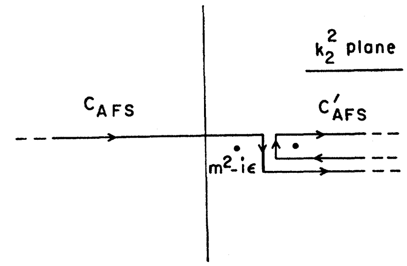

FIG. 6. The integration contour of (2.4) in the k_2^2 plane split up into two pieces, C_{AFS} and C_{AFS}', where the latter contour encircles all the singularities in k_2^2 of the Regge-pole amplitude (these singularities are symbolically denoted by the dot).

to give the leading contribution to (2.1) for $t\to\infty$; the approximation which has led us to (2.6) consisted in ignoring the singularities of $R(k_3^2,t;\,k_2^2,k_4^2)$ in the complex k_2^2 and k_4^2 planes. We now show that if we include the latter singularities, the integral (2.4) vanishes identically. To see this, one only has to realize that the singularities of a Feynman amplitude in any one of its external invariant masses must lie in the lower half of the complex plane, if the remaining variables are kept real (we are referring here to four-line connected parts).[15] Thus, if $R(k_3^2,t;\,k_2^2,k_4^2)$ vanishes as k_2^2 or k_4^2 becomes infinite, as we believe to be the case,[10] then the integral (2.4) will vanish identically, since the singularities of the integrand in k_2^2 (or k_4^2) are all located in the lower half of the complex plane. The two cancelling pieces of the amplitude (2.4) may be readily exhibited. Consider the k_2^2 integration for example; the contour integral along the real axis may be split up as shown in Fig. 6; the contribution to the amplitude coming from the contour C_{AFS} corresponds to an AFS type of approximation, while the integral along the contour C_{AFS}' (which encloses all the singularities of R in k_2^2) becomes the dispersion integral for the function $R(k_3^2,t;\,m^2,k_4^2)$; the two contributions mentioned above evidently cancel. If one continues to treat the two pieces separately and performs the remaining integrations, one finds that a

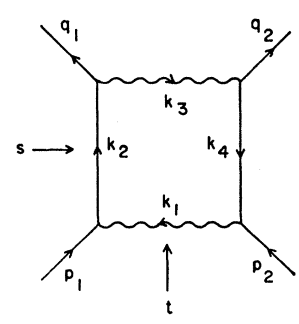

FIG. 7. Box diagram in which two of the elementary lines have been replaced by Regge poles.

[15] This follows from the fact that the coefficients of the invariant masses which appear in the Feynman denominator function, $D(\alpha,s,t,p_i^2)$, are nonnegative; see N. Nakanishi, Progr. Theoret. Phys. (Kyoto), Suppl. 18, (1961).

similar cancellation takes place within each of the separate pieces; the AFS approximation to (2.1) is obtained by consistently ignoring the C_{AFS}' integrations. The fact that the AFS-type singularities are absent in the full amplitude (2.1), but present in the two-particle discontinuity [which is given by (2.6)], shows that these singularities must be found present if the amplitude is continued into the second sheet of the 2-particle, t-channel branch point.

III. THE DOUBLE REGGE-POLE EXCHANGE DIAGRAM

A. The AFS Approximation

In this section we analyze the diagram of Fig. 7; for simplicity we consider the exchange of two identical Regge poles; the modifications that are required if this condition is relaxed are rather obvious and we state them at the end of the section.

Making the same approximations to the Jacobian (2.2, 3) as before, we arrive at the following expression for the leading contribution to the amplitude at large positive t, and $s<0$:

$$A(s,t)\approx\frac{C}{4t}\int dk_1^2dk_2^2dk_3^2dk_4^2\tau(k_1^2,k_3^2,s)\times\frac{1}{[k_2^2-m^2+i\epsilon][k_4^2-m^2+i\epsilon]}\times R(k_3^2,t;\,k_2^2,k_4^2)R(k_1^2,t;\,k_2^2,k_4^2)\,,\quad(3.1)$$

where $\tau(k_1^2,k_3^2,s)$ is the triangle function defined in (2.5), and where the functions R are the amplitudes associated with the two Regge poles. The AFS approximation corresponds, as before, to ignoring the singularities of R in k_2^2 and k_4^2; closing the k_2^2 and k_4^2 integration contours in the lower half planes, we pick up the following contribution coming from the poles of the two propagators:

$$[A(s,t)]_{\text{AFS}}=-\pi^2C\int dk_1^2dk_3^2\tau(k_1^2,k_3^2,s)\times\frac{\gamma(k_1^2)}{\sin\pi\alpha(k_1^2)}\frac{\gamma(k_3^2)}{\sin\pi\alpha(k_3^2)}\times C(k_1^2)C(k_3^2)t^{\alpha(k_1^2)+\alpha(k_3^2)-1}\,,\quad(3.2)$$

where we have substituted (2.7) for the on-the-mass-shell Regge-pole amplitudes (throughout this section we shall omit all signature factors and signature labels, since they are not pertinent to the discussion); the coefficients $C(k_1^2)$ and $C(k_3^2)$ are defined by

$$P_{\alpha(y)}(z)\underset{z\to\infty}{\longrightarrow}C(y)z^{\alpha(y)}\,.\quad(3.3)$$

