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Four-Dimensional Symmetry

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We study the symmetry of scattering amplitudes at vanishing momentum transfer. We show that the little group of the Poincaré group corresponding to vanishing four-momentum (isomorphic to the homogeneous Lorentz group, or, by analytic continuation, to the four-dimensional rotation group) is in general not a symmetry of the scattering amplitude. However, the spectrum of the amplitude is classified according to the larger symmetry. Regge poles occur in families; the members of the family follow the first one in integer steps at vanishing momentum transfer and are classified according to a new quantum number derived from the higher symmetry. We derive a one-parameter "mass formula" describing the deviation of the slopes of Regge trajectories from the value required by the higher symmetry. The theory is applied to the problem of the high-energy scattering of particles of arbitrary mass, and leads to an unambiguous asymptotic expression for the scattering amplitude. We analyze the implications of the new symmetry for the spectrum of hadrons. The predicted new Regge trajectories lead to observable particles and resonances; their coupling strength is determined by the symmetry in terms of the parameters of the "parent" trajectories. In particular we assign the $N'(1400)$ resonance to the second "daughter" of the N_α trajectory; the symmetry breaking turns out to be small, and the decay width of $N'(1400)$ is computed in satisfactory agreement with the experimental result.

1. INTRODUCTION

THE object of this paper is to study a new type of symmetry of scattering amplitudes. It is well known that a scattering amplitude at fixed and nonzero energy-momentum four-vector E_μ is invariant (or covariant) under the little group of the Poincaré group belonging to E_μ. For any nonvanishing timelike E_μ, the little group is known to be (or to be isomorphic to) the three-dimensional rotation group. If, however, $E_\mu=0$, the corresponding little group is much larger and in fact is isomorphic to the homogeneous Lorentz group.[1] It is not immediately evident that this larger group is of any significance for physical scattering amplitudes. To be sure, if one expects some immediate physical consequences, one must put the energy-momentum four-vector of a *crossed* channel equal to zero, corresponding to zero transferred momentum. (The point where E_μ of the "direct" channel vanishes is not in the physical region if at least one of the scattered particles has a nonvanishing mass.) Even so, as a scattering amplitude on the mass shell depends on two invariants (e.g., the Mandelstam invariants s and t) only, putting $t=0$ does not necessarily mean that in the crossed channel the energy-momentum four-vector vanishes but only that it is lightlike.

Nevertheless, it is well known that the Bethe-Salpeter amplitude describing the scattering of equal-mass particles *does* possess a larger symmetry at vanishing E_μ, at least as long as one considers the Bethe-Salpeter equation in the generalized ladder approximation.[2]

Some time ago we pointed out[2] that such a higher symmetry leads to very interesting physical consequences. In particular we found that every singularity in the angular momentum plane "induces" a series of other singularities of the same nature (poles induce poles, branch points induce branch points, etc.); moreover, the "induced" singularities follow the "primary" one at unit steps. This situation can be described more easily in the language of the four-dimensional symmetry: It simply means that there is *one* singularity in the complex four-dimensional angular momentum variable, which, when decomposed according to the "ordinary" angular momentum, gives rise to the series described above.

Soon after the paper[2] appeared, several authors investigated the possible consequences of such a higher symmetry on scattering amplitudes.[3] The difference between the latter approaches and our own is that whereas the authors mentioned in Ref. 3 worked directly with the homogeneous Lorentz group ($\mathfrak{L}$), in Ref. 2 we considered a scattering amplitude in the Euclidean region of momenta, thereby converting the higher-symmetry group into SO_4. The difference between these approaches is, however, only technical in nature, as will become evident from the exposition of the present paper.

Quite recently Freedman and Wang,[4] investigating the asymptotic behavior of the scattering amplitude of particles of unequal masses, found that the usual analyticity requirements are consistent with a Regge-type asymptotic behavior if one assumes a series of Regge poles instead of just one, the "daughters" of the leading pole following the "parent" pole at unit intervals. Freedman and Wang suggested that this striking analogy with the situation described in Ref. 2 is not accidental and that in fact a Regge-type behavior

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[1] E. P. Wigner, Ann. Math. **40**, 149 (1939).
[2] G. Domokos and P. Surányi, Nucl. Phys. **54**, 529 (1964); G. Domokos, Acta Phys. Austriaca, Suppl. 1 (1964).
[3] M. Toller, University of Rome, Report No. 84, 1965 (unpublished); Ya. A. Smorodinskij, M. Uhlir, and P. Winternitz, Dubna Report No. E-1591, 1964 (unpublished). References to earlier papers are quoted in these works.
[4] D. Z. Freedman and J. M. Wang, Phys. Rev. Letters **17**, 569 (1966); Phys. Rev. **153**, 1596 (1967).

is a consequence of the four-dimensional symmetry of the amplitude. We shall show that the conjecture of Freedman and Wang is indeed correct: Their result is an immediate consequence of the four-dimensional kinematics.

The approach adopted in Refs. 2 and 4 indicates already that in order to establish the invariance of a scattering amplitude under SO_4 (or $\mathfrak{L}$, respectively), one must make extensive use of its analyticity properties. Therefore, it seems, this symmetry is most naturally described in the framework of an analytic S-matrix theory. (We do not see, e.g., any immediate possibility of deriving it from a Lagrangian approach: The symmetry in question is not a straightforward consequence of the symmetries of the Lagrangian and the canonical commutation relations.) Thus this symmetry occupies a rather unique position in the family of the symmetries of the scattering matrix.

The difficulties in investigating the invariance are of two kinds. First of all, roughly speaking, "there are not enough variables left" in an amplitude on the mass shell: It depends on, say, the c.m. energy and the scattering angle only (more precisely on the direction of a three-dimensional unit vector and the energy), whereas in order to obtain the four-dimensional symmetry one expects a dependence on a four-dimensional unit vector. This extra degree of freedom is furnished by an analytic continuation of the amplitude in the masses of the external particles. The other problem can be illustrated on a familiar example.

Suppose we have an electron moving under the influence of a homogeneous external electric field $\mathbf{E}$ (along, say, the z direction) and of a spherically symmetric potential (Stark effect). In the presence of the electric field the Hamiltonian, and consequently the scattering amplitude, has an axial symmetry only, due to the presence of the term $\sim\mathbf{d}\cdot\mathbf{E}$ in the Hamiltonian. (Here $\mathbf{d}$ is the induced dipole moment of the atom.) We want to discover the spherical symmetry of the system in the limit as $|\mathbf{E}|\rightarrow 0$. First of all, we have to require that $\mathbf{E}\rightarrow 0$ *along a fixed direction*. Then, still with $\mathbf{E}\neq 0$, we subject the atom to a *general* rotation. Obviously, if the axis of the rotation is not parallel to $\mathbf{E}$, the scattering amplitude will not be invariant with respect to this transformation.

Technically, one anticipates the larger symmetry even in the presence of the electric field and considers the matrix elements of the various operators between angular momentum eigenstates. As long as the field $\mathbf{E}$ is present, the rotational symmetry is "broken" and there are nonvanishing matrix elements between different values of the angular momentum l. One has to show then that as $\mathbf{E}\rightarrow 0$, the off-diagonal matrix elements of the observables tend to zero; in particular that the spectrum of the Hamiltonian goes over smoothly to the degenerate one (levels with the same l but different values of the magnetic quantum number m become degenerate).

For "well-behaved" potentials this is indeed the case, in consequence of Riesz's celebrated theorem on analytic perturbations.

Our treatment of the broken four-dimensional symmetry follows the same pattern. Introducing a convenient set of variables, we investigate the pertinent analytic properties of the scattering amplitude of spinless particles. The analytic properties we find are sufficient to guarantee that in the limit of vanishing total energy-momentum vector the matrix elements of the scattering operator become diagonal in the four-dimensional angular momentum. However, as a consequence of the singular behavior of the transformation formulas between the standard kinematical variables and those adapted to the four-dimensional symmetry, the full amplitude is not invariant (or, in a different language, the amplitude which *is* invariant does not have the correct continuation to the mass shell) with the exception of the equal-mass case.

In the face of this difficulty, we start "experimenting" with models. We find that in spite of the trouble with the full amplitude, the *spectrum* as determined from a Bethe-Salpeter equation shows the larger symmetry as expected. In particular, we rederive and generalize our earlier result[2] that Regge poles occur in families, which are classified according to representations of the four-dimensional rotation group (or, equivalently, the homogeneous Lorentz group).

Next we turn to investigate some properties of the families of Regge trajectories, neglecting for the time being the effects of symmetry breaking on the trajectories. To this end we construct an expansion of the scattering amplitude adapted to the broken symmetry; roughly speaking we expand the amplitude so as to exhibit the structure of the spectrum, but do not expand the residues of the poles; this kind of expansion turns out to be identical to the one used in the second paper of Ref. 2 and also in the papers of Ref. 3. We find that the residues of the poles of odd order (as defined in that section) vanish in the equal-mass limit. Furthermore, we show (neglecting the symmetry breaking) that poles of odd order cannot correspond to real particles although they seem to play an important role in the crossed channel.

Encouraged by the previous results, we continue the "theoretical experiment" with the Bethe-Salpeter equation and find that one can derive a remarkably simple "mass formula" for the Regge trajectories; the four-dimensional symmetry to lowest order is broken by a "dipole term," affecting the slope of the trajectories in a definite way. The lowest-order symmetry-breaking term contains one free parameter, which has the same value for the whole family of trajectories generated by one four-dimensional pole.

As a first application of the theory, we show that the higher symmetry resolves the difficulties connected with the asymptotic behavior of unequal-mass scattering amplitudes, and thus verify the conjecture of Freedman and Wang.[4] We then investigate the implications of the higher symmetry, on the particle spectrum; the "induced" or "daughter" trajectories of even order should give rise to observable particles in the timelike region, with coupling strengths predictable from the parameters of the "parent trajectory." In particular, we find that the $N'(1400)$ resonance can be consistently assigned to the first even daughter of the nucleon (N_α) trajectory with a reasonably small symmetry breaking. The predicted decay width is in fair agreement with the experimental result.

In the last section we summarize our results and discuss briefly some questions related to the dynamics of hadrons.

In the following section we start by briefly summarizing without proofs the mathematical apparatus necessary for our investigations. We do not claim to present essentially new results there, although many of the results presented are scattered in the literature. As far as we know, however, the explicit expression of the matrix elements of the hyperspherical functions [our Eq. (2.13)] has not been given previously. The reader familiar with the representation theory of the Lorentz and four-dimensional rotation groups may proceed directly to Sec. 3.

