Configuration-Space Approach to Three-Particle Scattering

W. ZICKENDRAHT
Physikalisches Institut der Universität Marburg, Marburg, Germany
(Received 11 April 1966; revised manuscript received 16 December 1966)

A configuration-space approach to the three-particle problem which has been proposed for bound states in previous publications is generalized to include scattering states. It is illustrated for the case of nucleon-deuteron scattering. The coupling between the elastic and the inelastic channel can be treated in an elegant way. The problem is reduced approximately to a finite system of coupled differential equations, in which the unknown functions depend on a single variable only. An iteration procedure for the solution is proposed. The accuracy of the method is tested for a simple example.

I. INTRODUCTION

INTEREST in the three-particle problem has increased since the Faddeev equations[1–3] turned out to be a useful tool for studying three-particle scattering. Noyes has proposed a nonsingular integral equation for two-particle scattering,[4] by which the Faddeev equations can be reduced approximately to a system of coupled integral equations in a single variable which are soluble on a computer. However, recently Omnes[5] has drawn attention to the weak points of these methods using Faddeev equations.

The present paper offers a new method, which approaches the three-particle problem from quite a different point of view, and which also allows us to treat bound states as well as scattering states. This method deals with the three-particle problem in configuration space. It has already been shown that a whole class of bound three-particle states can be treated this way.[6] With the use of a special set of coordinates[7–9] and a special system of orthogonal functions[6,10] the Schrödinger equation is reduced to a coupled system of differential equations for functions which depend on a single variable only. The initially infinite number of coupled differential equations is approximately reduced to a finite system. However, the orthogonal system is not suited for scattering problems like nucleon-deuteron scattering where we have an elastic as well as an inelastic channel. We can use the functions for describing the inelastic channel, but not for the elastic one, as will be shown in Sec. II. The purpose of this paper is to describe a modification of the method which allows us to solve three-particle scattering problems also. With this modification it is again possible to reduce the Schrödinger equation to a system of coupled differential equations for functions which depend on a single variable only. This reduction is possible for any kind of interaction between the three particles (except interactions with a hard core). But for simplification the discussion below will be restricted to central forces. For further simplification we consider only nucleon-deuteron scattering, where we have only one bound state, and we do not antisymmetrize the wave function. This antisymmetrization can be included easily as shown in Ref. 6. The case of three particles with different masses is also discussed in Ref. 6.

The coordinates used are (in the center-of-mass system)

1. external coordinates: the three Euler angles which are defined by the three principal axes of the moment of inertia;
2. internal coordinates: a length y and two angles α and β which are related to the three distances r_{ik} by[6]

$$\begin{aligned} r_{12}&=y(1-\sin\alpha\,\sin\beta)^{1/2}/\sqrt{2}\\ r_{23}&=y[1-\sin\alpha\,\sin(\beta-\tfrac{2}{3}\pi)]^{1/2}/\sqrt{2}\\ r_{31}&=y[1-\sin\alpha\,\sin(\beta-\tfrac{4}{3}\pi)]^{1/2}/\sqrt{2}. \end{aligned} \tag{1}$$

For the case of a three-particle scattering state, α and β have a simple meaning: $\sin\alpha$ and β are plane polar coordinates in the Dalitz diagram.[9,10] The wave function for orbital angular momentum L can be written in the form

$$\Psi=\sum_{K=-L}^{+L}F_K{}^L(y,\alpha,\beta)D_{MK}{}^L(\phi,\theta,\psi). \tag{2}$$

The Schrödinger equation is reduced in this case to a system of coupled differential equations for the functions $F_K{}^L$. For further simplification the discussion below is restricted to s states, $L=0$:

$$\left\{\frac{h^2}{m}\left[\frac{\partial^2}{\partial y^2}+\frac{5}{y}\frac{\partial}{\partial y}+\frac{4}{y^2}\left(\frac{\partial^2}{\partial\alpha^2}+2\frac{\cos2\alpha}{\sin2\alpha}\frac{\partial}{\partial\alpha}+\frac{1}{\sin^2\alpha}\frac{\partial^2}{\partial\beta^2}\right)\right]+[E-V(y,\alpha,\beta)]\right\}F_0{}^0=0. \tag{3}$$

Asymptotically ($y\to\infty$), $F_0{}^0$ will consist of the elastic channel (nucleon+deuteron) and the inelastic channel

---

[1] L. I. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys. JETP **12**, 1014 (1961)].
[2] R. L. Omnes, Phys. Rev. **134**, B1358 (1964).
[3] C. Lovelace, in *Strong Interactions and High Energy Physics* edited by R. G. Moorhouse (Oliver and Boyd, London, 1964).
[4] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965).
[5] R. L. Omnes, Phys. Rev. Letters **17**, 775 (1966).
[6] W. Zickendraht, Ann. Phys. (N. Y.) **35**, 18 (1965).
[7] W. Zickendraht, Proc. Natl. Acad. Sci. U. S. **52**, 1565 (1964).
[8] V. Gallina, P. Nata, L. Bianchi, and G. Viano, Nuovo Cimento **24**, 835 (1962).
[9] A. J. Dragt, J. Math. Phys. **6**, 533 (1965).
[10] J. M. Lévy-Leblond and M. Lévy-Nahas, J. Math. Phys. **6**, 1571 (1965).

(three unbound nucleons). For this reason $F_0{}^0$ is expressed the following way:

$$F_0{}^0=F_1+F_2. \quad (4)$$

Here F_1 represents the elastic channel asymptotically, F_2 the inelastic channel:

$$F_1 \underset{y\to\infty}{\sim} \text{nucleon+deuteron}$$

$$F_2 \underset{y\to\infty}{\sim} \text{three unbound nucleons}.$$

Details on the functions F_1 and F_2 will be given in the following section.