Except for a trivial change, we treat formula (3.2) by the same recipe as was used in dealing with the single Regge-pole exchange diagram. Let us switch to a new set of integration variables, x and r, which are defined in terms of k_1^2 and k_3^2 by

$$x=k_3^2-(1/4s)(k_1^2-s-k_3^2)^2\,,$$

and

$$r=[(k_1^2-s-k_3^2)/2(-s)^{1/2}]-\tfrac{1}{2}(-s)^{1/2}.$$

In terms of x and r the integral (3.2) becomes

$$[A(s,t)]_{\text{AFS}}=-\pi^2C\int_{-\infty}^{0}\frac{dx}{(-x)^{1/2}}\int_{-\infty}^{+\infty}dr\left\{\frac{\gamma(x-[r+\frac{1}{2}\surd(-s)]^2)}{\sin\pi\alpha(x-[r+\frac{1}{2}\surd(-s)]^2)}\times\frac{\gamma(x-[r-\frac{1}{2}\surd(-s)]^2)}{\sin\pi\alpha(x-[r-\frac{1}{2}\surd(-s)]^2)}\tilde{C}(x,r;\,s)\right\}t^{\alpha(x-[r+\frac{1}{2}\surd(-s)]^2)+\alpha(x-[r-\frac{1}{2}\surd(-s)]^2-1)}\,,\quad(3.4\text{a})$$

where

$$\tilde{C}(x,r;\,s)=C(x-[r+\tfrac{1}{2}\surd(-s)]^2)\times C(x-[r-\tfrac{1}{2}\surd(-s)]^2)\,,\quad(3.4\text{b})$$

and $C(y)$ is defined by (3.3). Consider the analytic structure of the integrand of (3.4) in the r plane; there will be poles arising from the vanishing of the sine factors in the denominator, as well as branch points which arise from the normal threshold branch points of the Regge trajectory function, $\alpha(u)$, and reduced residue function, $\gamma(u)$; the location of the singularities in the r plane is thus given by

$$r_n=\pm\tfrac{1}{2}(-s)^{1/2}\pm i(u_n-x)^{1/2}\,,\quad(3.5)$$

where u_n stands for the square of the masses of the bound states and resonances lying on the trajectory $\alpha(u)$, and for the position of the normal threshold branch points of $\alpha(u)$ and $\gamma(u)$. If we define the "angular momentum" variable l,[16]

$$l=\alpha(x-[r+\tfrac{1}{2}\surd(-s)]^2)+\alpha(x-[r-\tfrac{1}{2}\surd(-s)]^2)-1\,,\quad(3.6)$$

then the integral (3.4) may be written in the form

$$[A(s,t)]_{\text{AFS}}=-\pi^2C\int_{-\infty}^{0}\frac{dx}{(-x)^{1/2}}\times\int_{C_l}\frac{dl}{\partial l/\partial r}B(l,s,x)t^l\,,\quad(3.7)$$

[16] The quantity l should not in general be identified with the total angular momentum j in the s channel; one may interpret it as the total angular momentum in the s channel, obtained by coupling the (complex) spins of the two Regge poles to a relative angular momentum $L=-1$.

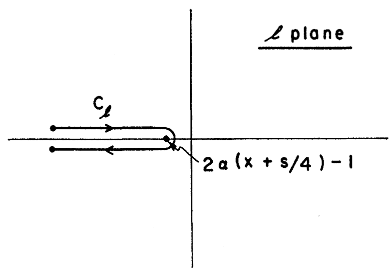

FIG. 8. The complex l plane showing the contour C_l of the integral (3.7) for the case where we exchange two Regge poles, together with the singularity of the integrand at $l=2\alpha(x+s/4)-1$.

where the contour C_l is shown in Fig. 8, and where $B(l,s,x)$ is the quantity appearing within braces in (3.4) expressed in terms of l and x through relation (3.6). To obtain the contour C_l of Fig. 8 we have used the fact that $\alpha(u)$ is a real analytic function of u, and have made the assumption that $d\alpha/du>0$ for $u<0$ [which is true if $\alpha(u)$ has only the right-hand normal threshold cuts and satisfies a dispersion relation with a possible subtraction]. Notice that the Jacobian for the transformation $dxdr\rightarrow dxdl$ is singular at $r=0$, since (3.6) is invariant under the transformation $r\rightarrow -r$. This manifests itself in formula (3.7) as a singularity of the integrand at $l=2\alpha(x+\frac{1}{4}s)-1$ (corresponding to $r=0$) which arises from the vanishing of $\partial l/\partial r$; in fact, it follows trivially from (3.6) that at $r=0$

$$\partial l/\partial r=0. \tag{3.8}$$

Thus, for any given x, the contour C_l of Fig. 8 is the "minimizing" contour [since any other contour obtained by distorting C_l must pass through a point l_0 for which $\mathrm{Re}l_0\geqslant 2\alpha(x+\frac{1}{4}s)-1$]; the asymptotic behavior of the integral over l in (3.7) will therefore be determined by the singularity at $l=2\alpha(x+\frac{1}{4}s)-1$.

In order to continue the expression (3.7) to positive values of s it is easiest to return to the form (3.4), since we have complete knowledge of the singularity structure of the integrand in the r plane. As we increase s from negative to positive values, the complex conjugate pairs of singularities (3.5) move onto the imaginary axis, and remain on that axis for $s>0$; for a fixed value of x, the r-integration contour of (3.4) [corresponding to the minimizing contour C_l] will then remain undistorted as long as $s\leqslant 4(M^2-x)$, where M is the mass of the lowest-lying bound state on the trajectory $\alpha(u)$; if no such state exists, then M is to be replaced by the energy corresponding to the lowest normal threshold branch point of the Regge pole. We shall assume for the present that there exists such a bound state; it then follows that for $s=4(M^2-x)$ the r-integration contour will be pinched by the pair of singularities located at $r=-(i/2)s^{1/2}+i(M^2-x)^{1/2}$ and $r=+(i/2)s^{1/2}-i(M^2-x)^{1/2}$; thus for $s>4(M^2-x)$ the contour will appear as shown in Fig. 9.

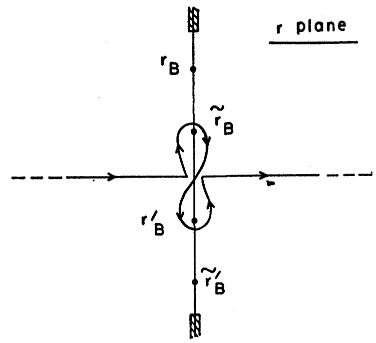

FIG. 9. The new integration contour of (3.4a) in the complex r plane as it appears for $s>4(M^2-x)$; r_B, r_B', $\tilde{r}_B$, and $\tilde{r}_B'$ give the position of the singularities arising from a bound state of mass $u_B^{1/2}$ lying on the trajectory $\alpha(u)$; they are given by: $r_B=i\frac{1}{2}s^{1/2}+i(u_B-x)^{1/2}$, $\tilde{r}_B=i\frac{1}{2}s^{1/2}-i(u_B-x)^{1/2}$, $r_B'=-i\frac{1}{2}s^{1/2}+i(u_B-x)^{1/2}$, and $\tilde{r}_B'=-i\frac{1}{2}s^{1/2}-i(u_B-x)^{1/2}$.