2. SUMMARY OF SOME MATHEMATICAL RESULTS

A. Representations; Clebsch-Gordan Series

The group SO_4 is compact, its unitary representations are finite-dimensional.[5] The six generators $M_{\mu\nu}$ can be grouped into two vectors:

$$M_1=M_{23}, \text{ (cyclic)},$$

and

$$N_i=M_{i4}.$$

It is customary to introduce the linear combinations

$${}^1\mathbf{V}=\tfrac{1}{2}(\mathbf{M}-\mathbf{N}),\quad {}^2\mathbf{V}=\tfrac{1}{2}(\mathbf{M}+\mathbf{N}),$$

that satisfy the independent SU_2 commutation relations:

$$[{}^rV_i,{}^{r'}V_k]=i\epsilon_{ikl}{}^rV_l\delta_{rr'}\quad (r=1,2).$$

Correspondingly the irreducible representations (I.R.) are classified according to the eigenvalues j_1, j_2 of ${}^1\mathbf{V}^2=j_1(j_1+1)\mathbf{1}$ and ${}^2\mathbf{V}^2=j_2(j_2+1)\mathbf{1}$ with $j_1, j_2=0, \frac{1}{2}, 1, \cdots$. The ${}^r\mathbf{V}^2$ are simply related to the Casimir operators $F=-M_{\mu\nu}M^{\mu\nu}$ and $G=\frac{1}{4}\epsilon^{\mu\nu\zeta\sigma}M_{\mu\nu}M_{\zeta\sigma}$; indeed,

$${}^1\mathbf{V}^2=-\tfrac{1}{2}(F+G),\quad {}^2\mathbf{V}^2=-\tfrac{1}{2}(F-G).$$

We introduce two other labels j_0, n so that acting on an I.R.

$$F=-\tfrac{1}{2}[n(n+2)+j_0^2]\mathbf{1},$$
$$G=(n+1)j_0\mathbf{1}.$$

Here $\mathbf{1}$ stands for the unit operator. In view of the previous equation, we have

$$j_1(j_1+1)=\tfrac{1}{4}[j_0^2+n(n+2)]-\tfrac{1}{2}(n+1)j_0,$$
$$j_2(j_2+1)=\tfrac{1}{4}[j_0^2+n(n+2)]+\tfrac{1}{2}(n+1)j_0,$$

which establishes the relation between the two ways of labeling the I.R.

The product of two I.R. reduces to a sum of I.R., each occurring with multiplicity one. The easiest way to analyze the contents of the product of two I.R. is by using the $SU_2\times SU_2$ labeling (j_1,j_2). We have, namely, in an obvious notation

$$(j_1',j_2')\otimes(j_1'',j_2'')=\sum\oplus(j_1,j_2),$$
$$|j_1'-j_1''|\leqq j_1\leqq j_1'+j_1'',$$
$$|j_2'-j_2''|\leqq j_2\leqq j_2'+j_2''.$$

We use a canonical basis for the I.R. and label the basis vectors within an I.R. by the three-dimensional "angular momentum" l and "magnetic quantum number" m. Given an I.R. (j_1,j_2) the possible values of l are

$$l=j_1+j_2,\ j_1+j_2-1,\ \cdots,\ |j_1-j_2|.$$

(The easiest, but not the only possible, way of proving this statement is through the use of spinor calculus.) In terms of the labels j_0, n we have

$$l=j_0,\ j_0+1,\ j_0+2,\ \cdots,n.$$

In this basis the Clebsch-Gordan coefficients (CGC) can be factorized according to the Wigner-Eckart theorem; in an obvious notation we have

$$\begin{pmatrix} n_1j_{01} & n_2j_{02} & \big| & nj_0 \\ l_1m_1 & l_2m_2 & \big| & lm \end{pmatrix}=(l_1m_1,l_2m_2|lm)$$
$$\times\begin{pmatrix} n_1j_{01} & n_2j_{02} & \big\| & nj_0 \\ l_1 & l_2 & \big\| & l \end{pmatrix},$$

the first factor being a CGC of SU_2. (In what follows we shall have to deal with I.R. with $j_0=0$ only; hence we simplify the notation by suppressing the label j_0 everywhere.) Familiar orthogonality properties of the CGC follow from their definition.

According to Weyl's unitary trick[6] we obtain the Lie algebra of the proper homogeneous Lorentz group $\mathfrak{L}_+(\uparrow)$ by putting $N_j\rightarrow iN_j$; the unitary I.R. of $\mathfrak{L}_+(\uparrow)$ are obtained by the replacement $n=-1-i\lambda$ and are classified as follows:

$j_0=0,\ \lambda=i$		trivial rep.,
$j_0=\frac{1}{2}, \frac{3}{2}, \cdots$	$-\infty<\lambda<\infty$	double valued,
$j_0=1, 2, 3, \cdots$	$-\infty<\lambda<\infty$	single valued,
$j_0=0$	$\lambda\geqq 0$	single valued,
$j_0=0$	$0\leqq -i\lambda\leqq 1$	single valued.

We mention already here and shall repeat occasionally the important fact that the spherical functions and their matrix elements of SO_4 and $\mathfrak{L}_+(\uparrow)$ can actually be continued into each other by the replacements indicated above. This forms the basis of the statement made in the Introduction, namely, that our results are equivalent to those obtained by working directly with $\mathfrak{L}_+(\uparrow)$ instead of SO_4.

[5] For the representation theory of SO_4 see, e.g., G. Racah, Nuovo Cimento, Suppl. 14, 75 (1959).

[6] H. Weyl, *Classical Groups* (Princeton University Press, Princeton, New Jersey, 1947), Chap. IX.

Knowing the analytic properties of the various matrix elements in question, the proof of equivalence in each case is an easy exercise and is left to the reader.

B. Spherical Functions

In a canonical basis, and for I.R. with $j_0=0$, the components of the four-dimensional angular momentum $M_{\mu\nu}$ (μ, $\nu=1, 2, 3, 4$) act as differential operators:

$$M_{\mu\nu}=i^{-1}(x_\mu\partial_\nu-x_\nu\partial_\mu). \qquad (2.1)$$

The functions of the four-dimensional sphere are eigenfunctions of the operators

$$-F=\tfrac{1}{4}M_{\mu\nu}M_{\mu\nu}, \quad \mathbf{J}^2=L_{ik}L_{ik} \quad \text{and} \quad L_{12}$$

($i, k=1, 2, 3$); the eigenvalue of G being zero. We introduce polar coordinates of the unit vector e_μ as follows:

$$\begin{aligned} e_4&=\cos\beta\,,\\ e_3&=\sin\beta\,\cos\theta\,, \qquad (2.2)\\ e_1&=\sin\beta\,\sin\theta\,\cos\phi\,,\\ e_2&=\sin\beta\,\sin\theta\,\sin\phi\,,\end{aligned}$$

the four-dimensional volume element becomes $d^4p = p^3dp\,\sin^2\!\beta d\beta\,\sin\theta d\theta d\phi$, with $p=\sqrt{(p^2)}$. Here $0\leqq p<\infty$, $0\leqq\beta\leqq\pi$, $0\leqq\theta\leqq\pi$, $0\leqq\varphi\leqq 2\pi$. Denoting the spherical harmonics by $Z_{nl}{}^m(\beta\theta\phi)\equiv Z_{nl}{}^m(e_\mu)$, we can write the first eigenvalue equation as follows:

$$\begin{aligned} &FZ_{nl}{}^m(\beta\theta\phi)+\tfrac{1}{2}n(n+2)Z_{nl}{}^m(\beta\theta\phi)\\ &\equiv\left(\frac{\partial^2}{\partial\beta^2}+2\cot\beta\frac{\partial}{\partial\beta}+\frac{1}{\sin^2\beta}\mathbf{J}^2\right)Z_{nl}{}^m(\beta\theta\phi)\\ &\qquad+\tfrac{1}{2}n(n+2)Z_{nl}{}^m(\beta\theta\phi)=0. \qquad (2.3)\end{aligned}$$

The set of orthonormal solutions of Eq. (2.3) thus can be written in the form of a product:

$$Z_{nl}{}^m(\beta\theta\phi)=p_{nl}(\beta)Y_l{}^m(\theta,\phi) \quad (n=0, 1, 2, \cdots), \qquad (2.4)$$

where the second factor is a three-dimensional spherical harmonic, while two equivalent forms of $p_{nl}(\beta)$ are the following:

$$p_{nl}(\beta)=\left(\frac{(n+1)\Gamma(n+l+2)}{\Gamma(n-l+1)}\right)^{1/2} \times(\sin\beta)^{-1/2}P_{n+1/2}{}^{-(l+1/2)}(\cos\beta) \qquad (2.5\text{a})$$

or

$$p_{nl}(\beta)=2^{l+1/2}\left(\frac{(n+1)\Gamma(n-l+1)}{\pi\Gamma(n+l+2)}\right)^{1/2} \times\Gamma(l+1)\,\sin\beta^l C_{n-l}{}^{l+1}(\cos\beta). \qquad (2.5\text{b})$$

Here $P_\nu{}^\lambda(x)$ and $C_\nu{}^\lambda(x)$ are Legendre and Gegenbauer functions, respectively, as defined, e.g., in Ref. 7. [The solutions (2.5a), (2.5b) are valid for any value of n and l; if n, l are integers, the Taylor series of $P_{n+1/2}{}^{-(l+1/2)}$ and $C_{n-l}{}^{l+1}$ terminate.]

From (2.5) we get the condition already known:

$$0\leqq l\leqq n \quad (n, l \text{ integers}). \qquad (2.6)$$

On setting $\beta=i\psi$ $(0\leqq\psi<\infty)$, $n=-1-i\lambda$ $(0<\lambda<\infty)$, we find the basis functions of unitary representations of the homogeneous Lorentz group.[8] This replacement is but a trivial generalization of Weyl's unitary trick.[6]

The normalization condition reads as follows:

$$\int_0^\pi \sin^2\!\beta d\beta\int d\Omega\, Z_{nl}{}^{m*}(\beta\Omega)Z_{n'l'}{}^{m'}(\beta,\Omega) =\delta_{nn'}\delta_{ll'}\delta_{mm'}, \quad (n=0, 1, 2, \cdots) \qquad (2.7)$$

($d\Omega$ being the surface element of the three-dimensional unit sphere). Of particular importance is the function $Z_{n0}{}^0$; its expression is very simple:

$$Z_{n0}{}^0(\beta)=(2\pi^2)^{-1/2}\frac{\sin(n+1)\beta}{\sin\beta}.$$

The addition theorem reads

$$Z_{n0}{}^0(\gamma)=\frac{2^{1/2}\pi}{n+1}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Z_{nl}{}^{m*}(\beta'\theta'\phi')Z_{nl}{}^m(\beta''\theta''\phi''), \qquad (2.8)$$

with

$$\begin{aligned}\cos\gamma&=\cos\beta'\,\cos\beta''+\sin\beta'\,\sin\beta''\,\cos\theta, \qquad (2.9)\\ \cos\theta&=\cos\theta'\,\cos\theta''+\sin\theta'\,\sin\theta''\,\cos(\phi'-\phi'').\end{aligned}$$

Equations (2.8) and (2.9) are equally valid for SO_4 and $\mathcal{L}$, with the replacements indicated above.