II. INELASTIC CHANNEL

For F_2 the following expansion is used:

$$F_2=\sum_{\lambda,\mu} h_{\lambda\mu}(y)e^{i\mu\beta}f_{\lambda\mu}(\alpha). \quad (5)$$

The functions $e^{i\mu\beta}f_{\lambda\mu}(\alpha)$ are eigenfunctions of the operator

$$\frac{\partial^2}{\partial\alpha^2}+2\frac{\cos 2\alpha}{\sin 2\alpha}\frac{\partial}{\partial\alpha}+\frac{1}{\sin^2\alpha}\frac{\partial^2}{\partial\beta^3},$$

occurring in Eq. (3). They form a complete orthogonal system and constitute a special classification of three-particle states.[6,9,11,12] The corresponding orthogonal systems for orbital angular momenta $L=1$, 2 are given in Ref. 6, along with a method for constructing the orthogonal systems for higher orbital angular momenta. Asymptotically the equations for the different $h_{\lambda\mu}$ decouple ($y\rightarrow\infty$):

$$\left[\frac{\hbar^2}{m}\left(\frac{\partial^2}{\partial y^2}+\frac{5}{y}\frac{\partial}{\partial y}-\frac{4\lambda(\lambda+2)}{y^2}\right)+E\right]h_{\lambda\mu}=0. \quad (6)$$

The term $4\lambda(\lambda+2)/y^2$ represents a three-particle "centrifugal" barrier and has the same effect as the centrifugal barrier in the ordinary two-body problem: it subordinates the importance of functions with high values of λ. The asymptotic expressions for $h_{\lambda\mu}$ are (for $E>0$)

$$h_{\lambda\mu}\rightarrow B_{\lambda\mu}e^{-iky}/y^{5/2}+C_{\lambda\mu}e^{iky}/y^{5/2}, \quad (7)$$

$$k=(mE/\hbar^2)^{1/2}. \quad (8)$$

In the case of nucleon-deuteron scattering we will have no incoming wave in the asymptotic expression for F_2; thus

$$B_{\lambda\mu}=0, \quad (9)$$

$$F_2\rightarrow (e^{iky}/y^{5/2})\sum_{\lambda\mu} C_{\lambda\mu}e^{i\mu\beta}f_{\lambda\mu}(\alpha). \quad (10)$$

If the energies of the three particles are plotted in a Dalitz diagram, the density of points is simply given by

$$\rho(\alpha,\beta)=\left|\sum_{\lambda\mu} C_{\lambda\mu}e^{i\mu\beta}f_{\lambda\mu}(\alpha)\right|^2. \quad (11)$$

The functions $e^{i\mu\beta}f_{\lambda\mu}(\alpha)$ are not suited for describing the elastic channel. To see this we consider for the moment an unbound nucleon which is not interacting with the deuteron. Then the wave function for an s wave is proportional to

$$\chi_d(r_{12})(\sin k_3r_3)/r_3. \quad (12)$$

χ_d is the deuteron wave function, r_{12} the neutron-proton distance, and r_3 the distance of the unbound nucleon from the center-of-mass of the deuteron. Expressed with the new coordinates y, α, β, Eq. (12) has the form

$$\chi_d[y(1-\sin\alpha\sin\beta)^{1/2}/2^{1/2}] \times\frac{\sin[k_3(\frac{3}{8})^{1/2}y(1+\sin\alpha\sin\beta)^{1/2}]}{(\frac{3}{8})^{1/2}y(1+\sin\alpha\sin\beta)^{1/2}}. \quad (13)$$

If the nucleon and deuteron are far apart, that is for large y, the function χ_d is different from zero only for values of α and β around $\pi/2$. To describe this property with the complete orthogonal system $e^{i\mu\beta}f_{\lambda\mu}(\alpha)$, one would need the whole series $0\leq\lambda\leq\infty$, which would make the method useless for any practical application. This is the reason for the term F_1 in Eq. (4). A possible ansatz for F_1 will be given in the following section.

III. ELASTIC CHANNEL

The Schrödinger equation for the three-nucleon problem has the following form

$$\{(\hbar^2/m)(\Delta_1+\Delta_2)+E-V\}\Psi=0. \quad (14)$$

Δ_1 and Δ_2 belong to the vectors

$$\mathbf{x}_1=\mathbf{r}_{12}$$
$$\mathbf{x}_2=(\tfrac{4}{3})^{1/2}\mathbf{r}_3; \quad (15)$$

V is the interaction:

$$V=V_{12}(r_{12})+V_{23}(r_{23})+V_{31}(r_{31}).$$

It should be emphasized again that two-nucleon potentials with a hard core are not suited for this method; rather, soft-core potentials should be used.[13] V can be written as a sum of two terms, the first causing elastic scattering only, the second giving inelastic scattering also:

$$V=[V_{12}(r_{12})+V_{23}(r_3)+V_{31}(r_3)] +[V_{23}(r_{23})-V_{23}(r_3)+V_{31}(r_{31})-V_{31}(r_3)]. \quad (16)$$

A possible starting point is to solve the elastic scattering problem first [the first term in Eq. (16)]. In the second step the second term in Eq. (16) is taken into account.

[11] P. Kramer, Z. Naturforsch **18a**, 260 (1963).
[12] E. Chacon and M. Moshinsky, Rev. Mex. Fis. **14**, 119 (1965).
[13] L. Bystritzkii, F. Legar, and I. Ulegla, Phys. Letters **20**, 186 (1966).