B. Asymptotic Behavior of the AFS Amplitude, and Location of the *j*-plane Singularities

The asymptotic behavior of the amplitude (3.4) will be determined by the above-mentioned singularities in the r plane, and by the upper limit of the x integration (i.e., $x=0$). Thus for $s<4M^2$ the large-t behavior of (3.7) is controlled by the singularity of the integrand at $l=2\alpha(x+s/4)-1$, and $x=0$ (corresponding to $r=x=0$), while for $s>4M^2$ it is controlled by the singularity at $l=\alpha(M^2)+\alpha(s+M^2-2s^{1/2}(M^2-x)^{1/2})-1$, and $x=0$ [corresponding to $r=\pm(i/2)s^{1/2}\mp i(M^2-x)^{1/2}$, and $x=0$]; that the latter singularity will dominate over the former for $s>4M^2$, follows from the fact that $\partial^2 l/\partial\eta^2>0$, where $i\eta=r$, and l is defined by (3.6). To find the precise form for the asymptotic behavior of (3.4), we expand the various Regge-trajectory functions around the above-mentioned points in the x and r planes keeping only the linear terms, and ignore the variation of all slowly varying functions in the integrand; one readily finds that

$$[A(s,t)]_{\mathrm{AFS}}\underset{t\rightarrow\infty}{\longrightarrow} D(s)t^{2\alpha(s/4)-1}/\ln t,\quad \text{for}\quad s<4M^2, \tag{3.9a}$$

where

$$D(s)=-\pi^3C[\gamma(s/4)/\sin\pi\alpha(s/4)]^2\times\tilde{C}(0,0,s)/2\alpha'(s/4), \tag{3.9b}$$

and

$$[A(s,t)]_{\mathrm{AFS}}\underset{t\rightarrow\infty}{\longrightarrow} H(s)t^{l_B+\alpha[(s^{1/2}-M)^2]-1}/(\ln t)^{1/2},\quad \text{for}\quad s>4M^2, \tag{3.10a}$$

where

$$H(s)=2\pi^{3/2}C\frac{\gamma((s^{1/2}-M)^2)}{\sin\pi\alpha((s^{1/2}-M)^2)}\times\frac{\gamma(M^2)\tilde{C}(0,iM-is^{1/2}/2;s)}{(-1)^{l_B}M\alpha'(M^2)[(s^{1/2}/M)\alpha'((s^{1/2}-M)^2)]^{1/2}} \tag{3.10b}$$

and l_B is the spin of the bound state of mass M. Assuming that the amplitude (3.4) has a Sommerfeld-Watson representation, we then conclude from (3.9) that the continued partial-wave amplitude associated with the s reaction must have a logarithmic branch point at $j=2\alpha(s/4)-1$ which for $s<4M^2$ is the leading singularity in the j plane; similarily, one may conclude from (3.10) that for $s>4M^2$ the leading singularity is a branch point of the inverse-square-root type located at $j=l_B+\alpha((s^{1/2}-M)^2)-1$; since for $s<4M^2$ the latter singularity has no effect on the asymptotic behavior of the amplitude (3.4), it must appear on an unphysical sheet of the logarithmic branch point.

One can readily generalize the above discussion to the case where we exchange two different trajectories α_1 and α_2. The analog of the singularity at $j=2\alpha(s/4)-1$ is still determined by (3.8) and by the upper limit of the x integration in (3.7). The position of the singularity is readily found to be

$$j=\alpha_1(u)+\alpha_2((s^{1/2}-u^{1/2})^2)-1\,, \qquad (3.11a)$$

where u is a solution to

$$\alpha_1'(u)-\alpha_2'((s^{1/2}-u^{1/2})^2)(s^{1/2}-u^{1/2})/u^{1/2}=0\,. \quad (3.11b)$$

Let us suppose, for simplicity, that only one of the trajectory functions, say α_1, passes through a physical bound state of mass m_1 and spin l_1. In the r plane of the integrand of (3.4) this gives rise to a pair of singularities which for $s>0$, and fixed negative x, appear on the imaginary axis at $r=-i\frac{1}{2}s^{1/2}\pm i(m_1{}^2-x)^{1/2}$. We are forced to distort the minimizing contour associated with the vanishing of $\partial l/\partial r$ (where $l\equiv\alpha_1+\alpha_2-1$) when the singularity at $r=-i\frac{1}{2}s^{1/2}+i(m_1{}^2-x)^{1/2}$ coincides with $r=r_0$, where r_0 is a solution to (3.8); in the l plane this corresponds to the coincidence of the bound-state singularity at $l=l_1+\alpha_2(s+m_1{}^2-2s^{1/2}(m_1{}^2-x)^{1/2})-1$ and the singularity arising from the vanishing of $\partial l/\partial r$ in (3.7); now with regard to the x integration, the asymptotic behavior of (3.7) is determined by its upper limit; furthermore, the above-mentioned singularities in the l plane will coincide at $x=0$, when s is a solution to

$$\alpha_1'(m_1{}^2)-\alpha_2'((s^{1/2}-m_1)^2)(s^{1/2}-m_1)/m_1=0\,. \quad (3.12)$$

Let s_c be the critical value of s; it then follows that (except for logarithmic factors) the asymptotic behavior of the amplitude will be of the form $t^{\lambda(s)}$, where $\lambda(s)$ is given by (3.11) if $s<s_c$, and by $l_1+\alpha_2((s^{1/2}-m_1)^2)-1$, if $s>s_c$. (The latter behavior is determined by the above-mentioned bound-state singularity.) The general picture in the angular-momentum plane which is suggested by the analysis of this section is summarized below. Finally we wish to remark that none of the above singularities will be present in the complete amplitude (3.1); the mechanism responsible for their cancellation is, of course, of the same type as the one discussed in Sec. II in connection with the single Regge pole exchange diagram.