Making use of the second solution D_n of the equation for the Gegenbauer functions, we can construct the expansion

$$\frac{1}{z-t}=\sum_{n=0}^{\infty}Z_{n0}{}^0(t)D_n(z), \qquad (2.10)$$

with $t=\cos\beta$ and $D_n(z)=2^{3/2}\pi[z-(z^2-1)^{1/2}]^{n+1}$. (We could construct a more general expansion involving $Z_{nl}{}^m$, but we shall not need it.) The expansion (2.10) (being an analog of the familiar Heine's expansion) is convergent in an ellipse with foci at $\cos\beta=\pm1$. [In the standard notation, our $Z_{n0}{}^0(x)$ and $D_n(z)$ are proportional to the Jacobi functions of the first and second kind, $P_n{}^{(1/2,1/2)}(x)$ and $(z^2-1)^{1/2}Q_n{}^{(1/2,1/2)}(z)$, respectively.] An analytic continuation can be obtained by the Watson-Sommerfeld method, i.e., by writing (2.10) in the form of a contour integral:

$$\frac{1}{z-t}=\frac{1}{2i}\int_c d\nu\,\cot\nu\pi Z_{\nu0}{}^0(t)D_\nu(z) \qquad (2.11)$$

[7] A. Erdelyi, W. Magnus, H. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I; I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press Inc., New York, 1965).

[8] See, e.g., H. Joos, Fortschr. Physik **10**, 65 (1962).

and suitably deforming the contour. Asymptotic estimates of the functions of the first and second kind are the same as in the case of the familiar functions P_l and Q_l. In fact it follows from the general theory of hyperspherical functions[9] (of which the functions of the three- and four-dimensional spheres are special cases) that the exponent of the leading term in their asymptotic series both for $|v|\to\infty$ and $|z|\to\infty$ is independent of λ. (The asymptotic behaviors of $Z_{n0}{}^0$ and D_n can be most easily read off from their definitions. We shall not need the asymptotic behavior of the more complicated functions.)

C. Matrix Elements of Spherical Harmonics

In order to find the matrix elements of the spherical harmonics of SO_4, we remark that the Wigner-Eckart theorem allows us to factor out a Clebsch-Gordan coefficient of the three-dimensional rotation group. Moreover, from the definition of the $Z_{nl}{}^m$, we see that even the reduced matrix elements can be factorized. Thus using the well-known expression

$$(Y_l{}^m)_{l_1m_1}{}^{l_2m_2}\equiv\int d\Omega\, Y_{l_1}{}^{m_1*}Y_l{}^mY_{l_2}{}^{m_2}$$

$$=(-1)^{l_{\max}-m_1}i^l\begin{pmatrix} l_1 & l & l_2\\ -m_1 & m & m_2\end{pmatrix}(\pm i)^{l_1+l_2}$$

$$\times\begin{pmatrix} l_1 & l & l_2\\ 0 & 0 & 0\end{pmatrix}\left[\frac{(2l+1)(2l_1+1)(2l_2+1)}{4\pi}\right]^{1/2},$$

we can write

$$(n'l'm';n''l''m''|nlm)$$

$$\equiv\int\sin^2\beta d\beta\int d\Omega Z_{n'l'}{}^{m'}Z_{n''l''}{}^{m''}Z_{nl}{}^{m*}$$

$$=(Y_{l''}{}^{m''})_{lm}{}^{l'm'}(n'l';n''l''\|nl),$$

so that we have to compute the last factor only. For the representations considered in this paper, the latter is given by the expression

$$(n'l';n''l''\|nl)$$

$$=\int_0^\pi\sin^2\beta d\beta p_{n'l'}(\beta)p_{n''l''}(\beta)p_{nl}(\beta).\quad(2.12)$$

Equation (2.12) is obviously symmetric in the variables $(n'l')$, $(n''l'')$, (nl). The integral in (2.12) can be easily evaluated with the help of the following trigonometric expansion[7] of the Gegenbauer functions:

$$C_{n-l}{}^{l+1}(\cos\beta)=\sum_{k=0}^{\infty}\frac{\Gamma(l+1+k)\Gamma(n+1-k)}{k!\Gamma(n-l+1-k)[\Gamma(l+1)]^2}\times\cos(2k-n+l)\beta.$$

Inserting this expansion into (2.12), the integral can be evaluated in an elementary way. The result stated in a manifestly symmetric form reads:

$$(n',l';n''l''\|nl)=(2\pi)^{-1/2}\frac{[(n'+1)(n''+1)(n+1)\Gamma(n'-l'+1)\Gamma(n''-l''+1)\Gamma(n-l+1)]^{1/2}}{\Gamma(l'+1)\Gamma(l''+1)\Gamma(l+1)[\Gamma(n'+l'+2)\Gamma(n''+l''+2)\Gamma(n+l+2)]^{1/2}}$$

$$\times\Gamma(l+l'+l''+3)\sum_{k',k'',k=0}^{\infty}\frac{1}{k'!k''!k!}\frac{\cos\pi(k+k'+k''-\frac{1}{2}(n+n'+n'')+\frac{1}{2}(l+l'+l''))}{\Gamma(k'+k''+k-\frac{1}{2}(n'+n''+n)+l'+l''+l+2)}$$

$$\times\frac{\Gamma(l'+1+k')\Gamma(l''+1+k'')\Gamma(l+1+k)\Gamma(n'+1-k')\Gamma(n''+1-k'')\Gamma(n+1-k)}{\Gamma(\frac{1}{2}(n'+n''+n)-(k'+k''+k)+2)\Gamma(n'-l'+1-k')\Gamma(n''-l''+1-k'')\Gamma(n-l+1-k)}.\quad(2.13)$$

Finally let us record the important selection rule

$$(n'l';n''l''\|nl)=0$$

unless

$$n'+n''+n\equiv0\quad(\mathrm{mod}\ 2).\quad(2.14)$$

This follows immediately from (2.12), taking into account that p_{nl} is an even (odd) function of $\cos\beta$ if n is even (odd).

For integer values of (n',l''), (n'',l''), (n,l) the series (2.13) terminates. n', n'', n have to satisfy a triangular inequality, otherwise the matrix element (2.12) vanishes. Quite often, if at least one of the quantum numbers is small, it is easier to evaluate the integral in (2.12) directly rather than to use Eq. (2.13).

[9] G. Szegö, Am. Math. Soc. Colloq. Publ. No. XIX (1938).

3. THE PROBLEM OF THE INVARIANCE OF THE SCATTERING AMPLITUDE

We consider the scattering of spinless particles. The truncated Green's function T describing the scattering process is an invariant function of three independent momenta. As we have already mentioned in the Introduction, we shall work in the Euclidean region $\mathcal{E}$ of the momenta, i.e., replace $p_1{}^0$ by $p_{14}=ip_1{}^0$, etc. The possibility of continuing the Green's function to $\mathcal{E}$ has been widely discussed in the literature and we take it for granted. As in $\mathcal{E}$ the symmetry group of T becomes compact, this procedure simplifies the form of most of the expressions. Let $p_1, q_1; p_2, q_2$ be the momenta of the incoming and outgoing particles, respectively. On the mass shell let $p_1^2=p_2^2=-M^2$, $q_1^2=q_2^2=-\mu^2$. For the sake of brevity, sometimes we shall call the particles the "nucleon" and "pion," respectively. If the momenta of the external particles are on the mass shell, T is equal to the scattering amplitude. We conveniently choose the three independent momenta E, p, p' according to the definition

$$p_1=\tfrac{1}{2}E-p\,,\quad q_1=\tfrac{1}{2}E+p\,,\qquad (3.1)$$
$$p_2=\tfrac{1}{2}E-p'\,,\quad q_2=\tfrac{1}{2}E+p'\,.$$

In order to investigate the analytic properties of T that are relevant for the symmetry under consideration, we introduce polar coordinates according to (2.2) and investigate T as a function of E^2, p^2, p'^2 and the two unit vectors $e_\mu=p_\mu/\sqrt{(p^2)}$, $e_\mu'=p_\mu'/\sqrt{(p'^2)}$.

Let us choose a coordinate system where $E_4=E$, $E_k=0$ ($k=1, 2, 3$). (In the Minkowskian region of momenta this choice corresponds to the center-of-mass frame.) Let us quote first of all the necessary kinematic formulas that follow from Eq. (3.1) and connect the variables used in this section with the standard kinematical invariants. We identify the square of E_μ with the Mandelstam invariant u, by putting $-E^2=u$. Then from (3.1) we have

$$p^2=\tfrac{1}{2}(p_1^2+q_1^2+\tfrac{1}{2}u)\,,$$
$$p'^2=\tfrac{1}{2}(p_2^2+q_2^2+\tfrac{1}{2}u)\,,$$
$$\cos\beta=\frac{q_1^2-p_1^2}{2p\sqrt{(-u)}}\,,\qquad (3.2)$$
$$\cos\beta'=\frac{q_2^2-p_2^2}{2p'\sqrt{(-u)}}\,,$$

where $p=\sqrt{(p^2)}$ etc. is the modulus of the four-vector p_μ. In $\mathcal{E}$ both p and $\sqrt{(-u)}$ are real.[10]

With the definition of u as given above, the other two Mandelstam invariants s and t are

$$-s=(p+p')^2\,,\quad -t=(p-p)^2\,,$$

and the cosine of the angle between the unit vectors e_μ and e_μ' becomes

$$\cos\gamma\equiv(e_\mu\cdot e_\mu')=\frac{t-s}{\{[2(p_1^2+q_1^2)+u][2(p_2^2+q_2^2)+u]\}^{1/2}}\,.$$

After this preparation we can proceed to investigate the invariance of T under SO_4 at the point $u=0$.

(a) First we have to prove that T, considered as a function of e_μ and e_μ', is expandable into a series of the form

$$T(e_\mu',e_\mu)=\sum_{n'nlm}\langle n'|T(l)|n\rangle Z_{n'l}^{m*}(e_\mu')Z_{nl}^{m}(e_\mu)\,.\qquad (3.3)$$

(We adopt the convention that variables in which the amplitude is diagonal are written as an argument; further, we suppress variables which have the same values on both sides of the equation.) To this end we have to investigate the analytic properties of T in the components of e_μ and e_μ'.