The solution of the elastic-scattering problem will be called

$$\chi_d(x_1)F(x_2). \tag{17}$$

It has the asymptotic form $(y\to\infty)$

$$\chi_d(x_1)F(x_2)\to e^{-i\delta}\chi_d(x_1)[\sin(\kappa x_2+\delta)]/x_2 \tag{18}$$

$$\kappa=(\tfrac{3}{4})^{1/2}k_3=[(m/\hbar^2)(E+E_d)]^{1/2}.$$

E_d is the binding energy of the deuteron. The expression (18) is approximately equal to

$$e^{-i\delta}\chi_d(x_1)[\sin(\kappa y+\delta)]/y, \tag{19}$$

because $\chi_d(x_1)$ is different from zero only for $\alpha\approx\beta\approx\pi/2$, and thus

$$x_2=y[1+\sin\alpha\,\sin\beta]^{1/2}/2^{1/2}\approx y. \tag{20}$$

There will be a second term in F_1 due to the inelastic scattering:

$$F_1=\chi_d(x_1)\{F(x_2)+\Phi(y)\}. \tag{21}$$

$\Phi(y)$ will have outgoing waves only, and will occur together with the functions $h_{\lambda\mu}$ as a consequence of the second term in Eq. (16). Instead of Eq. (21) one could start with another function, which we will call $\bar{F}_1$ and which makes use of the fact expressed by Eq. (20). Equations (4), (5), and (21) are replaced by

$$F_0{}^0=\bar{F}_1+\bar{F}_2, \tag{22}$$

$$\bar{F}_1=\chi_d(x_1)\Phi(y), \tag{23}$$

$$\bar{F}_2=\sum_{\lambda,\mu}\bar{h}_{\lambda\mu}(y)e^{i\mu\beta}f_{\lambda\mu}(\alpha). \tag{24}$$

$\bar{\Phi}(y)$ will have an ingoing and an outgoing wave, the amplitude of the ingoing wave being equal asymptotically to the amplitude of the ingoing wave in Eq. (18). The asymptotic Eqs. (6) and (7) are unchanged for the functions $\bar{h}_{\lambda\mu}$. The respective advantages of the functions F_1 and $\bar{F}_1$ will be discussed later.

IV. DERIVATION OF THE COUPLED SYSTEM OF DIFFERENTIAL EQUATIONS

So far no approximations have been made. The hypothesis which makes a solution of the problem possible is that only a few terms of the sums in $h_{\lambda\mu}$ or $\bar{h}_{\lambda\mu}$, Eqs. (5) and (24), respectively, are of importance. It is the centrifugal barrier in Eq. (6) which justifies this approximation. Only low values of λ are of importance. (An example will be given later.) The maximum value of λ will be called λ_m:

$$F_2=\sum_{\lambda=0}^{\lambda_m}\sum_{\mu}h_{\lambda\mu}(y)e^{i\mu\beta}f_{\lambda\mu}(\alpha). \tag{25}$$

We have a corresponding equation for $\bar{F}_2$. The system of coupled differential equations for the functions $h_{\lambda\mu}(\lambda\leq\lambda_m)$ and $\Phi(y)$ could be derived now. But it has turned out that the functions $h_{\lambda\mu}$ (or $\bar{h}_{\lambda\mu}$) are not the most practical ones, and that the problem can be reduced further by defining new functions $H_{\lambda\mu}$ and $\bar{H}_{\lambda\mu}$ instead of $h_{\lambda\mu}$ and $\bar{h}_{\lambda\mu}$, respectively. [In the following, $e^{i\lambda\beta}f_{\lambda\mu}(\alpha)$ will be abbreviated as $g_{\lambda\mu}(\alpha,\beta)$; the approximate functions for $F_0{}^0$ will be called $F_{0a}{}^0$ and $\bar{F}_{0a}{}^0$.]

$$\begin{aligned}F_{0a}{}^0&=\chi_d(x_1)F(x_2)+\chi_d(x_1)\Phi(y)+\sum_{\lambda=0}^{\lambda_m}\sum_{\mu}h_{\lambda\mu}(y)g_{\lambda\mu}(\alpha,\beta)\\&=\chi_d(x_1)F(x_2)+\Phi(y)\left[\chi_d(x_1)-\sum_{\lambda=0}^{\lambda_m}\sum_{\mu}g_{\lambda\mu}(\alpha,\beta)\int d\tau_{\alpha'\beta'}g_{\lambda\mu}{}^*(\alpha',\beta')\chi_d(x_1')\right]\\&\qquad+\sum_{\lambda=0}^{\lambda_m}\sum_{\mu}g_{\lambda\mu}(\alpha,\beta)\left[h_{\lambda\mu}(y)+\Phi(y)\int d\tau_{\alpha'\beta'}g_{\lambda\mu}{}^*(\alpha'\beta')\chi_d(x_1')\right]\\&=\chi_d(x_1)F(x_2)+\Phi(y)\left[\chi_d(x_1)-\sum_{\lambda=0}^{\lambda_m}\sum_{\mu}g_{\lambda\mu}(\alpha,\beta)\int d\tau_{\alpha'\beta'}g_{\lambda\mu}{}^*(\alpha'\beta')\chi_d(x_1')\right]+\sum_{\lambda=0}^{\lambda_m}\sum_{\mu}g_{\lambda\mu}(\alpha,\beta)H_{\lambda\mu}(y).\end{aligned} \tag{26}$$

$$d\tau_{\alpha'\beta'}=\sin\alpha'\,\cos\alpha'd\alpha'd\beta', \tag{27}$$

$$x_1'=y[1-\sin\alpha'\,\sin\beta']^{1/2}/2^{1/2}. \tag{28}$$

The range of the variables is

$$0\leq\alpha'\leq\pi/2,\quad 0\leq\beta\leq2\pi.$$

Functions $\bar{H}_{\lambda\mu}(y)$ are defined correspondingly. The asymptotic expressions for $H_{\lambda\mu}$ and $h_{\lambda\mu}$ are identical. The advantage of this new form of $F_{0a}{}^0$ is the orthogonality of the functions multiplying $\Phi(y)$ and $g_{\lambda\mu}(\alpha,\beta)$ in α, β space. Thus the functions $\Phi(y)$ and $H_{\lambda\mu}(y)$ are coupled over the potentials only, while for $\Phi(y)$ and $h_{\lambda\mu}(y)$ one would have coupling terms in the differential equations which do not contain any potential.