C. Concluding Remarks and Summary

Let us summarize the situation for the case where the two exchanged trajectories are identical. Assuming that the AFS approximation to the diagram of Fig. 7 has a Sommerfeld-Watson representation, we are led, on the basis of the asymptotic expressions (3.9) and (3.10), to the following picture in the angular-momentum plane of the s reaction: If there exists a bound state of mass M lying on the trajectory $\alpha(u)$, then for $s<4M^2$ (that is below the threshold corresponding to the two-particle intermediate state formed by the bound states of the Regge-pole amplitudes) the leading singularity in the j plane is located at

$$j=2\alpha(s/4)-1\,. \qquad (3.13)$$

All other singularities which lie to the right of (3.13) hence must appear on an unphysical sheet. As we increase s above $4M^2$, a new singularity emerges onto the physical sheet via the branching at (3.13) and controls the asymptotic behavior of the amplitude; its position is given by the formula

$$j=\alpha((s^{1/2}-M)^2)+l_B-1\,, \qquad (3.14)$$

where l_B is the spin of the bound state; (3.14) is the analog of the moving singularity $j=\alpha((s^{1/2}-m)^2)-1$ found in the single Regge-pole exchange case. If, on the other hand, $\alpha(u)$ has no bound state, then (3.13) remains the leading singularity for $s<16m^2$, that is, below the four-particle production threshold (corresponding to two-particle intermediate states for each of the Regge-pole amplitudes). For $s>16m^2$ a new singularity then appears on the physical j sheet via the branching at (3.13); its location is given by

$$j=\alpha((s^{1/2}-2m)^2)+\alpha(4m^2)-1\,. \qquad (3.15)$$

Now there exists no essential difference between the various types of singularities in the r plane of the integrand of (3.4); the general picture thus suggested by the above analysis is that as we keep increasing s through positive values, all the mass-dependent singularities (which arise from the bound states and normal thresholds of the Regge-pole amplitudes) will appear in turn on the physical sheet via the mass-independent branch point at $j=2\alpha(s/4)-1$ whenever s has the appropriate value for the coincidence of the singularities of type (3.14) and (3.15) with the singularity (3.13); for any given s, the rightmost singularity will then determine the asymptotic behavior of the amplitude.

So far we have not specified which of the two s-channel partial-wave amplitudes of definite signature carries the above-mentioned branch points; to answer this question we notice that

$$\xi_\pm(\alpha_1)\xi_\pm(\alpha_2)t^{\alpha_1+\alpha_2}=[(-t)^{\alpha_1}\pm t^{\alpha_1}][(-t)^{\alpha_2}\pm t^{\alpha_2}]\,.$$

Now for $t\to+\infty$ the amplitude (3.1) presumably approaches (3.4a); the corresponding limit for $t\to-\infty$ would then be given by the negative of this expression (the reasoning is of course identical to that given in connection with the single Regge-pole exchange diagram). We hence conclude that the amplitude is even or odd under the transformation $t\to -t$, depending on whether we exchange two Regge poles of the same or opposite signatures, respectively. [The signature factors had been omitted in (3.4a).] Similar reasoning to that used in connection with the single-Regge-pole exchange diagram then leads us to conclude that the above-mentioned branch points in the j plane will appear in the even or odd signature partial-wave amplitudes depending on whether the two exchanged trajectories have equal or opposite signatures (in that order).

Concerning the normal threshold singularities of $[A(s,t)]_{\mathrm{AFS}}$ in s, they are generated in a similar way to those of the single Regge-pole exchange amplitude; thus, let us suppose that there exist two physical bound states of masses m_1 and m_2 which lie on the trajectories α_1 and α_2, respectively. For $s>0$ the pairs of singularities of the integrand of (3.4a) in the r plane which arise from each of the bound state poles of the Regge amplitudes [their locations are given by (3.5) with $u_n\to m_1^2$ and m_2^2] lie on the imaginary axis; as we keep increasing s, two of the four singularities (one from each pair) will pinch the r-integration contour when $s^{1/2}-(m_1^2-x)^{1/2}-(m_2^2-x)^{1/2}=0$. Performing the x integration then generates an end-point singularity of $[A(s,t)]_{\mathrm{AFS}}$ at $s=(m_1+m_2)^2$. This is the two-body normal threshold branch point corresponding to the intermediate state formed by the bound states of mass m_1 and m_2. As we keep increasing s above $s=(m_1+m_2)^2$, the r-integration contour will again be pinched between two of the bound-state singularities and one from each pair of threshold singularities (which arise from the normal threshold branch points of the Regge-pole amplitudes) when either

$$s^{1/2}-(m_1^2-x)^{1/2}-(4m^2-x)^{1/2}=0$$

or

$$s^{1/2}-(m_2^2-x)^{1/2}-(4m^2-x)^{1/2}=0;$$

[we have assumed that $u=4m^2$ is the lowest-normal threshold of $R(\alpha_1(u),t)$ and $R(\alpha_2(u),t)$]. Subsequent integration over x then produces the corresponding three-body normal threshold branch points at $s=(m_1+2m)^2$, and $s=(m_2+2m)^2$, respectively; the generalization of the above results to include higher normal threshold singularities (which will be complex if they arise from resonances lying on the trajectories α_1 and α_2) is self-evident.

In concluding we wish to add one further remark; let us suppose that we exchange two identical trajectories α; the singularities (3.13–15) then appear in the even-signature–partial-wave amplitude; one may now easily convince oneself that for j even [$b^+(j,s)$ then coincides with the physical partial-wave amplitude] they give rise to unphysical singularities in the s plane; in complete analogy to the single Regge-pole exchange case one can however show that the discontinuity across these singularities of $b^+(j,s)$ vanishes at the even integers since the AFS amplitude was shown above to have only physical branch points.