(b) Next we should show that the matrix element $\langle n'|T(l)|n\rangle$ becomes diagonal in n and independent of l at $u=0$, so that the expansion (3.3) would degenerate into

$$T(u=0,\,e_\mu',\,e_\mu)=\sum_n T(u=0,\,n)Z_{n0}^{0}(\gamma)\qquad (3.3')$$

(after using the addition theorem for the hyperspherical functions). At this point we could repeat all the familiar steps of the three-dimensional Regge theory: show that $T(u=0,n)$ defines an analytic function of n, rewrite (3.3′) as a Watson-Sommerfeld integral, isolate the contributions of the poles, etc. We shall see, however, that although the partial-wave amplitudes become diagonal, the expansion (3.3) *does not go over* to (3.3′), so that the amplitude—with the notable exception of the equal-mass case—*is not invariant under* SO_4 at $u=0$. Interestingly enough, the higher symmetry still manifests itself in the spectrum: A study of the Bethe-Salpeter equation shows that at $u=0$ the spectrum (i.e., the poles or branch cuts) appear as simple singularities in the n rather than in the l plane; or (in slightly a different language) in the l plane they show the pattern required by the higher symmetry.

We outline the steps to be followed.

(a) In order to prove the existence of the expansion (3.3), we have to locate the singularities of T in the components of e_μ and e_μ'. As far as the dependence on the angles Θ and φ is concerned, the situation is well known; indeed it is connected with the existence of a three-dimensional partial-wave expansion.

Therefore let us focus our attention on the dependence on β and β'. As Eqs. (3.2) show, when we vary β with p^2 and keep u fixed, this amounts to saying that we vary the external masses so that the sum of their squares remains constant.

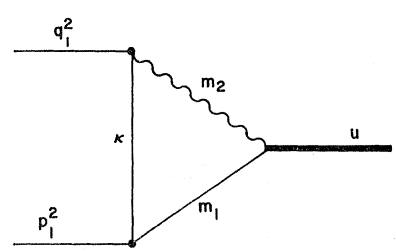

FIG. 1. Reduced triangular diagram for the determination of the singularities in cosβ.

[10] The external "masses," i.e., the quantities $(-p_1^2)^{1/2}$, $\cdots(-q_2^2)^{1/2}$ are pure imaginary in $\mathcal{E}$.

The location of the nearest singularities of T in p_1^2 and q_1^2 is determined by the reduced diagram shown in Fig. 1. The wavy line represents a two-pion state, with a continuous "mass," m_2; the mass of the internal "nucleon" line is m_1 and the mass of the internal "pion" line is κ. In general, the internal lines represent any state with the same quantum numbers as the corresponding particles, thus

$$\mu \leqq \kappa < \infty\,,$$
$$M \leqq m_1 < \infty\,,$$
$$2\mu \leqq m_2 < \infty\,.$$

The diagram of Fig. 1 first of all generates normal thresholds in the external masses, located at $-q_1^2 = 9\mu^2$ and $-p_1^2 = (M+\mu)^2$, respectively. Correspondingly in the $\cos\beta$ plane we have two branch points. It is convenient to introduce the variable z by the definition: $z = p^{1/2}(-u)\cos\beta$. The normal thresholds in z are located at the points

$$z_1 = \tfrac{1}{4}u - p^2 - 9\mu^2$$

and

$$z_2 = -\tfrac{1}{4}u + p^2 + (M+\mu)^2. \qquad (3.4)$$

The next step is to investigate whether there are anomalous thresholds in the z plane. Consider the discontinuity of the amplitude corresponding to the diagram in Fig. 1 across the normal cut starting at z_2. As we know that a three-dimensional partial-wave expansion exists, it is sufficient to consider the S-wave part of that discontinuity. A straightforward calculation shows that this is proportional to the function

$$\ln\frac{2p_1^2(m_2^2+q_1^2-\kappa^2)-(p_1^2+q_1^2+u)(p_1^2+m_1^2-\kappa^2)+[\Delta(-p_1^2-q_1^2u)\Delta(-p_1^2\kappa^2m_1^2)]^{1/2}}{2p_1^2(m_2^2+q_1^2-\kappa^2)-(p_1^2+q_1^2+u)(p_1^2+m_1^2-\kappa^2)-[\Delta(-p_1^2-q_1^2u)\Delta(-p_1^2\kappa^2m_1^2)]^{1/2}},$$

with

$$\Delta(a,b,c) = a^2+b^2+c^2-2ab-2ac-2bc.$$

We can insert here the *minimal* allowed internal masses, i.e., $\kappa=\mu$, $m_1=M$, $m_2=2u$.

The singularities are determined by the vanishing of either the numerator or the denominator in the argument of the logarithm. p_1^2 and q_1^2 should be expressed in terms of z, p, u. Introducing the auxiliary variable $x = p^2 - \frac{1}{4}u$, after some manipulation we find that the condition for a singularity is

$$\begin{aligned} F(z,x) = [2(M^2+\mu^2)-u]z^2 + ux^2 + 2xz(M^2-4\mu^2) \\ + x[u(M^2+6\mu^2)-(M^2-4\mu^2)^2] \\ - z(M^2-4\mu^2)(u-M^2-4\mu^2) \\ + u\mu^2[u+3(M^2-\mu^2)] = 0. \quad (3.5) \end{aligned}$$

The singularity described by (3.5) starts out on the second sheet in z. The necessary condition for it to appear on the first sheet is that $\partial F/\partial x = 0$ be satisfied[11] besides (3.5) for some x and $z = x + (M+\mu)^2$. Differentiating (3.5) and solving for x we find

$$x = -\frac{(M^2-4\mu^2)[(M+2\mu)^2+2\mu^2]+u(M^2+6\mu^2)}{2(u+M^2-4\mu^2)}.$$

For sufficiently small values of u, this requires $p^2<0$ which is outside $\mathcal{E}$. (Actually we find that u should lie in the interval $-1.2M^2 < u < 2.6M^2$.)

We can treat the discontinuity across the cut starting at z_1 in a similar way. Thus we have established that *if u is in a sufficiently small neighborhood of zero, there are no anomalous singularities in the z (or $\cos\beta$) plane.* Consequently, the Gegenbauer expansion in $\cos\beta$ converges[9] in an ellipse with foci at $\cos\beta = \pm 1$ and semimajor axis equal to $(p^2+9\mu^2-\frac{1}{4}u)/2p\sqrt{(-u)}$ (for $u<0$), determined by the nearest normal threshold. Moreover, the expansion in $\cos\beta$ can be continued into the hyperbolic region up to the physical values of the external masses.

Evidently, the analysis of the singularities in $\cos\beta'$ can be performed in the same way.

Having shown that an expansion in the "mass-type" variables exists, we proceed to the next step and (b) investigate the invariance of T under SO_4 at $u=0$. The crucial point in the demonstration is the well-known theorem stating that if a function is nonincreasing at infinity and has no singularities in any finite part of the complex plane, then it is a constant.

Let us observe that (suppressing the variables that are kept constant) T being an invariant function, it depends on $E_\mu p_\mu = z$, $E_\mu p_\mu' = z'$, and on the invariant momentum transfers. As we have just seen, the singularities in z and z' are located at finite points in the complex plane. We *assume that the discontinuities of T across the cuts in z and z' decrease at infinity.* (This is equivalent to the usual assumption that the amplitude possesses unsubtracted spectral representations in the external masses.)

We now construct the matrix element $\langle n_2|T(l)|n_1\rangle$. First of all we write a dispersion relation for fixed u. We shall assume that no subtractions are necessary; subtracted dispersion relations can be treated in a well-known way.

$$T(s,t,z,z') = \frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{T_t(t',z,z')dt'}{t'-t} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{T_s(s',z,z')ds'}{s'-s}.$$

[11] We remind the reader that if the left-hand singularity on an unphysical sheet encircles the normal branch point, then it certainly has an extremum there: $dz/dx=0$. Differentiating Eq. (3.5): $dF/dx = \partial F/\partial x + (\partial F/\partial z)dz/dx$, the condition stated in the text follows.

(Note that the thresholds are determined by the *internal* masses, so they are not changed.) Using the formulas

$$-s=p^2+p'^2+2pp'\cos\gamma\,,$$
$$-t=p^2+p'^2-2pp'\cos\gamma\,,$$

we introduce the even and odd parts, $T^{(\pm)}$, in $\cos\gamma$ of the amplitude T and expand it into partial waves using Eq. (2.10):

$$T^{(\pm)}(\cos\gamma,z,z')=\sum_{n'''}T_{n'''}{}^{(\pm)}(z,z')Z_{n'''0}{}^{0}(\gamma)\,, \quad (3.6)$$

with

$$T_{n'''}{}^{(\pm)}(z,z')=\frac{1\pm(-1)^n}{2\pi pp'}\int_{4\mu^2}^{\infty}dx\,T_x{}^{(\pm)}(x,z,z')D_{n'''}\times\left(\frac{x-p^2-p'^2}{2pp'}\right) \quad (3.7)$$

and

$$T_x{}^{(\pm)}(x,z,z')=T_s(x,z,z')\pm T_t(x,z,z')\,.$$

Equation (3.7) is the analog of the well-known Froissart-Gribov formula.

We now proceed by expanding $T_{n'''}$ according to z and z'. We write

$$T_{n'''}{}^{(\pm)}(x)=\frac{-\pi^{-2}}{pp'u}\left(\int_{-\infty}^{z_1}+\int_{z_2}^{\infty}\right)d\eta\left(\int_{-\infty}^{z_1}+\int_{z_2}^{\infty}\right)d\eta'\times\frac{T_{n\eta\eta'}(\eta,\eta')}{\left(\dfrac{\eta}{p\sqrt{(-u)}}-\cos\beta\right)\left(\dfrac{\eta'}{p'\sqrt{(-u)}}-\cos\beta'\right)} \quad (3.8)$$

and apply Eq. (2.10) to each of the denominators. [The energy u is supposed to be chosen small enough, so that the anomalous thresholds have retreated to unphysical sheets and the integration regions $(-\infty,z_1)$ and (z_2,∞) do not overlap.] Thus we obtain the triple series

$$T(\gamma,\beta,\beta')=\sum_{n'n''n'''}T_{n'n''n'''}Z_{n'0}{}^{0}(\beta)Z_{n''0}{}^{0}(\beta')Z_{n'''0}{}^{0}(\gamma)\,.$$

In order to obtain the matrix element of interest, we apply the addition theorem (2.8) to the third hyperspherical function and couple together the functions of the "initial" and "final" states. Thus we get

$$\langle n_2|T_{(l)}{}^{(\pm)}|n_1\rangle=\sum_{n'n''n'''}\begin{pmatrix}n' & n''' & \| & n_1\\ 0; & l & \| & l\end{pmatrix}\times\begin{pmatrix}n'' & n''' & \| & n_2\\ 0; & l & \| & l\end{pmatrix}T_{n'n''n'''}{}^{(\pm)} \quad (3.9)$$

with the Clebsch-Gordan coefficients defined in Sec. 2 B.