The following abbreviations will be used:

$$\begin{aligned}F_{0a}{}^0&=G_1+G_2,\\\bar{F}_{0a}{}^0&=\bar{G}_1+\bar{G}_2,\end{aligned} \tag{29}$$

$$G_1=\chi_d(x_1)F(x_2)+\Phi(y)\bar{\chi}_d(y,\alpha,\beta), \tag{30}$$

$$\bar{\chi}_d(y,\alpha,\beta)=\chi_d(x_1)-\sum_{\lambda=0}^{\lambda_m}\sum_{\mu} g_{\lambda\mu}(\alpha,\beta)\times\int d\tau_{\alpha'\beta'}g_{\lambda\mu}^*(\alpha'\beta')\chi_d(x_1'), \quad (31)$$

$$G_2=\sum_{\lambda=0}^{\lambda_m}\sum_{\lambda} g_{\lambda\mu}(\alpha,\beta)H_{\lambda\mu}(y), \quad (32)$$

$$\bar{G}_1=\bar{\Phi}(y)\bar{\chi}_d(y,\alpha,\beta), \quad (33)$$

$$\bar{G}_2=\sum_{\lambda=0}^{\lambda_m}\sum_{\mu} g_{\lambda\mu}(\alpha,\beta)\bar{H}_{\lambda\mu}(y). \quad (34)$$

The coupled differential equations for the functions Φ and $H_{\lambda\mu}$ (or $\bar{\Phi}$ and $\bar{H}_{\lambda\mu}$) are obtained the following way: 1. Multiplication of the Schrödinger equation by $\bar{\chi}_d^*(y,\alpha,\beta)$ and integration over α and β. 2. Multiplication of the Schrödinger equation by $g_{\lambda\mu}^*(\alpha,\beta)$ and integration over α and β (for all values of λ and μ with $\lambda\leq\lambda_m$). The result is

$$\frac{\hbar^2}{m}\left[\frac{\partial^2\Phi}{\partial y^2}+\left(\frac{5}{y}+2D(y)\right)\frac{\partial\Phi}{\partial y}\right]+[E+E_d-V_3(y)]\Phi=\sum_{\lambda=0}^{\lambda_m}\sum_{\mu}V_{\lambda\mu}(y)H_{\lambda\mu}+V_E(y), \quad (35)$$

$$\frac{\hbar^2}{m}\left[\frac{\partial^2 H_{\lambda\mu}}{\partial y^2}+\frac{5}{y}\frac{\partial H_{\lambda\mu}}{\partial y}-\frac{4\lambda(\lambda+2)}{y^2}H_{\lambda\mu}\right]+EH_{\lambda\mu}=\sum_{\lambda'=0}^{\lambda_m}\sum_{\mu'}V_{\lambda\mu\lambda'\mu'}(y)H_{\lambda'\mu'}+V_{1\lambda\mu}(y)\Phi+V_{\lambda\mu E}(y). \quad (36)$$

The following definitions are used in Eqs. (35) and (36):

$$D(y)=\int d\tau_{\alpha'\beta'}\bar{\chi}_d^*\frac{\partial\bar{\chi}_d}{\partial y}\bigg/\int d\tau_{\alpha'\beta'}\bar{\chi}_d^*\bar{\chi}_d, \quad (37)$$

$$V_3(y)=\int d\tau_{\alpha'\beta'}\bar{\chi}_d^*[V\bar{\chi}_d-V_{12}\chi_d]\bigg/\int d\tau_{\alpha'\beta'}\bar{\chi}_d^*\bar{\chi}_d, \quad (38)$$

$$V_{\lambda\mu}(y)=\int d\tau_{\alpha'\beta'}\bar{\chi}_d^*Vg_{\lambda\mu}\bigg/\int d\tau_{\alpha'\beta'}\bar{\chi}_d^*\bar{\chi}_d, \quad (39)$$

$$V_E(y)=\int d\tau_{\alpha'\beta'}\bar{\chi}_d^*\chi_dF\Delta V\bigg/\int d\tau_{\alpha'\beta'}\bar{\chi}_d^*\bar{\chi}_d. \quad (40)$$

ΔV is the second bracket in Eq. (16) written as a function of y, α', β'.

$$V_{\lambda\mu\lambda'\mu'}=\int d\tau_{\alpha'\beta'}g_{\lambda\mu}^*g_{\lambda'\mu'}V, \quad (41)$$

$$V_{1\lambda\mu}=\int d\tau_{\alpha'\beta'}g_{\lambda\mu}^*V\bar{\chi}_d, \quad (42)$$

$$V_{\lambda\mu E}=\int d\tau_{\alpha'\beta'}g_{\lambda\mu}^*\Delta V\chi_dF. \quad (43)$$

The integrands in Eqs. (37) to (43) are functions of y, α', β'. The equations for $\bar{\Phi}$ and $\bar{H}_{\lambda\mu}$ do not contain the terms $V_E(y)$ and $V_{\lambda\mu E}(y)$; otherwise they are identical to Eqs. (35) and (36).

V. DISCUSSION OF THE PROPERTIES OF THE DIFFERENTIAL EQUATIONS

In this section the asymptotic behavior of Eqs. (35) and (36) will be discussed. Afterwards the expressions which express the conservation of current will be derived.

(a) Asymptotic behavior of the differential equations: In the asymptotic region ($y\rightarrow\infty$) Eqs. (35) and (36) decouple, as will be shown now, $\Phi(y)$ describing the relative motion of the nucleon and the deuteron. y is proportional asymptotically to the nucleon-deuteron distance [Eq. (20)]. The coefficient $D(y)$ is equal asymptotically to $-3/2y$. To show this the integrands in Eq. (37) are transformed to new coordinates

$$\begin{aligned}\sin\alpha'\sin\beta'&=\cos\theta',\\ \sin\alpha'\cos\beta'&=\sin\theta'\sin\phi',\\ \cos\alpha'&=\sin\theta'\cos\phi'.\end{aligned} \quad (44)$$