IV. DIAGRAMS THAT HAVE THE MANDELSTAM SINGULARITIES

So far we have dealt with a set of Feynman graphs which in an AFS-type of approximation gave rise to angular-momentum branch points which are, however, absent in the full amplitude. Nevertheless we have studied them in great detail for two reasons: (a) We wished to obtain a clearer understanding of the mechanism responsible for the cancellation of the singularities (which presumably is not in operation for such diagrams as shown in Figs. 10 and 11). (b) We expect that the location and nature of the singularities found in an AFS-type of approximation to the diagrams of Figs. 1 and 7 is the same as that found for the full amplitudes associated with the diagrams of Figs. 10 and 11. It is clear that the complexity of the latter diagrams will make it impossible to carry out as careful an analysis as was made for their simpler versions, and we will have to sacrifice a certain amount of rigor in favor of simplicity.

A. The Single Regge-Pole Exchange Diagram

Consider the Feynman amplitude corresponding to the diagram of Fig. 10 which is expected to have the angular-momentum branch points that we found in Sec. II in connection with the AFS approximation to the diagram of Fig. 1:[2]

$$A(s,t)=-i\left(\frac{g}{4\pi^2}\right)^6\int d^4\xi_1 d^4k_1 d^4\eta_1\frac{1}{\eta_1^2-m^2+i\epsilon}\times\prod_{i=1}^{4}\frac{1}{[\xi_i^2-m^2+i\epsilon][k_i^2-m^2+i\epsilon]}\times R(\eta_2^2,U;\,\xi_3^2,\xi_4^2,k_3^2,k_4^2)\,,\quad(4.1)$$

where R is the amplitude associated with the Regge pole,[17] U being the invariant-momentum-transfer squared

$$U=(\xi_3+k_3)^2=(\xi_4+k_4)^2.\quad(4.2)$$

Let $s'=\eta_2^2$, $s''=\eta_1^2$, $t'=(q_1+\eta_2)^2$, and $t''=(q_2-\eta_2)^2$. The components of the four-vector η_1 may be expressed in terms of the invariants s', s'', t', and t''; the Jacobian for the transformation is given as before by (2.2, 3) with the replacements: $k_1^2\to s''$, $k_3^2\to s'$, $k_2^2\to t'$, and $k_4^2\to t''$; proceeding as in Sec. II, we shall keep for $t\to\infty$ and small momentum transfers s only the first two terms in

[17] The amplitudes A and R are related to the T-matrix element by

$$T_{fi}=(2\pi)^4\delta^4(P_f-P_i)\frac{F_{fi}}{[(2\pi)^{3/2}]^4\prod_{i=1}^{4}(2\omega_i)^{1/2}}\,,$$

where $F=A$, or R.

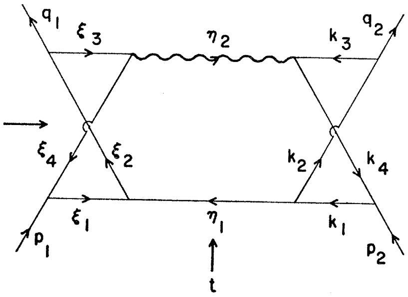

FIG. 10. Single Regge-pole exchange diagram which has singularities of the AFS type.

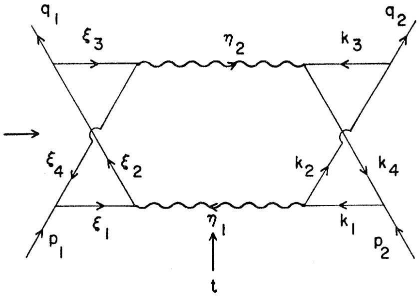

FIG. 11. Double Regge-pole exchange diagram which has singularities of the AFS type.

the expression for D [see Eq. (2.3)]. With this approximation the integral (4.1) becomes

$$A(s,t) \underset{t\to\infty}{\longrightarrow} -\frac{i}{4}(g/4\pi^2)^6\frac{1}{t}\int ds'ds''\frac{\tau(s,s',s'')}{s''-m^2+i\epsilon}$$
$$\times\left[\int_{-\infty}^{+\infty}dt'\int_{-\infty}^{+\infty}dt''F(s',s'',t',t'';s,t)\right], \quad (4.3)$$

where $\tau(s,s',s'')$ is the triangle function defined in (2.5), and where

$$F(s',s'',t',t'';s,t)$$
$$=\int d^4\xi_1 d^4k_1\prod_{i=1}^{4}\frac{R(s',U;\xi_3^2,\xi_4^2,k_3^2,k_4^2)}{[\xi_i^2-m^2+i\epsilon][k_i^2-m^2+i\epsilon]}. \quad (4.4)$$

Included among the singularities of F there are those pertaining to the integral (4.4) with R replaced by a constant; i.e., it has the singularities of $\tilde{F}$, where

$$\tilde{F}(s',s'',t',t'';s)=A^C(s,t';s',s'')A^C(s,t'';s',s'');$$

here $A^C(s,t;s',s'')$ is the invariant amplitude associated with the "cross" in Fig. 10:

$$A^C(s,t';s',s'')=\int d^4\xi_1\prod_{i=1}^{4}\frac{1}{\xi_i^2-m^2+i\epsilon}. \quad (4.5)$$

Now $A^C(s,t';s',s'')$ has normal threshold branch points at $t'=4m^2-i\epsilon$ and $u'\equiv(\eta_2-p_1)^2=4m^2-i\epsilon$, where t' and u' are related according to: $t'+u'+s=2m^2+s'+s''$. It follows that (4.4) will be singular at