Let us look at the expression of $T_{n'n''n'''}{}^{(\pm)}$. From (3.8) we have

$$T_{n'n''n'''}{}^{(\pm)}=\frac{-\pi^{-2}}{pp'u}\int d\eta\int d\eta'\,D_{n'}\left(\frac{\eta}{p\sqrt{(-u)}}\right)\times D_{n''}\left(\frac{\eta}{p'\sqrt{(-u)}}\right)T_n{}^{(\pm)}(\eta,\eta')\,. \quad (3.10)$$

As $|u|\to 0$,

$$D_n\left(\frac{\eta}{p\sqrt{(-u)}}\right)\sim 2^{3/2}\pi\left(\frac{p\sqrt{(-u)}}{\eta}\right)^{n+1},$$

thus only the term $T_{00n'''}{}^{(\pm)}$ survives. Taking into account the obvious property of the Clebsch-Gordan coefficients

$$\begin{pmatrix}0 & n & \| & n'\\ 0 & l & \| & l\end{pmatrix}=\delta_{nn'}\,,$$

this shows that

$$\lim_{u\to 0}\langle n'|T(u,l)|n\rangle=\delta_{nn'}T(u=0,\,n)\,.$$

At this point, however, we encounter an infamous difficulty.

The formulas (3.2) connecting the pairs of variables $(p,\cos\beta)$ and (p_1^2,q_1^2) are singular at $u=0$. Consequently, when we want to go to the mass shell, we have to let $p_1^2\to -M^2$, $q_1^2\to-\mu^2$, etc., and pass to the limit $u=0$ afterwards. (This requires the continuation of the expansion by means of a Watson-Sommerfeld transformation; this fact is, however, irrelevant to the following discussion.)

It is evident from Eq. (3.2) that on doing so, $\cos\beta$ and $\cos\beta'$ tend to infinity as $u\to 0$. Using the asymptotic formula[7]

$$C_n{}^{\lambda}(\cos\beta)\sim\frac{2^n\Gamma(\lambda+n)}{\Gamma(n+1)\Gamma(\lambda)}(\cos\beta)^n\,,\quad (|\cos\beta|\to\infty)$$

one can check that

$$Z_{nl}{}^{m}(\beta,\Theta,\varphi)=O(|u|^{-n/2})\quad(u\to 0)\,.$$

Thus the expansion (3.3) could degenerate into (3.3′) only if the off-diagonal elements $\langle n'|T|n\rangle$ satisfied the inequality

$$|\langle n'|T|n\rangle|<A|u|^{(n+n')/2}\quad(n\neq n',\,u\to 0)\,, \quad (3.11)$$

where A is some constant. The inequality (3.11) can, however, never be satisfied (excluding some pathological cases). Indeed, using (3.9) and (3.10) we can easily see that as $u\to 0$

$$\langle n'|T|n\rangle=O(u^{(n+n')/2})\,.\quad(n\neq n')$$

Thus the contribution of the off-diagonal elements to (3.3) does not vanish, so the amplitude *is not* invariant.

The *equal-mass case* is obviously exceptional: On the mass shell $\cos\beta=0$; the functions Z_{nl}^m do not blow up, so $\langle n'|T(l)|n\rangle \to \delta_{nn'}T(n)$ is sufficient to guarantee the invariance of the full amplitude.

This situation can be illustrated on a simple example. Suppose we want to expand the function $(p_1^2+m^2)^{-1}$. Using (3.2) we rewrite it as $[p\sqrt{(-u)}]^{-1}(x-\cos\beta)^{-1}$, with $x=(p^2+m^2-\frac{1}{4}u)\times[p\sqrt{(-u)}]^{-1}$. Expanding, we obtain

$$\frac{1}{p_1^2+m^2}=\frac{2}{p\sqrt{(-u)}}\sum_{n=0}^{\infty}C_n'(\cos\beta)[x-(x^2-1)^{1/2}]^{n+1}.$$

If $u\to 0$ with p_1^2, q_1^2 fixed, then $\cos\beta\to\infty$ and $x\to\infty$. However, using the asymptotic formulas

$$C_n'(\cos\beta)\sim 2^n(\cos\beta)^n,$$
$$[x-(x^2-1)^{1/2}]^{n+1}\sim(2x)^{-n-1},$$

we obtain the formal expansion

$$\frac{1}{p_1^2+m^2}\sim\frac{1}{xp\sqrt{(-u)}}\sum_{n=0}^{\infty}\left[\frac{\cos\beta}{x}\right]^n \quad (u\to 0).$$

The series standing on the right-hand side is at least summable, e.g., in the Abel sense and its sum equals to $(1-\cos\beta/x)^{-1}$. Inserting this expression and using the definitions of x and p, we see that the function $(p_1^2+m^2)^{-1}$ *is* reproduced by the expansion. However, if we keep the $n=0$ term only, we get at $u=0$

$$\frac{1}{\frac{1}{2}(p_1^2+q_1^2)+m^2}$$

instead of the original expression. Obviously, this is correct on the mass shell, if there $p_1^2=q_1^2$, which is the equal-mass case.

4. A "THEORETICAL EXPERIMENT": SYMMETRY OF THE SPECTRUM AT $u=0$

We have just seen that (perhaps apart from some pathological cases) the full scattering amplitude is not invariant under SO_4 unless $\cos\beta=\cos\beta'=0$ on the mass shell, i.e., the masses of the particles are equal both in the initial and final states.

It is reasonable to expect, however, that although the amplitude itself is not invariant, the spectrum may show the higher symmetry. (By spectrum we mean the positions of the Regge trajectories at $u=0$, i.e., the spectrum of the angular momentum.) Evidently, in order to investigate this problem, we need some dynamical scheme which determines the Regge trajectories. Let us observe first of all that some of the familiar dynamical schemes can be excluded *a priori*, as not having a chance to produce the four-dimensional symmetry. The best known of these "bad" schemes is the on-shell N/D system of equations in the two-particle approximation.

It is instructive to see why this scheme is a "bad" one. As is well known, one starts by writing down a dispersion relation for the amplitude at constant momentum transfer or scattering angle and obtains the dynamical equations by relating the right-hand discontinuity of the amplitude to the amplitude itself by the use of unitarity in the two-particle approximation. Now, in the two-particle approximation, for a suitably normalized amplitude A the unitarity condition reads:

$$A(u+i0,\zeta)-A(u-i0,\zeta)$$
$$=i\frac{\Delta(u,M^2,\mu^2)}{u}\int d\zeta' d\zeta''\,K(\zeta,\zeta',\zeta'')$$
$$\times A(u+i0,\zeta')A(u-i0,\zeta''),$$

where $\Delta(x,y,z)=x^2+y^2+z^2-2xy-2xz-2yz$, ζ, ζ', ζ'' are the cosines of scattering angles, and K is the well-known symmetric kernel. From the geometrical point of view $d\zeta' d\zeta'' K(\zeta,\zeta',\zeta'')$ is proportional to the surface element of the three-dimensional sphere. Consequently, although one could continue the unitarity relation below threshold, due to the mass-shell restriction, the "two-particle unitarity" can never become form invariant under the four-dimensional group. In other words, the mass-shell restriction precludes the introduction of a sufficient number of variables, thus producing the first difficulty mentioned in the Introduction.

A scheme which *does* work is the Bethe-Salpeter equation, where the four-dimensional symmetry had originally been discovered. A detailed treatment of the problem has been given long ago (cf. Ref. 2 and the literature quoted there, see also Ref. 4). Here we repeat the essential points of the argument; we concentrate on the problem of the Regge trajectories, and apply a method which permits an immediate generalization to treat the breaking of the symmetry.

Consider the Bethe-Salpeter equation for the off-shell amplitude of spinless particles. In the momentum representation it can be written as

$$T(p',p;E)=\lambda K(p,p';E)$$
$$+\lambda\int\frac{dkK(p',k;E)T(k,p;E)}{[(\frac{1}{2}E+k)^2+M^2][(\frac{1}{2}E-k)^2-\mu^2]}. \quad (4.1)$$

If we take the ladder approximation to the Born term λK, with exchanged mass m, we get

$$\lambda=g^2/(2\pi)^4 i,$$
$$K(p,p')=1/[(p-p')^2+m^2]. \quad (4.2)$$

In the Euclidean metric Eq. (4.1) can be transformed to an ordinary Fredholm equation with a Hilbert-Schmidt kernel, so the singularities correspond to bound-state poles.[12] Let us introduce the notation

$$L(p,p')=\frac{K(p,p')}{[(\frac{1}{2}E+k)^2+M^2][(\frac{1}{2}E-k)^2+\mu^2]}; \quad (4.3)$$

[12] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).

then the solution can be written in the well-known way as

$$T(p',p;E)=\frac{1}{D(U)}\int d^4k\,\frac{\delta D(U)}{\delta L^T(p',k)}\lambda K(k,p)\,,\quad (4.4)$$

where L^T is the transpose of L and $D(E)$ is the Fredholm determinant:

$$D(E)=1-\frac{\lambda}{1!}\int L(k,k)d^4k+\frac{\lambda^2}{2!}\times\int d^4k_1d^4k_2\begin{vmatrix}L(k_1,k_1) & L(k_1,k_2)\\ L(k_2,k_1) & L(k_2,k_2)\end{vmatrix}+\cdots.$$

On introducing the notation for the traces:

$$\langle\langle A\rangle\rangle\equiv\int d^4k\,A(k,k)\,,$$

$$\langle\langle AB\rangle\rangle\equiv\int d^4k_1d^4k_2\,A(k_1k_2)B(k_2k_1)\,,\text{ etc.,}$$

we can rewrite the Fredholm determinant in a more compact form:

$$D(u)=1-\left[\frac{\lambda}{1!}\langle\langle L\rangle\rangle+\frac{\lambda^2}{2!}(\langle\langle LL\rangle\rangle-\langle\langle L\rangle\rangle^2)+\cdots\right].\quad (4.5)$$

Let us expand now the kernel L into four-dimensional partial waves. Denoting the product of one-particle propagators by F, we have, by a straightforward application of the technique developed in Sec. 2,

$$\langle n''l''m''p''|F|n'l'm'p'\rangle=-\frac{\delta(p''-p')}{p'^5u}\sum(-1)^{n_2}D_{n_1}\left(\frac{p'^2-\frac{1}{4}u+M^2}{p'\sqrt{(-u)}}\right)D_{n_2}\left(\frac{p'^2-\frac{1}{4}u+\mu^2}{p'\sqrt{(-u)}}\right)$$
$$\times(n_1'l_1';n_10\|n_1''l_1'')(n_2'l_2';n_20\|n_2''l_2')\begin{pmatrix}n_1' & n_2' & \| & n'\\ l_1' & l_2' & \| & l'\end{pmatrix}\begin{pmatrix}n_1'' & n_2'' & \| & n''\\ l_1' & l_2' & \| & l''\end{pmatrix}\delta_{l',l''}\delta_{m',m''}\,,\quad (4.6)$$