The second term in $\bar{\chi}_d$ [Eq. (31)] will not give any contribution to the asymptotic form of $D(y)$. This can be shown by the same method which is applied below to the other terms;

$$\begin{aligned}\int d\tau_{\alpha'\beta'}\chi_d^*\chi_d&=\int_{-\pi/2}^{+\pi/2}d\phi'\cos\phi'\int_0^{\pi}d\theta'\sin^2\theta'\chi_d^*(y\sin\tfrac{1}{2}\theta')\chi_d(y\sin\tfrac{1}{2}\theta')\\ &=16\int_0^{\pi/2}d\theta\sin^2\theta\cos^2\theta\,\chi_d^*(y\sin\theta)\,1\,\chi_d(y\sin\theta)\\ &=(16/y^3)\int_0^{y}du\,u^2(1-u^2/y^2)^{1/2}\chi_d^*(u)\chi_d(u).\\ &\underset{y\rightarrow\infty}{\simeq}16/y^3.\end{aligned} \quad (45)$$

Similarly one finds

$$\int d\tau_{\alpha'\beta'}\chi_d^*\partial\chi_d/\partial y \sim -24/y^4. \quad (46)$$

Thus

$$D(y)\sim -3/2y. \quad (47)$$

The asymptotic behavior of the other coefficients in Eqs. (35) can be derived also with the transformation (44). One obtains (only nuclear forces are considered, no Coulomb forces)

$$\begin{aligned} V_3(y)&\sim v_3/y^3, \\ V_{\lambda\mu}(y)&\sim v_{\lambda\mu}, \\ V_{\lambda\mu\lambda'\mu'}(y)&\sim v_{\lambda\mu\lambda'\mu'}/y^3, \\ V_{1\lambda\mu}(y)&\sim v_{1\lambda\mu}/y^3. \end{aligned} \quad (48)$$

$V_E(y)$ and $V_{\lambda\mu E}(y)$ decrease exponentially for large y. The coefficients v_3, etc., in Eq. (48) are constants. The asymptotic forms of Eqs. (35) and (36) can be written down now;

$$\frac{\hbar^2}{m}\left(\frac{\partial^2\Phi}{\partial y^2}+\frac{2}{y}\frac{\partial\Phi}{\partial y}\right)+(E+E_d)\Phi=\sum_{\lambda=0}^{\lambda_m}\sum_{\mu} v_{\lambda\mu}H_{\lambda\mu}(y),$$

$$\frac{\hbar^2}{m}\left(\frac{\partial^2 H_{\lambda\mu}}{\partial y^2}+\frac{5}{y}\frac{\partial H_{\lambda\mu}}{\partial y}-\frac{4\lambda(\lambda+2)}{y^2}H_{\lambda\mu}\right)+EH_{\lambda\mu}=\frac{v_{1\lambda\mu}}{y^3}\cdot\Phi. \quad (49)$$

From Eqs. (49) it can be derived easily that the functions $H_{\lambda\mu}$ show the asymptotic behavior of Eq. (7) and

$$\Phi(y)\sim Be^{-i\kappa y}/2iy+Ae^{i\kappa y}/2iy. \quad (50)$$

The solutions which one is looking for in the case of nucleon-deuteron scattering will have $B=0$ and $B_{\lambda\mu}=0$; or $B=1$, $B_{\lambda\mu}=0$ when using $\bar{\Phi}$ and $\bar{H}_{\lambda\mu}$ instead of Φ and $H_{\lambda\mu}$.

(b) Conservation of current is expressed by the equation

$$y^5\int_0^{2\pi}d\beta\int_0^{\pi/2}d\alpha \sin\alpha\cos\alpha \times(F_0^{0*}\partial F_0^0/\partial y-F_0^0\partial F_0^{0*}/\partial y)=0. \quad (51)$$

It can be derived from Eq. (3). Asymptotically Eq. (51) connects the coefficients $C_{\lambda\mu}$ of Eq. (7) ($B_{\lambda\mu}=0$) and A of Eq. (50). One obtains

$$\frac{k}{4\kappa}\sum_{\lambda=0}^{\infty}\sum_{\mu}|C_{\lambda\mu}|^2+|A-e^{-2i\delta}|^2=1 \quad (52)$$

when using Φ and $H_{\lambda\mu}$, and

$$\frac{k}{4\kappa}\sum_{\lambda=0}^{\infty}\sum_{\mu}|C_{\lambda\mu}|^2+|A|^2=1 \quad (53)$$

when using $\bar{\Phi}$ and $\bar{H}_{\lambda\mu}$.

From Eqs. (35) and (36) one obtains the same Eqs. (52) and (53) for the approximate solutions of the problem. The infinite sum over λ is then replaced by the corresponding finite sum. This way one can check the accuracy of numerical solutions of Eqs. (35) and (36). But it cannot be checked from Eqs. (52) and (53) whether or not one has to take into account higher values of λ.

VI. METHODS FOR SOLVING THE DIFFERENTIAL EQUATIONS

Two methods for solving the differential equations will be discussed shortly in this section without going into mathematical details.

A. Numerical Solution

One could solve the problem by integrating Eqs. (35) and (36) numerically starting at small y. After constructing the maximum number of linearly independent solutions, one has to combine these to get only outgoing waves for $H_{\lambda\mu}$ or $\bar{H}_{\lambda\mu}$.

B. Iteration Procedure

The case of the functions $\Phi(y)$, $H_{\lambda\mu}(y)$ will be discussed first. The first step of the iteration procedure is taken by putting

$$\Phi=H_{\lambda\mu}=0 \quad (54)$$

on the right-hand sides of Eqs. (35) and (36) and solving with the appropriate Green's functions, giving solutions Φ_1, $H_{1\lambda\mu}$. In the second step Φ and $H_{\lambda\mu}$ on the right-hand sides are replaced by Φ_1 and $H_{1\lambda\mu}$. One finds solutions Φ_2, $H_{2\lambda\mu}$, and so on. This procedure does not converge for proton-deuteron scattering, as a consequence of the Coulomb potential. For this case it will be useful to introduce a cutoff radius y_c, outside of which one sets all coupling terms equal to zero. y_c should be large compared to the size of the deuteron. That part of $V_{\lambda\mu\lambda\mu}(y)$ originating from the Coulomb interaction has to be transferred to the left-hand side of Eq. (36) and included in the Green's function. There is a big disadvantage of this procedure using the functions Φ and $H_{\lambda\mu}$: The quantities $V_E(y)$, $V_{\lambda\mu E}(y)$ (as well as the Green's functions) of Eq. (35) are energy-dependent. The functions $V_E(y)$ and $V_{\lambda\mu E}(y)$ are double integrals over the α, β space. Computing them for every energy will take quite a bit of machine time. Therefore it will be better to use the functions $\bar{\Phi}$, $\bar{H}_{\lambda\mu}$ instead. Moreover, part of the left-hand side of Eq. (35) is transferred to the right-hand side, to get a simple Green's function:

$$\frac{\partial^2\bar{\Phi}}{\partial y^2}+\frac{2}{y}\frac{\partial\bar{\Phi}}{\partial y}+\kappa^2\bar{\Phi}=-\left[\frac{3}{y}+2D(y)\right]\frac{\partial\bar{\Phi}}{\partial y}+\frac{m}{\hbar^2}V_3(y)\bar{\Phi}+\frac{m}{\hbar^2}\sum_{\lambda=0}^{\lambda_m}\sum_{\mu} V_{\lambda\mu}(y)\bar{H}_{\lambda\mu}, \quad (55)$$

$$\frac{\partial^2\bar{H}_{\lambda\mu}}{\partial y^2}+\frac{5}{y}\frac{\partial\bar{H}_{\lambda\mu}}{\partial y}-\frac{4\lambda(\lambda+2)}{y^2}\bar{H}_{\lambda\mu}+k^2\bar{H}_{\lambda\mu}=\frac{m}{\hbar^2}\sum_{\lambda'=0}^{\lambda_m}\sum_{\mu}V_{\lambda\mu\lambda'\mu'}(y)\bar{H}_{\lambda'\mu'}+V_{1\lambda\mu}(y)\bar{\Phi}. \quad (56)$$

The right-hand sides of Eqs. (55) and (56) are treated as the inhomogeneous parts in the iteration procedure. The iteration procedure is started by replacing on the right-hand sides of Eqs. (55) and (56)

$$\bar{\Phi}=\bar{\Phi}_0,\quad \bar{H}_{\lambda\mu}=0, \quad (57)$$

where $\bar{\Phi}_0$ is a solution of the equation

$$\frac{\hbar^2}{m}\left(\frac{\partial^2\bar{\Phi}_0}{\partial y^2}+\frac{2}{y}\frac{\partial\bar{\Phi}_0}{\partial y}+\kappa^2\bar{\Phi}_0\right)=0, \quad (58)$$

corresponding to a plane wave and yielding the Born approximation in the first step of the iteration procedure. The quantities $D(y)$, $V_3(y)$, $V_{\lambda\mu}(y)$, $V_{\lambda\mu\lambda'\mu'}(y)$, and $V_{1\lambda\mu}(y)$ do not depend on the energy. Thus one has to compute them only once and can use them for all energies for which one would like to have the cross sections. The Green's functions for Eqs. (55) and (56) have the following form:

$$G_\Phi(y,y')=(\pi/2y^{1/2})H_{1/2}(\kappa y_>)y'^{3/2}J_{1/2}(\kappa y_<), \quad (59)$$

$$G_{\lambda\mu}(y,y')=(\pi/2y^2)H_{2\lambda+2}(ky_>)y'^3J_{2\lambda+2}(ky_<); \quad (60)$$

$$y_<=y',\quad y_>=y,\quad \text{for}\quad y>y';$$
$$y_<=y,\quad y_>=y',\quad \text{for}\quad y<y';$$
$$H_n=J_n+iY_n.$$

J_n and Y_n are Bessel functions of first and second kind.[14] Eqs. (55) and (56) are now

$$\bar{\Phi}=\int_0^\infty dy' G_\Phi(y,y')I_\Phi(y')+\bar{\Phi}_0 \quad (61)$$

$$\bar{H}_{\lambda\mu}=\int_0^\infty dy' G_{\lambda\mu}(y,y')I_{\lambda\mu}(y'). \quad (62)$$

Here the right-hand sides of (55) and (56) have been abbreviated as I_Φ and $I_{\lambda\mu}$, respectively. The Green's functions (57) and (58) depend only on κy and ky, and can thus be used for all energies. Things will be more complicated again for proton-deuteron scattering, where one has to include the Coulomb potential. This is done by transferring the Coulomb potentials included in $V_3(y)$ and $V_{\lambda\mu\lambda\mu}(y)$ to the left-hand sides and including them in G_Φ and $G_{\lambda\mu}$. The coupling terms will be set equal to zero for $y>y_c$.

The functions Φ and $H_{\lambda\mu}$ will in general be more accurate than the functions $\bar{\Phi}$ and $\bar{H}_{\lambda\mu}$ (after the same number of iteration steps). However, when computing $\bar{\Phi}$ and $\bar{H}_{\lambda\mu}$ one saves the computation of the energy-dependent functions $V_E(y)$ and $V_{\lambda\mu E}(y)$, and thus can do a few more steps of the iteration procedure to get better accuracy. And one will need less machine time for this than for computing the double integrals of $V_E(y)$ and $V_{\lambda\mu E}(y)$. To improve $\bar{\Phi}$ and $\bar{H}_{\lambda\mu}$ one can also start with the elastic-scattering solution for Φ_0 and use the corresponding Green's function.

[14] Erdélyi, Magnus, Oberhettinger, and Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, New York, 1953), Vol. 2.