$$t'=4m^2-i\epsilon \quad (4.6)$$

and

$$t'=s'+s''-s-2m^2+i\epsilon, \quad (4.7)$$

with an identical set of singularities of F in the variable t'' which arise from the normal threshold singularities of $A^C(s,t'';s',s'')$. The essential feature to be noticed about the singularities (4.6, 7) is that they appear on opposite sides of the t' integration contour, $C_{t'}$ (the same, of course, applies to the singularities of F in t''); the integration contours of (4.3) in the t' and t'' planes are thus forced to cross the real axis somewhere between t' (or t'') $=4m^2$ and t' (or t'') $=s'+s''-s-2m^2$.[18] Now from the Landau equations for the normal threshold singularities of the two crosses in Fig. 10 it follows that when $t'=4m^2-i\epsilon$ the ξ_1 integration contours in (4.4) cannot be distorted so as to avoid the region where $\xi_2=-\xi_4$, and $\xi_2^2=\xi_4^2=m^2$; similarily, when $t''=4m^2-i\epsilon$, we cannot avoid the integration region where $k_2=-k_4$, and $k_2^2=k_4^2=m^2$; from this it follows that when $t'=t''=4m^2-i\epsilon$, $U=t/4$, where U is defined by (4.2). Similarly, from the Landau equations for the u-channel threshold singularities at $u'=4m^2-i\epsilon$ and $u''=4m^2-i\epsilon$ (corresponding to $t'=s'+s''-s-2m^2+i\epsilon$ and $t''=s'+s''-s-2m^2+i\epsilon$), one finds that there exist corresponding unavoidable integration regions in (4.4) where $\xi_1=-\xi_3$, $\xi_1^2=\xi_3^2=m^2$, and $k_1=-k_3$, $k_1^2=k_3^2=m^2$, respectively; once again one may verify that at $t'=t''=s'+s''-s-2m^2+i\epsilon$, $U=t/4$. Next we notice that the s' and s'' integrations in (4.3) include the boundary point $s'=0$, $s''=s$ (the boundary of the s', s'' integrations being given by $s^2+s'^2+s''^2-2ss'-2ss''-2s's''=0$); furthermore, we cannot distort the integration contours so as to avoid this point; now at $s'=0$, $s''=s$, the singularities of F in t' and t'' are located at $t'=4m^2-i\epsilon$, $t''=4m^2-i\epsilon$, and $t'=-2m^2+i\epsilon$, $t''=-2m^2+i\epsilon$, so that the t' and t'' integration contours are forced to cross the real axis somewhere between t' (or t'') $=4m^2$ and t' (or t'') $=-2m^2$; thus one might expect that in this region of the "approximate pinch," $\xi_i^2\approx k_i^2\approx m^2$, and $U\approx t/4$. In order to get an estimate of the contribution to the amplitude at large t coming from the angular-momentum cut, we shall make the (not totally unreasonable) assumption that the major contribution to the quantity appearing within brackets in (4.3) comes from the above-mentioned region of the "approximate pinch" in the t' and t'' planes, and from the integration region of (4.4) where all four-momenta squared are close to their mass-shell value; thus one might attempt to approximate the function $R(s',U;\xi_3^2,\xi_4^2,k_3^2,k_4^2)$ appearing

[18] Our qualitative reasoning is that of C. Wilkin (see Ref. 3) and of E. S. Abers, H. Burkhardt, V. L. Teplitz, and C. Wilkin, Nuovo Cimento 42, 365 (1966). Our goal is however a more ambitious one in that we wish to arrive at a quantitative estimate for the contribution to the amplitude coming from the angular-momentum cut.

in (4.4) by its value at $U=t/4$, and $\xi_3^2=\xi_4^2=k_3^2=k_4^2=m^2$ (we certainly cannot consider the present discussion as rigorous; however, the approximations will at least give us some kind of estimate for the strength of the cut); with this approximation the integral (4.3) becomes

$$A(s,t)\underset{t\to\infty}{\longrightarrow}-\frac{i}{4}\left(\frac{g}{4\pi^2}\right)^6\frac{1}{t}\int ds'ds''\frac{\tau(s,s',s'')}{s''-m^2+i\epsilon}R(\alpha(s'),t/4)\times\left\{\int_{-\infty}^{+\infty}dt'A^C(s,t';s',s'')\right\}^2, \quad (4.8)$$

where $A^C(s,t';s',s'')$ is given by (4.5).

Before proceeding with the analysis of (4.8) we wish to emphasize once more that the existence of the "approximate pinch" in the t' *and* t'' planes was essential in deriving the expression; this in turn requires that both the right *and* left-hand portions of the diagram must have a third double spectral function with respect to the s reaction; in fact, it has been shown by Wilkin[3] that if either the right or left-hand portion of the diagram does not have a third double spectral function, one can distort the integration contours of the Feynman amplitude (4.1) in such a way that the Regge pole will not assume its characteristic asymptotic form anywhere along the paths of integration.

We now return to formula (4.8) and extract from it the leading term for $t\to\infty$ which comes from the integration region $s'\simeq 0$, $s''\simeq s$. Approximating $A^C(s,t';s',s'')$ by $A^C(s,t';0,s)$, and proceeding as in Sec. II, we obtain, upon substituting (2.7) for R,

$$A(s,t)\to-\frac{i}{16}\left(\frac{g}{4\pi^2}\right)^6\frac{\gamma(0)}{\sin\pi\alpha(0)}[K(s)]^2C(\alpha)\xi_\pm(\alpha)\left(\frac{t}{4}\right)^{\alpha(0)-1}\times\int_{-\epsilon}^{0}ds'\left(\frac{t}{4}\right)^{\alpha'(0)s'}\int ds''\frac{\tau(s,s',s'')}{s''-m^2+i\epsilon},$$

$$\approx\frac{i\pi}{16}\left(\frac{g}{4\pi^2}\right)^6\frac{\gamma(0)}{\sin\pi\alpha(0)}\frac{[K(s)]^2}{m^2-s}C(\alpha)\xi_\pm(\alpha)\times\left(\frac{t}{4}\right)^{\alpha(0)-1}\Big/\alpha'(0)\ln(t/4),\quad\text{for}\quad s<0, \quad (4.9a)$$

where $C(\alpha)\equiv C(\alpha(0))$, $\xi_\pm(\alpha)=\xi_\pm(\alpha(0))$, and

$$K(s)=\int_{-\infty}^{+\infty}dt'A^C(s,t';0,s). \quad (4.9b)$$

Except for the factor $[K(s)]^2$ and the replacement $t\to t/4$, formula (4.9a) is identical to that obtained for the AFS approximation to the single Regge-pole exchange diagram treated in Sec. II [see (2.18)]; thus once again we conclude that, for $s<0$, the leading branch point in the angular-momentum plane of the s-channel partial-wave amplitude is located at $j=\alpha(0)-1$, and that it is of the logarithmic type.