(every dummy index is to be summed over), whereas the matrix element of the kernel in the ladder approximation has the following simple expression:

$$\langle n''l''m''p''|K|n'l'm'p'\rangle=\delta_{n',n''}\delta_{l',l''}\delta_{m'm''}\frac{1}{2p'p''}D_{n'}\left(\frac{p^2+p'^2+m^2}{2pp'}\right)\quad (4.7)$$

and finally

$$\langle n''l''m''p''|L|n'l'm'p'\rangle=\sum_{nlm}\int_0^\infty p^3dp\,\langle n''l''m''p''|K|nlmp\rangle\times\langle nlmp|F|n'l'm'p'\rangle.\quad (4.8)$$

The expression (4.8) is of course diagonal in l, m and independent of m. Inserting (4.8) into (4.5) and using the orthogonality of the hyperspherical functions, we have for the Fredholm determinant:

$$D(u)=\prod_l D_l(u)\,,$$

with

$$D_l(u)=1-\left\{\frac{\lambda}{1!}\sum_n\int p^3dp\,\langle n_p|L(u,l)|np\rangle+\frac{\lambda^2}{2!}\times\left[\sum_{n'n''}\int p'^3dp'\int p''^3dp''\,\langle n'p'|L(u,l)|n''p''\rangle\times\langle n''p''|L(u,l)|n'p'\rangle-\sum_n\int p^3dp\,(\langle np|L(u,l)|np\rangle)^2\right]+\cdots\right\}\quad (4.9)$$

corresponding to the fact that l is conserved. Exactly as in the previous section, we can see that the kernel becomes diagonal in the limit as $u\to 0$:

$$\langle n''p''|L(u,l)|n'p'\rangle\to\delta_{n'n''}\langle p''|L(n)|p'\rangle\,,$$

so $D(u)$ now is split into factors according to n rather than l:

$$D(u=0)=\prod_n D_n\,,$$

with

$$D_n=1-\left[\frac{\lambda}{1!}\langle L(n)\rangle+\frac{\lambda^2}{2!}(\langle L(n)L(n)\rangle-\langle L(n)\rangle^2)+\cdots\right]\quad (4.10)$$

and where we introduced the notation

$$\langle L(n)\rangle\equiv\int_0^\infty p^3dp\,\langle p|L(n)|p\rangle\,,\text{ etc.}$$

This is exactly what we wanted to show: The intercepts of the Regge poles given by the zeros of the Fredholm determinant at $u=0$ *are grouped according to the higher symmetry.* (In fact, we shall see later that one pole in the n plane corresponds to an infinite series of poles in the l plane.)

Let us reemphasize: The full amplitude *is not* invariant under the four-dimensional group, although the spectrum (at least in the present model) is. A moment of thinking reveals the deeper reason of this remarkable phenomenon.

Why is the scattering amplitude not invariant even at $u=0$? We have seen why in the previous section: We cannot continue the amplitude onto the mass shell and go to zero with u afterwards. Thus, loosely speaking, it is the external mass dependence that prevents the amplitude from becoming invariant. However, the Fredholm determinant $D(u)$ does not depend on the external masses at all. (Indeed, we integrate over the relative momenta.) Consequently, there cannot arise the convergence problem encountered in Sec. 3—and so it is perfectly natural that $D(u)$ (and its roots as well) show the higher symmetry in its "full power."

We conjecture now that this is not a feature of the particular model considered, but rather a special case of the general rule. Let us remark immediately that, knowing the analyticity properties exhibited in the previous section, it is a matter of simple exercise (although of a rather tiresome one$\cdots$) to show that the general Bethe-Salpeter equation (with the kernel consisting of an arbitrary number of Feynman diagrams and corrections to the one-particle propagators included) obeys the rule just found. Moreover, the bound-state spectrum has a certain "universality" property. Whether one considers the energy levels or the Regge trajectories, one finds that their characteristics do not depend on the quantum numbers and, in particular, the masses of the states they are coupled to. (In fact, it is precisely this universality property which makes, *inter alia,* a particle interpretation possible.)

To conclude, we feel that the property of the spectrum we just discovered is a general one and thus we proceed by assuming that at $u=0$ the Regge trajectories show the pattern required by the SO_4 symmetry.

5. FAMILIES OF REGGE TRAJECTORIES AND HIGH-ENERGY SCATTERING

The four-dimensional angular momentum is, strictly speaking, conserved at $u=0$ only. Thus, in order to be able to use our formulas, even at $u\neq 0$, we have to analyze the "ordinary" angular momentum contents of an eigenstate of the four-dimensional angular momentum.

The transformation function we need can be simply found from the addition theorem (2.8). Using the addition theorem for the three-dimensional spherical harmonics, we can rewrite (2.8) as follows:

$$Z_{n0}{}^0(\gamma)=\frac{2^{1/2}\pi}{n+1}\sum_l p_{nl}(\beta)p_{nl}(\beta')Y_{l0}(\cos\theta) \quad (2.8')$$

and hence, by the orthogonality of the Y_{l0},

$$\langle l|n\rangle\equiv\int d\Omega\; Y_{l0}{}^*Z_{n0}{}^0=\frac{2^{1/2}\pi}{n+1}p_{nl}(\beta)p_{nl}(\beta'). \quad (5.1)$$

We introduce the quantum number κ with the definition

$$n=l+\kappa. \quad (5.2)$$

The significance of κ from a group-theoretical point of view is evident: The angular momentum states contained in an I.R. (n,j_0) of SO_4, as we mentioned in Sec. 2, are

$$l_{\max}=n=j_1+j_2,\quad n-1,\;\cdots,\quad j_0=|j_1-j_2|=l_{\min}$$

or in other words,

$$l=n-\kappa,$$

with

$$\kappa=0,\,1,\,2,\,\cdots,\,n-j_0. \quad (5.3)$$

We "Reggeize" n (and consequently l), but κ always remains an integer; this is again evident from the Bethe-Salpeter equation, cf. Cutkosky[13] and Ref. 2. However, if n is "Reggeized," and so not an integer, the restriction (5.3) should not be imposed on κ. Indeed, (5.3) follows from the expressions (5.1) and (2.5) only if n and l are integers, otherwise the functions p_{nl} do not vanish, however big is κ.

In terms of κ the selection rule (2.14) can be rewritten as

$$\kappa'+\kappa''+\kappa\equiv 0 \pmod 2. \quad (2.14')$$

This follows immediately from the definition of κ and (2.14) upon observing that conservation of parity requires $l'+l''+l\equiv 0 \pmod 2$.

Finally, in order to elucidate further the physical meaning of κ, let us notice[13] that in the equal-mass case the Bethe-Salpeter wave function of index κ of two spinless particles is multiplied by $(-1)^\kappa$ upon inverting the relative time, so it is justified to call it the "relative-time parity." Next we isolate the contribution of one four-dimensional pole from the scattering amplitude. To this end we take the expansion (3.6) in its Watson-Sommerfeld form [cf. (2.10)]:

$$T^{(\pm)}(\gamma,z,z';u)=(2i)^{-1}\left(\frac{2}{\pi}\right)^{1/2}\int T_n{}^{(\pm)}(z,z';u)\times\cot n\pi\frac{\sin(n+1)\gamma}{\sin\gamma}dn, \quad (3.6')$$

but unlike Sec. 3, we *do not* expand the function $T_n{}^{(\pm)}(z,z';u)$ further.

Again invoking the results of the "theoretical experiment"[14] of the previous section, we assume that $T_n{}^{(\pm)}$ can be continued in n beyond the region given by a Froissart-Gribov formula like (3.7), so that we can isolate the contribution of a pole from (3.6′). Using the

[13] R. E. Cutkosky, Phys. Rev. **96**, 1124 (1952).

[14] Details of the continuation procedure can be found in the first paper of Ref. 2.

factorization theorem of the residues, we find the contribution of a four-dimensional pole with trajectory $\alpha(u)$:

$$T_{4\text{-pole}}^{(\pm)}(\gamma,z,z';u)=\left(\frac{2}{\pi}\right)^{1/2}\frac{g(z,u)\bar{g}'(z',u)}{i}(1\pm e^{i\pi\alpha(u)})\times\cot(\alpha(u)\pi)\frac{\sin(\alpha(u)+1)\gamma}{\sin\gamma}. \quad (5.4)$$

In terms of the quantum numbers l and κ the explicit expression of the transformation function (5.1) is the following:

$$\langle l|n\rangle \equiv F_{l\kappa}(\beta)F_{l\kappa}(\beta'),$$

where $F_{l\kappa}(\beta)$ can be found from (2.5):

$$F_{l\kappa}(\beta)=2^{-\kappa}(2\pi)^{1/4}\left(\frac{\kappa!}{\Gamma(\kappa+l+1)\Gamma(l+\kappa+3/2)}\right)^{1/2}\times\Gamma(l+1)(\sin\beta)^{l}C_{\kappa}^{l+1}(\cos\beta). \quad (5.5)$$

(We used the duplication formula for the Γ function.) Thus using (2.8′) we can rewrite (5.4) so as to exhibit the "ordinary Regge-pole contents" of a four-dimensional pole:

$$T_{4\text{-pole}}^{(\pm)}(\gamma,z,z';u)=\sum_{\kappa=0}^{\infty}\frac{1\pm e^{i\pi\alpha_\kappa(u)}(-1)^\kappa}{2i}g_\kappa(\cos\beta,u)\times\bar{g}_\kappa'(\cos\beta',u)\cot(\alpha_\kappa(u)+\kappa)\pi P_{\alpha_\kappa(u)}(\cos\Theta), \quad (5.6)$$

where we have introduced the notation

$$\alpha_\kappa(u)=\alpha(u)-\kappa, \quad (5.7)$$

giving the positions of the poles in the l plane, and with (5.5)

$$g_\kappa(\beta,u)=g(z,u)2^{1/2-\kappa}\left(\frac{\kappa!}{\Gamma(\alpha_\kappa+\kappa+1)\Gamma(\alpha_\kappa+\kappa+3/2)}\right)^{1/2}\times\Gamma(\alpha_\kappa+1)(\sin\beta)^{\alpha_\kappa}C_\kappa^{\alpha_\kappa+1}(\cos\beta), \quad (5.8)$$

while $\bar{g}_\kappa(\cos\beta,u)=-g_\kappa^*(-\cos\beta,u)$ from time-reversal invariance.

Equations (5.4) to (5.8) are exact at $u=0$ and at small but nonvanishing values of u hold presumably to a good approximation (they imply that the SO_4 symmetry even at $u\neq 0$ is broken only by the mass difference $M^2-\mu^2$ in the residue functions but not by the trajectories). The expansion as we use it is in fact a "broken symmetry expansion" tailored to the nature of the problem: It goes over to the expansion (3.3′) in the exact symmetry limit: $u=0$ and $p_1^2=q_1^2$, $p_2^2=q_2^2$.