VII. CONTRIBUTIONS OF TERMS WITH HIGH VALUES OF λ

The contributions of functions with high values of λ to the cross section can be ignored. This is partly a consequence of the "centrifugal" barrier [Eq. (6)]. An estimate of this decrease with increasing λ will be given in this section. We consider the first step of the iteration procedure, Eq. (62). The asymptotic expression for $\bar{H}_{\lambda\mu}$ is

$$\bar{H}_{\lambda\mu}\to[m/\hbar^2(2\pi k)^{1/2}y^{5/2}]e^{iky}e^{-i\pi(\lambda+5/4)}\int_0^\infty dy'y'^3\int_0^\pi d\zeta \times\cos[ky'\sin\zeta-(2\lambda+2)\zeta]V_{1\lambda\mu}(y')\bar{\Phi}_0(y'). \quad (63)$$

The effect of the "centrifugal" term is now contained in the cosine function in the integrand. For high values of λ this cosine function is rapidly oscillating and causes strong cancellations in the integral. For the same reason the function $V_{1\lambda\mu}$ defined in Eq. (42) will decrease with increasing λ, because the functions $g_{\lambda\mu}$ are rapidly oscillating in (α,β) space. To show this in more detail, the special case $\mu=\lambda$ is considered now. The normalized function $g_{\lambda\lambda}$ is

$$g_{\lambda\lambda}=[(\lambda+1)/\pi]^{1/2}\sin^\lambda\alpha e^{i\lambda\beta}. \quad (64)$$

For $V\bar{\chi}_d$ in $V_{1\lambda\lambda}$ the abbreviation V_d is used:

$$V_{1\lambda\lambda}=[(\lambda+1)/\pi]^{1/2}\int_0^{2\pi}d\beta\int_0^{\pi/2}d\alpha \times\sin^{\lambda+1}\alpha\cos\alpha e^{i\lambda\beta}V_d(y,\alpha,\beta). \quad (65)$$

The real part of $V_{1\lambda\lambda}$ is considered now (the proof for the imaginary part is quite analogous):

$$\mathrm{Re}(V_{1\lambda\lambda})=[(\lambda+1)/\pi]^{1/2}\int_0^{2\pi}d\beta\int_0^{\pi/2}d\alpha \times\sin^{\lambda+1}\alpha\cos\alpha\cos\lambda\beta V_d(y,\alpha,\beta). \quad (66)$$

The integration over β is broken up into small intervals now;

$$n\pi/\lambda\le\beta\le(n+1)\pi/\lambda,\quad 0\le n\le 2\lambda-1. \quad (67)$$

If λ is sufficiently large, V_d can be replaced approximately by a linear function in such a small interval:

$$V_d(y,\alpha,\beta)\approx V_d(y,\alpha,n\pi/\lambda)+(\lambda/\pi)\{V_d(y,\alpha,(n+1)\pi/\lambda)-V_d(y,\alpha,n\pi/\lambda)\}\{\beta-n\pi/\lambda\}. \quad (68)$$

The integration over β will give then

$$\int_0^{2\pi} d\beta \cos\lambda\beta V_d(y,\alpha,\beta) \approx (2/\pi\lambda) \sum_{n=0}^{2\lambda-1} [V_d(y,\alpha,(n+1)\pi/\lambda) - V_d(y,\alpha,n\pi/\lambda)](-)^{n+1}$$

$$\approx (2/\lambda^2) \sum_{n=0}^{2\lambda-1} (-)^{n+1} \left[\frac{\partial}{\partial\beta} V_d(y,\alpha,\beta)\right]_{\beta=n\pi/\lambda}$$

$$= (2/\lambda^2) \sum_{n=0}^{\lambda-1} \left\{ \left[\frac{\partial}{\partial\beta} V_d(y,\alpha,\beta)\right]_{\beta=(2n+1)\pi/\lambda} - \left[\frac{\partial}{\partial\beta} V_d(y,\alpha,\beta)\right]_{\beta=2n\pi/\lambda} \right\}$$

$$\approx (2\pi/\lambda^3) \sum_{n=0}^{\lambda-1} \left[\frac{\partial^2}{\partial\beta^2} V_d(y,\alpha,\beta)\right]_{2n\pi/\lambda}$$

$$\approx (1/\lambda^2) \int_0^{2\pi} d\beta \left[\frac{\partial^3}{\partial\beta^3} V_d(y,\alpha,\beta)\right]. \tag{69}$$

With respect to α, the integrand in Eq. (66) will be different from zero only in a narrow neighborhood of $\alpha = \pi/2$. The quantity $V_d(y,\alpha,\beta)$ can be assumed to be a smooth function of α. Hence we replace in the integrand

$$V_d(y,\alpha,\beta) \rightarrow V_d(y,\pi/2,\beta). \tag{70}$$

So one has as the approximate result for (66):

$$\mathrm{Re}(V_{1\lambda\lambda}) \approx [(\lambda+1)/\lambda]^{1/2} [1/\lambda^2(\lambda+2)] \int_0^{2\pi} d\beta \left[\frac{\partial^3}{\partial\beta^3} V_d(y,\alpha,\beta)\right]$$

$$\approx (1/\lambda^{5/2}\pi^{1/2}) \int_0^{2\pi} d\beta \left[\frac{\partial^3}{\partial\beta^3} V_d(y,\alpha,\beta)\right]. \tag{71}$$

Similarly the integration over y' in Eq. (63) can be approximated, and an additional decrease of $\bar{H}_{\lambda\mu}$ with $1/\lambda^2$ is found. So altogether one has for large λ

$$\bar{H}_{\lambda\lambda} \sim C e^{iky} / y^{5/2} \lambda^{9/2}. \tag{72}$$

Here C is everything in Eq. (63) which is not λ-dependent. The arguments for functions with $\mu \neq \lambda$ will be the same, as all these functions are rapidly oscillating. Assuming that C does not depend on μ, one finds that the contribution of functions with high λ to the cross section is proportional to

$$[C \sum_{\lambda \text{ large}} (1/\lambda^{7/2})]^2. \tag{73}$$

$\bar{H}_{\lambda\lambda}$ [Eq. (72)] was multiplied by λ because the number of functions for a given λ is proportional to λ. Thus one can conclude that functions with high values of λ contribute very little to the cross section. Nothing has been said so far about the values of λ which can be considered as large. For high energies there are other parts of the integrand in Eq. (63) which are rapidly oscillating. Thus the estimate of this section will be correct only for $\lambda \geq \lambda_{\min}$, where $\lambda_{\min}$ depends on the energy and will increase with the energy. Fortunately $\lambda_{\min}$ will have rather low values in general. Let us consider nucleon-deuteron scattering again to see this. Only energies below the threshold for meson production are considered, $E < 150$ MeV. Even at the upper end of this range the nucleon wavelength is comparable to the range of the potential. Thus the functions $\sin(ky' \sin\Phi)$, $\cos(ky' \sin\Phi)$, and $\bar{\Phi}_0$ in Eq. (63) are rather smooth and do not show rapid oscillations, which will occur only at higher energies. In the following section a simple example will be considered, which shows indeed that only functions with low values of λ will contribute to the cross section.