B. The Double Regge-Pole Exchange Diagram

The diagram involving the exchange of two Regge poles (see Fig. 11) can be dealt with in exactly the same way as above; for simplicity we shall consider the case of two identical Regge poles. Making the same type of approximations as before, one arrives at the following expression for the amplitude:

$$A(s,t)\underset{t\to\infty}{\approx}\frac{i}{4}\left(\frac{g}{8\pi^3}\right)^4\frac{1}{t}\int\int ds'ds''\tau(s,s',s'')R(\alpha(s'),t/4)\times R(\alpha(s''),t/4)\left\{\int_{-\infty}^{+\infty}dt'A^C(s,t';s',s'')\right\}^2; \quad (4.10)$$

the leading contribution to (4.10) comes from the integration region $s'\approx s''\approx s/4$, so that we may approximate $A^C(s,t';s',s'')$ by $A^C(s,t';s/4,s/4)$ in this domain. If we then substitute

$$R(\alpha(s),t/4)=\tilde\gamma(s)C(\alpha)\xi_\pm(\alpha)(t/4\bar t)^{\alpha(s)}/\sin\pi\alpha(s) \quad (4.11)$$

into Eq. (4.10) (here $\bar t$ is a reference energy to be specified below) and make the change of variables $\xi=s'+s''$, $\eta=s'-s''$, we arrive at the following formula for the asymptotic contribution to $A(s,t)$:

$$A(s,t)\approx\frac{i}{32}\left(\frac{g}{8\pi^3}\right)^4\left(\frac{\tilde\gamma(s/4)H(s)C(\alpha)\xi_\pm(\alpha)}{\sin\pi\alpha}\right)^2\times\frac{1}{\bar t}\left(\frac{t}{4\bar t}\right)^{2\alpha(s/4)-1-\frac12 s\alpha'}\int_{s/2-\epsilon}^{s/2}d\xi\left(\frac{t}{4\bar t}\right)^{\alpha'\xi}\times\int d\eta\frac{\theta(2s\xi-s^2-\eta^2)}{(2s\xi-s^2-\eta^2)^{1/2}}$$

$$\approx\frac{i\pi}{32}\left(\frac{g}{8\pi^3}\right)^4\left(\frac{\tilde\gamma(s/4)H(s)C(\alpha)\xi_\pm(\alpha)}{\sin\pi\alpha}\right)^2\times\left(\frac{t}{4\bar t}\right)^{2\alpha(s/4)-1}\Big/\bar t\alpha'\ln(t/4\bar t), \quad (4.12a)$$

where $\alpha\equiv\alpha(s/4)$, $\alpha'\equiv\alpha'(s/4)$, and

$$H(s)=\int_{-\infty}^{+\infty}dt'A^C(s,t';s/4,s/4). \quad (4.12b)$$

From formula (4.12) we see that for $s<0$ the position of the leading singularity in the angular-momentum plane of the s reaction is a logarithmic branch point located at $j=2\alpha(s/4)-1$.

Finally we wish to cast (4.12b) into a more convenient form for computational purposes. Since we shall be interested in the value of (4.12b) at small momentum transfers s, we will approximate the integrand by $A^C(s,t';0,0)$. Now, on account of the many approximations made in deriving formula (4.10), we can only hope to obtain a very rough estimate of the contribution to the amplitude coming from the cut. For practical reasons we shall therefore make a further approximation and replace $A^C(s,t;0,0)$ by $A^C(s,t)$, where the latter is the amplitude associated with the "cross" with all external masses taken equal to m^2. Now, $A^C(s,t)$ is known to have the spectral representation

$$A^C(s,t)=\frac{1}{\pi^2}\int dt'\int du'\frac{\rho(t',u')}{(t'-t)[u'-(4m^2-s-t)]}, \quad (4.13)$$

where $\rho(t,u)$ is the well-known Mandelstam double spectral function for the box diagram;[19] the boundary of the region where $\rho(t,u)\neq 0$ is given by $(t-4m^2)(u-4m^2)-4m^4=0$; from here it follows that, for fixed s, (4.13) defines an analytic function of t in the t plane cut from $t=4m^2$ along the positive t axis, and from $t=-s$ along the negative axis. The singularities at $t=4m^2$ and $t=-s$ are the ones responsible for the approximate pinch discussed previously (where the limit $\epsilon\to 0$ has been taken); the contour $C_{t'}$ of the integral (4.12b) extends just above and just below the right- and left-hand cuts, respectively. Now for fixed s, $A^C(s,t)$ vanishes like $1/t^2$ for large t; we therefore may distort the contour $C_{t'}$ around the right-hand cut of $A^C(s,t)$ and rewrite the integral (4.12b) in the form

$$H(s)\approx 2i\int_{4m^2}^{\infty}dt'A_t^C(s,t'), \quad (4.14)$$

where $A_t^C(s,t)$ is the t-channel absorptive part of $A^C(s,t)$, which, in the notation of Ref. 19 is given by

$$A_t^C(s,t)=\frac{-i\pi^3}{[K(t,u)]^{1/2}}\times\ln\left(\frac{\alpha(t,u)+(q_t/t^{1/2})[K(t,u)]^{1/2}}{\alpha(t,u)-(q_t/t^{1/2})[K(t,u)]^{1/2}}\right), \quad (4.15a)$$

where

$$K(t,u)=4tu[tu-4m^2(t+u)+12m^4],$$
$$\alpha(t,u)=tu-2m^2t-4m^2u+6m^4, \quad (4.15b)$$
$$q_t^2=-m^2+t/4,$$

and

$$u=4m^2-s-t.$$

If, in (4.12), α is taken to be the Pomeranchuk trajectory, then we obtain for $s=0$

$$[A(0,t)]_{\rm cut}=\frac{\pi}{128}\left(\frac{g}{8\pi^3}\right)^4\tilde{\gamma}(0)[H(0)]^2\times R(\alpha(0),t)/\bar{t}\alpha'(0)\ln(t/4\bar{t}), \quad (4.16a)$$

where

$$R(\alpha(0),t)=-i\tilde{\gamma}(0)\left(\frac{t}{\bar{t}}\right). \quad (4.16b)$$