Let us now analyze our results.

(a) We find (as already stated in Ref. 2) that the one four-dimensional pole is equivalent to a superposition of poles in the l plane with definite phase relations. At $u=0$, where the spectrum exhibits the SO_4 symmetry exactly, the family of Regge poles are spaced at integer values following the leading pole $\alpha_0(0)\equiv\alpha(0)$. (Freedman and Wang[4] call the poles with $\kappa\neq 0$ the "daughters" of the leading or "parent" pole α_0.) The quantum number κ is identical to κ of Ref. 13 labeling the "anomalous" solutions of the Bethe-Salpeter equation and n_r of Ref. 2.)

(b) The even- (odd-) κ poles have equal (opposite) signature to the parent pole. This is evident from (5.6).

(c) The residues of the odd-κ poles vanish for $M^2=\mu^2$.

In fact, for the Gegenbauer functions of integer order we have[7]

$$C_{2k}^{\lambda}(t)=\frac{(-1)^k}{(\lambda+k)B(\lambda,k+1)}F(-k,k+\lambda;\tfrac{1}{2};t^2),$$
$$C_{2k+1}^{\lambda}(t)=\frac{(-1)^k 2t}{(B(\lambda,k+1))}F(-k,k+\lambda+1;\tfrac{3}{2};t^2), \quad (5.9)$$
$$(k=0,1,2,\cdots),$$

where $B(x,y)$ is an Euler integral of the first kind and $F(a,b;c;z)$ is a hypergeometric function. In the equal-mass limit, $t=\cos\beta$ and/or $t=\cos\beta'$ vanishes; this together with $F(a,b;c;0)=1$ proves the statement.

(d) The residue functions g_κ contain the "threshold factor" $(\Delta(u,M^2,\mu^2)/4u)^{\alpha_\kappa/2}$ automatically as a consequence of the four-dimensional kinematics.

Proof: From (5.8) we see that g_κ is proportional to $(\sin\beta)^{\alpha_\kappa}$. It follows from (3.2) that

$$\sin\beta=(1-\cos^2\beta)^{1/2}=\left(\frac{u^2+2(p_1^2+q_1^2)u+(q_1^2-p_1^2)^2}{4p^2u}\right)^{1/2}.$$

On the mass shell ($p_1^2=-M^2$, $q_1^2=-\mu^2$) this becomes

$$(\sin\beta)_{\text{m.sh.}}=\left(\frac{\Delta(u,M^2,\mu^2)}{4u}\right)^{1/2}\frac{2^{1/2}}{(\frac{1}{2}u-M^2-\mu^2)^{1/2}};$$

hence the statement follows.

(e) If at $u=0$ (or, in the present approximation, anywhere below its threshold) the parent pole α_0 passes through an integer value, the residues of all but a finite number of daughters vanish. This is evident, e.g., from (5.9) and the definition of the hypergeometric function. In particular, for the Pomeranchuk pole, as $\alpha_0^{(p)}(0)=1$, only the daughters $\kappa=0$, 1, or in the case of forward elastic scattering [because of (c)] only $\kappa=0$ contributes. The $\kappa=1$ trajectory could, in principle, be observed in the reaction $\pi+N\to K+Y_{(1/2)}^{**}$ where $\alpha^{(p)}$ can be exchanged, and the masses are different. Many of the corollaries (a)–(e) of Eqs. (5.6)–(5.8) have been derived earlier, mostly in a more complicated way; (e) has been observed independently by Pignotti.[15] We have listed

[15] A. Pignotti (private communication).

these properties partly for the sake of completeness, partly to demonstrate the power of the present approach.

As an immediate application of the formalism, let us consider the asymptotic behavior of the scattering amplitude T as the energy approaches infinity when the external masses take on arbitrary values. It has been known for some time that, owing to the singularity of the kinematic formulas, the asymptotic behavior of the full amplitude is ambiguous even if the partial-wave amplitude $T(l)$ could be proven to be a meromorphic function of the angular momentum l. Solutions to this problem have been proposed recently by Goldberger and Jones[16] and Freedman and Wang.[4] (It was in fact this problem which led Freedman and Wang to rediscover the four-dimensional symmetry, independently of earlier approaches.) In order to simplify the writing, let us concentrate on unequal-mass elastic scattering, i.e., the particles in the initial and final states are the same (of masses M and μ), but the particles coupled to the vertices are different. Physically, this is the case of near-backward pion-nucleon scattering with the exchange of the N and/or Δ trajectories.

Assuming that the four-dimensional pole $\alpha(u)$ dominates, we have from (5.4)

$$T_{(s\to\infty)}{}^{(\pm)}(\gamma,z,z';u)\sim\left(\frac{2}{\pi}\right)^{1/2}\frac{g(z,u)\bar{g}(z',u)}{i}(1\pm e^{i\pi\alpha(u)})\times\cot(\alpha(u)\pi)\frac{\sin(\alpha(u)+1)\gamma}{\sin\gamma}. \quad (5.10)$$

The s dependence comes in through γ, defined in Sec. 3. On the mass shell,

$$\cos\gamma=\frac{t-s}{u-2(M^2+\mu^2)}=-\frac{4M\nu}{u-2(M^2+\mu^2)},$$

with $\nu=(s-t)/4M$, the familiar kinematic variable, and $z=z'=M^2-\mu^2$. Expressing further t through s and the cosine of the scattering angle, we have[17] near $u=0$:

$$\cos\gamma\sim\frac{2s}{2(M^2+\mu^2)-u}\quad(s\to\infty).$$

Thus inserting into (5.10) we obtain finally

$$T^{(\pm)}(\gamma,M^2-\mu^2,M^2-\mu^2;u)\sim\left(\frac{2}{\pi}\right)^{1/2}\frac{-|g(M^2-\mu^2,u)|^2}{i}(1\pm e^{i\pi\alpha(u)})\times\cot(\alpha(u))\pi\left(\frac{2s}{2(M^2+\mu^2)-u}\right)^{\alpha(u)}. \quad (5.11)$$

In other words, the fact that the spectrum at (and, approximately, near to) $u=0$ exhibits the SO_4 symmetry automatically eliminates the ambiguity in the asymptotic behavior of the scattering amplitude. (Mathematically, this is due to the fact that $\cos\gamma$ tends to infinity uniformly in the neighborhood of $u=0$, whereas $\cos\theta$ does not.) Equation (5.11) shows that the usual analyticity properties in u (in particular, T should be regular at $u=0$) are compatible with a four-dimensional pole, but not with a single three-dimensional one.[4]

6. VIOLATION OF THE SYMMETRY: THE "MASS FORMULA"

The four-dimensional symmetry is exact at $u=0$ only. We ask now the question: How is the symmetry violated by the Regge trajectories at $u\neq0$? We see from (5.7) that in the symmetry limit the trajectories run parallel to each other at unit intervals. We shall restrict ourselves to the lowest nonvanishing order in the symmetry breaking and in most cases to the linear approximation in the Regge trajectories. In order to motivate the mass formula, we again return to the Bethe-Salpeter equation as discussed in Sec. 4.

If $u\neq0$, the kernel is, of course, still diagonal in l, although not in n. Therefore it is sufficient to consider the "partial" Fredholm determinants D_l; the implicit equation $D_l(u)=0$ determines the Regge trajectories. In the ladder approximation only the matrix elements of F contain the total energy. Remembering the asymptotic behavior of the functions $D_n(x)$, we see that near $u=0$ the expression (4.6) gives essentially a power-series expansion in powers of $\sqrt{u}$. More precisely, using the orthogonality relations of the Clebsch-Gordan coefficients, we find

$$\langle n''p''|F(l',u)|n'p'\rangle=\frac{\delta(p''-p')}{p'^3}\frac{1}{(p'^2+M^2)(p'^2+\mu^2)}\Bigg\{\delta_{nn'}+4(2\pi)^{3/2}p'\sqrt{(-u)}\sum_{n_i',n_i'',l_i'}\left[\frac{1}{p'^2+M^2}\delta_{n_2'n_2''}\times(n_1'l_1';1,0\|n_1''l_1')-\frac{1}{p'^2+\mu^2}\delta_{n_1'n_1''}(n_2'l_2';1,0\|n_2''l_2')\right]\begin{pmatrix}n_1' & n_2' & \| & n'\\ l_1' & l_2' & \| & l'\end{pmatrix}\begin{pmatrix}n_1'' & n_2'' & \| & n''\\ l_1' & l_2' & \| & l'\end{pmatrix}+\cdots\Bigg\}. \quad (6.1)$$

[16] M. L. Goldberger and C. E. Jones, Phys. Rev. Letters 17, 105 (1966); Phys. Rev. 150, 1269 (1966).
[17] We can put $\cos\theta=-1$ to leading order in s.

Thus the lowest-order symmetry-breaking term is a matrix element of a tensor operator with $n=1$ (i.e., of a vector). We now show that the lowest-order correction to the Regge trajectories vanishes. Again it is sufficient to treat the ladder approximation; the general proof follows the same pattern. First of all let us observe that if we rewrite (6.1) symbolically as

$$\langle n''p''|F(l,u)|n'p'\rangle = \delta(p''-p')[\delta_{nn'}F_0+\sqrt{(-u)}\langle n''|F_1|n'\rangle+\cdots],$$

as a consequence of the selection rule (2.14) or (2.14′) the nonvanishing matrix elements of F_1 are $\langle n'\pm 1|F_1|n'\rangle$; correspondingly, the kernel L of the Bethe-Salpeter equation has an expansion of the form [cf. (4.8)]

$$\langle n''|L(l,u)|n'\rangle = \delta_{n'n''}L_0(n)+\sqrt{(-u)}\langle n''|L_1|n'\rangle-u\langle n''|L_2|n'\rangle+\cdots$$

with the same selection rule. Inserting this into the expression (4.9) of the Fredholm determinant, the first-order correction turns out to be

$$D_l^{(1)}(u)=\sqrt{(-u)}\sum_n\left\{-\frac{\lambda}{1!}\int_0^\infty dk\, k^3\langle nk|L_1|nk\rangle+\frac{\lambda^2}{2!}\int_0^\infty k_1^3dk_1\int_0^\infty k_2^3dk_2[\langle k_1|L_0(n)|k_1\rangle \times\langle nk_2|L_1|nk_2\rangle-\langle k_1|L_0(n)|k_2\rangle\langle nk_2|L_1|nk_1\rangle]+\cdots\right\}.$$

This expression, however, vanishes identically, as L_1 does not have diagonal matrix elements. Using the form (2.14′) of the selection rule, we see that the result remains true if l is complex.