The arguments of this section can be applied to all matrix elements, which occur in the second and consecutive steps of the iteration procedure.

VIII. SIMPLE EXAMPLE

Equation (63) is evaluated in this section for the two limiting cases of large and small k and for a simple example concerning the interaction potential and the deuteron wave function. The interaction is

$$V = a(e^{-\Gamma r_{12}^2} + e^{-\Gamma r_{23}^2} + e^{-\Gamma r_{31}^2}). \tag{74}$$

The deuteron wave function is determined approximately by a Ritz variational procedure;

$$\chi_d(r_{12}) = N e^{-\gamma r_{12}^2}. \tag{75}$$

γ is a parameter that is a function of a and Γ, and

N is the normalization constant. a and Γ in the potential are chosen to give the correct binding energy and (for mathematical simplicity) $\Gamma=\gamma$. In addition, the following approximations were made to facilitate the integrations:

$$\bar{\Phi}_0(y)=(\sin\kappa y)/y, \quad (76)$$

$$J_{2\lambda+2}=(-)^{\lambda+1}(2/\pi y)^{1/2}\cos(ky-\pi/4) \quad \text{for } E\gg E_d, \quad (77)$$

$$J_{2\lambda+2}=(ky)^{2\lambda+2}/(2\lambda+2)!2^{2\lambda+2} \quad \text{for } E\ll E_d. \quad (78)$$

The quantities of interest are the relative magnitudes of the coefficients $C_{\lambda\mu}$. In the high-energy limit the following results were obtained:

$$\begin{aligned}
E\gg E_d \quad |C_{1,\pm1}/C_{00}| &= 7\times10^{-2}\\
|C_{2,0}/C_{00}| &= 10^{-1}\\
|C_{2,\pm2}/C_{00}| &= 1.8\times10^{-1}\\
|C_{3,\pm1}/C_{00}| &= 2.2\times10^{-3}\\
|C_{3,\pm3}/C_{00}| &= 5\times10^{-2}\\
|C_{40}/C_{00}| &= 6\times10^{-4}\\
|C_{4,\pm2}/C_{00}| &= 7\times10^{-4}\\
|C_{4,\pm4}/C_{00}| &= 7\times10^{-4}\\
|C_{5,\pm1}/C_{00}| &= 1.8\times10^{-3}\\
|C_{5,\pm3}/C_{00}| &= 1.8\times10^{-3}\\
|C_{5,\pm5}/C_{00}| &= 1.8\times10^{-3}\\
|C_{60}/C_{00}| &= 4\times10^{-4}\\
|C_{6,\pm2}/C_{00}| &= 3\times10^{-4}\\
|C_{6,\pm4}/C_{00}| &= 3\times10^{-4}\\
|C_{6,\pm6}/C_{00}| &= 4\times10^{-4}.
\end{aligned}$$

In the low-energy limit it was found that

$$\begin{aligned}
E\ll E_d \quad |C_{1,\pm1}/C_{00}| &= 3\times10^{-2}E/E_d\\
|C_{20}/C_{00}| &= 7\times10^{-3}(E/E_d)^2\\
|C_{2,\pm2}/C_{00}| &= 10^{-2}(E/E_d)^2.
\end{aligned}$$

For higher values of λ, the factor E/E_d occurs with a higher power.

IX. THE LIMIT OF ELASTIC SCATTERING

The limit of elastic scattering was $\Phi=H_{\lambda\mu}=0$. This limit must be contained in the function $\bar{\Phi}$ too. It is obtained by treating the deuteron as a point particle. This is formally done by letting γ go to infinity in Eq. (75). It can easily be shown that the coupling terms in Eqs. (35) and (36) vanish in this case, and that the exact equation for $\bar{\Phi}$ has the well-known form

$$\frac{\hbar^2}{m}\left(\frac{\partial^2\bar{\Phi}}{\partial y^2}+\frac{2}{y}\frac{\partial\bar{\Phi}}{\partial y}\right) +[E+E_d-V_{23}((\tfrac{3}{4})^{1/2}y)-V_{31}((\tfrac{3}{4})^{1/2}y)]\bar{\Phi}=0. \quad (79)$$

X. CONCLUDING REMARKS

For actual problems one has to use properly symmetrized or antisymmetrized wave functions. This is easily done because the coordinates y, α, and β have very simple symmetry properties.[6] The inclusion of higher orbital angular momenta is no problem either. In this case one has to transform eigenfunctions of the orbital angular momentum $\Psi_{LM}(\theta_1,\phi_1,\theta_2,\phi_2)$, where θ_1, ϕ_1, θ_2, ϕ_2 are polar angles of $\mathbf{x}_1$ and $\mathbf{x}_2$, to the new coordinates. The formulas for this transformation are given in Ref. 6, pp. 22 and 26.

For nuclear reactions in which one has three outgoing particles, one can always use the complete orthogonal system of Ref. 6 to describe the asymptotic behavior of these three particles. By the same arguments which were given in this paper, one can expect that only functions with low values of λ are of importance. It was possible to show this for the decay of C^{12} into three α particles, for which Dalitz diagrams were measured at two different energies by Dehnhard *et al.*[15] The results have been published recently.[16] This decay of C^{12} is the first example for which the hypothesis of this paper could be confirmed. We may hope that there will be more Dalitz diagrams for three-particle reactions soon.

[15] D. Dehnhard, D. Kamke, and P. Kramer, Ann. Phys. (N. Y.) **14**, 201 (1964).
[16] W. Zickendraht, Z. Physik **200**, 194 (1962).