We now wish to obtain a numerical estimate for the right-hand side of (4.16a). From formulas (4.14) and (4.15a, b) one finds, after some algebra,

$$H(0)=-\frac{\pi^3}{m^2}\int_0^\infty dz\frac{1}{(z+2)(z^2+4z)^{1/2}}\times\ln[1+z(z+2)]\approx -1.53\pi^3/m^2.$$

Next we shall assume that the value of $\tilde{\gamma}(0)$ is approximately given by the corresponding residue function associated with the coupling of the Pomeranchuk trajectory to the π-π system; the latter has been estimated in Ref. 20; taking into account that the Regge-pole amplitude R used in this section is related to that of Ref. 20 (call it R') by $R=16\pi R'$, we find that $\tilde{\gamma}(0)\approx -16\pi$, if the reference energy $\bar{t}$ in (4.11) is chosen to be $\bar{t}=1.87$ (BeV)^2. Finally, to obtain an estimate of the coupling strength g, we take recourse to the following model: consider the amplitude for scattering of two scalar particles in the ladder approximation to the Bethe-Salpeter equation (all particles involved in the ladder are taken to have mass m); in this approximation an estimate of the coupling strength may be obtained by requiring that the leading Regge trajectory shall pass through unit angular momentum at zero energy. The calculations Ref. 21 show that the required value of g is approximately given by $g=(16\pi)m$. (This corresponds to $\lambda=16$ in Ref. 21.) Substituting the values for $H(0)$, $\tilde{\gamma}(0)$, and g into formula (4.16a), we find

$$A(0,t)/R(0,t)\approx -4.7/\bar{t}\alpha'(0)\ln(t/4\bar{t}), \quad (4.17)$$

where we have written $R(0,t)\equiv R(\alpha(0),t)$. Now, there are indications that the Pomeranchuk trajectory is rather flat; if we take, for example, its slope to be $\frac{1}{3}$ that of the P' trajectory (which we shall assume to go through angular momentum 2 at the mass of the f_0, and through $\frac{1}{2}$ at zero energy), then we find, using formula (4.17), that the ratio becomes unity at an energy around 140 BeV. This dominance of the cut over the pole would become even stronger as we moved away from the forward direction. Expanding the trajectory function $\alpha(s/4)$ appearing in (4.12a) around $s=0$, one obtains, for the ratio A/R at small momentum transfers s

$$A(s,t)/R(s,t)\approx -4.7\exp[-\tfrac{1}{2}s\lambda(t)]/\bar{t}\lambda(t), \quad (4.18a)$$

where

$$\lambda(t)=\alpha'(0)\ln(t/4\bar{t}). \quad (4.18b)$$

[19] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

[20] H. J. Rothe, Phys. Rev. **140**, B1421 (1965).

[21] For a discussion of the coupling strength required to produce a bound state of zero mass and unit spin in the ladder approximation to the Bethe-Salpeter equation, see C. Schwartz, Phys. Rev. **137**, B717 (1965). It has been shown subsequently by W. B. Kaufmann (private communication) that this bound state lies on the leading Regge trajectory.

The above-obtained results should, of course, not be taken at their face value, in view of the numerous approximations made in the derivation of (4.17) and (4.18); even if all parameters appearing in (4.16a) were known, it would not be surprising if the true result differed from the one obtained above by an order of magnitude, or even more.

V. CONCLUSION

The considerations of the preceding section indicate that the location and nature of the angular-momentum branch points associated with the diagrams for which the cancellation of the cuts does not occur is the same as that for the AFS approximation to their simpler versions, considered in detail in Secs. II and III. The role of the third double spectral function associated with the cross in the diagrams of Figs. 10 and 11 thus appears to be essentially that of preventing the above-mentioned cancellation from occurring; the latter diagrams have been studied in much more detail in Refs. 2 and 4 via s-channel unitarity, and the results support the above conclusions. Concerning our estimate of the contribution to the amplitude coming from the Mandelstam singularity associated with the diagram of Fig. 11, it can, of course, not be taken very seriously; it does, however, suggest that at moderate energies, the cut and pole contributions might conceivably be of the same order of magnitude. The method used in the analysis of Figs. 1 and 7 had been originally adapted to the purpose of exposing in as clear a way as possible the cancellation mechanism of the Amati, Fubini, Stanghellini cuts; this mechanism has been found to be extremely simple. The same method also led to a relatively simple analysis of the singularities in the angular-momentum plane of the s reaction; we found them to be of two general types: those that are independent of particle masses, and those which depend on them. Only the former ones remain on the physical j sheet at negative momentum transfers; their positions in the j plane are given by $j=\alpha(0)-1$ and $j=2\alpha(s/4)-1$ for the single and double Regge-pole exchange diagrams, respectively. It is interesting to note that both these singularities are of the logarithmic type and are a consequence of the singular nature of the mapping of the k_z plane into the complex l plane, where l is the angular momentum in the s channel obtained by coupling the (complex) spins of the exchanged systems to a relative orbital angular momentum $L=-1$. The analog of the singularity at $j=\alpha((s^{1/2}-m)^2)-1$ for the single Regge-pole exchange diagram is the singularity at $j=\alpha((s^{1/2}-M)^2)+l_B-1$ associated with the diagram involving the exchange of two identical Regge poles; these singularities appear on the physical j sheet via the particle-mass independent branch points for $s>m^2$ and $s>4M^2$, respectively; furthermore, both are of the inverse-square-root type. The similarity between the amplitudes (4.8) and (2.6), and, (4.10) and (3.2), suggests that the above picture in the j plane remains the same for the diagrams of Figs. 10 and 11.

In conclusion, the analysis presented in this paper indicates that everything we wish to know regarding the location and nature of the angular-momentum branch points associated with the diagrams in which the singularities are not cancelled, can be learned by investigating the corresponding simpler versions of these diagrams in an AFS-type of approximation; thus it appears that the additional complexity of the former diagrams, aside from modifying the strength of the singularities, merely serves to prevent the cancellation of the cuts from occurring.

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