The reader should notice the analogy with the Stark effect, mentioned in the Introduction: In the absence of an "accidental" degeneracy, there is no first-order Stark effect; the reason is the same, viz., the selection rule governing electric dipole transitions. One can show[18] that if we take into account spin, then the first-order correction to the fermion Regge trajectories does not vanish, because of the doubling of states. At any rate it is reassuring to find that there is no first-order correction to the trajectories: It would give rise to a dependence $\sim\sqrt{(-u)}$, leading to an unwanted branch point at $u=0$.

Let us turn now to the second-order correction, i.e., we take into account terms up to $0(u)$. The selection rule (2.14) tells us that $\langle n''|L_2|n'\rangle$ has nonvanishing matrix elements for $n''=n'$, $n'\pm 2$. The Fredholm determinant D_l now couples different values of n; up to order u the nonvanishing elements are on the main diagonal and along two lines parallel to it. Thus the structure of D_l is the following:

$$D_l=\begin{vmatrix} & \cdots & \cdots & \cdots & \cdots & \\ \cdots & D(n-2)-u\langle n-2|L_2|n-2\rangle & (-u)^{1/2}\langle n-2|L_1|n-1\rangle & -u\langle n-2|L_2|n\rangle & 0 & \cdots \\ \cdots & (-u)^{1/2}\langle n-1|L_1|n-2\rangle & D(n-1)-u\langle n-1|L_2|n-1\rangle & (-u)^{1/2}\langle n-1|L_1|n\rangle & -u\langle n-1|L_2|n+1\rangle & \cdots \\ \cdots & -u\langle n|L_2|n-2\rangle & (-u)^{1/2}\langle n|L_1|n-1\rangle & D(n)-u\langle n|L_2|n\rangle & (-u)^{1/2}\langle n|L_1|n+1\rangle & \cdots \\ \cdots & 0 & -u\langle n+1|L_2|n-1\rangle & (-u)^{1/2}\langle n+1|L_1|n\rangle & D(n+1)-u\langle n+1|L_2|n+1\rangle & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \end{vmatrix}. \quad (6.2)$$

Here $D(n)$ has the same meaning as in Sec. 4. Now to exhibit a trajectory, we write: $D(n)\equiv D(l,\kappa)=\bar{D}(l+\kappa-\alpha_0)$, exhibiting the zero intercept of the trajectory α. The factor $\bar{D}$ is finite if we assume that the poles of the amplitude are simple ones. Inserting into (6.2), we have a secular determinant for the trajectory $\alpha_\kappa(u)$ in the linear approximation. The solution to order u is the following:

$$\alpha_\kappa(u)=\alpha_0-\kappa-\frac{u}{\bar{D}}\langle\alpha_0+\kappa|L_2|\alpha_0+\kappa\rangle-\frac{u}{\bar{D}^2\alpha_0}\times\sum_{\kappa'=\kappa\pm1}\langle\alpha_0+\kappa|L_1|\alpha_0+\kappa'\rangle\langle\alpha_0+\kappa'|L_1|\alpha_0+\kappa\rangle. \quad (6.3)$$

(The off-diagonal elements of L_2 do not contribute to order u.) The second term is just the diagonal matrix element of the product $L_1\cdot L_1$. In the notation (n,j_0) for the I.R. of SO_4, L_1 is a tensor operator $\sim(1,0)$; thus according to the reduction formula

$$(1,0)\otimes(1,0)=(0,0)\oplus(2,0),$$

the second term is the sum of a scalar and a tensor term, the latter transforming in the same way as L_2. Thus we can rewrite the expression (6.3) as follows:

$$\alpha_\kappa(u)=\alpha_0+\alpha_1u-\kappa+\beta_1u\begin{pmatrix}\alpha_0+\kappa & 2 \| \alpha_0+\kappa\\ \alpha_0 & 0 \| \alpha_0\end{pmatrix}.$$

We have used the Wigner-Eckart theorem to write the matrix element of an irreducible tensor operator as a reduced matrix element (β_1) times a CGC, and discarded the three-dimensional part of the CGC. Furthermore we replaced n by $\alpha_0+\kappa$, l by α_0, which is correct to the

[18] G. Domokos (to be published).

order in u considered. α_1 is the reduced matrix element of the scalar operator divided by $\alpha_0\bar{D}^2$.

It is slightly more convenient to use Eq. (2.13) expressing the matrix element of a spherical harmonic directly. We find

$$(nl;20\|nl)=\left(\frac{2}{\pi}\right)^{1/2}\left[1-2\frac{l(l+1)}{n(n+2)}\right],$$

so that introducing some other convenient constants, the expression of the trajectory α_κ finally becomes

$$\alpha_\kappa(u)=\alpha_0-\kappa+\gamma u+\epsilon(\alpha_0-\kappa)(\alpha_0-\kappa+1)u. \quad (6.4)$$

We recognize that the first three terms in Eq. (6.4) give the trajectory $\alpha_\kappa(u)$ in the symmetry limit and linear approximation in u, while the term proportional to ϵ breaks the symmetry. We now argue that the form (6.4) is the most general expression of a trajectory if we restrict ourselves to the lowest order in the symmetry breaking. In fact, as ordinary rotational symmetry is not violated, the only quantity available which breaks the SO_4 symmetry, but not the rotational invariance, is the Casimir operator of the rotation group, i.e., $l(l+1)\mathbf{1}$ for the representations considered—and this consideration leads immediately to (6.4). The reader should notice the analogy between these considerations and those leading to the Gell-Mann–Okubo mass formula, the SU_6 mass formula,[19] or the general formula for the Stark splitting of spectral lines.[20] It is interesting to notice that if α_0 is an integer, the first-order symmetry breaking vanishes for some of the daughters. In particular, the observable daughter of Pomeranchuk is symmetric to first order in u. Let us emphasize the fact that the parameter ϵ characterizing the symmetry breaking is common for the whole family of trajectories generated by one four-dimensional parent trajectory, $\alpha_0(u)$.

7. PARTICLE SPECTRUM

What are the implications of the four-dimensional symmetry on the spectrum of "elementary" particles? In order to get a qualitative insight into the problem, let us neglect the symmetry-breaking term in (6.4). In that approximation the family of Regge trajectories consists of an infinite "bunch" of trajectories, running parallel to each other at integer intervals.

All the internal quantum numbers including parity and C or G parity are the same for the whole family of trajectories. This is a straightforward consequence of the fact that the complete symmetry group of the scattering matrix element at a fixed total four-momentum is a direct product of the little group of the Poincaré group with the group of reflections and internal symmetries.

There is, however, an important difference between the trajectories with even and odd κ. The signature of the odd-κ trajectories is opposite to the parent ($\kappa=0$) and its daughters with even κ. Thus if the parent develops a pole in the amplitude (5.6) at $\alpha_0(u)=s$, $s+2,\cdots$, the first odd daughter gives poles at $\alpha_1(u)=s+1, s+3, \cdots$ and so on. Moreover, as is evident from Sec. 5 the odd daughters are not coupled to a two-particle channel containing particles of equal mass at physical values of the angular momentum. In a Bethe-Salpeter theory the odd-κ solutions do not seem to have a nonrelativistic limit and are not normalizable in the usual sense. (For a recent work on this subject with references to earlier results see Ref. 21.)

In the *symmetry limit* it is immediately evident from the previous considerations that the trajectories of *odd* order give rise to ghosts. Let us consider elastic scattering; using the formula $C_n^\lambda(x)=(-1)^n C_n^\lambda(-x)$, valid for integer n, we have from (5.6) the contribution of the pole α_κ

$$T^{(\pm)}\sim|G(z,u)|^2\frac{\kappa!(\sin^2\beta)^{\alpha_\kappa}}{4^\kappa}\frac{1\pm(-1)^\kappa e^{i\pi\alpha_\kappa}}{2i}\times\cot((\kappa+\alpha_\kappa)\pi)[\Gamma(\alpha_\kappa+1)]^2(-1)^\kappa\times[C_\kappa^{\alpha_\kappa+1}(\cos\beta)]^2P_{\alpha_\kappa}(\cos\theta). \quad (7.1)$$

We introduced the abbreviation

$$G(z,u)=\frac{2^{1/2}g(z,u)}{[\Gamma(\alpha_0(u)+1)\Gamma(\alpha_0(u)+3/2)]^{1/2}}.$$

Now let us suppose that at some value of u, say u_N, the parent trajectory passes through an integer: $\alpha_0(u_N)=N$. (For the sake of simplicity we treat the case of a stable bound state; resonances can be treated in the same way.)

Expanding the trajectory

$$\alpha_0(u)=N+\alpha'(u-u_N)+\cdots,$$

($\alpha'>0$), we obtain the pole contribution in the partial wave $l_0=N-\kappa$

$$T_{l_0,\kappa}{}^{(\pm)}\sim\frac{-i(-1)^\kappa}{\pi\alpha'}\times\frac{|G|^2\kappa!(\sin^2\beta)^{l_0}[l_0!]^2[C_\kappa^{l_0+1}(\cos\beta)]^2}{u_N-u}. \quad (7.2)$$

Here the superscript ($\pm$) denotes the "four-dimensional signature" of the amplitude, so that $N=2k$ for $T^{(+)}$,

[19] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964).

[20] See, e.g., L. D. Landau and E. M. Lifschitz, *Quantum Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), Chap. X.

[21] M. Ciafaloni and P. Menotti, Phys. Rev. **140**, B929 (1965).

$N=2k+1$ for $T^{(-)}$, so we have the result stated previously:

Bound states and/or resonances appear

$$\text{in } T^{(+)}: \quad \text{for } l_0 \begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix} \text{ if } \kappa \text{ is } \begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix},$$

$$\text{in } T^{(-)}: \quad \text{for } l_0 \begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix} \text{ if } \kappa \text{ is } \begin{pmatrix}\text{odd}\\ \text{even}\end{pmatrix}.$$

However, the residues are proportional to $(-1)^\kappa$ times a positive definite quantity. Thus the odd-κ poles in the physical region violate unitarity, so evidently must be compensated by some, so far unknown, mechanism.[22]

We thus suggest that the odd daughters do not give rise to observable particles although they play a role in the crossed channel. Let us concentrate instead on the low-order even daughters, in particular, $\kappa=2$. They have the same signature as the parent and in the symmetry limit run parallel to it. As soon as they appear on the right half of the angular momentum plane, they should give rise to observable particles. It follows in particular, that every resonance with sufficiently high orbital momentum (starting with $l=2$) should be accompanied by lower spin resonances close in mass to the "parent" (if the symmetry breaking is weak). A quantitative prediction of the theory (again in the symmetry limit) is a simple expression, immediately following from (7.2) and relating the coupling strengths (decay widths, if the particle is unstable) to the corresponding quantity of the parent pole. If the parent gives a resonance in $l=L_0$, then the "daughter resonances" occur at $l=l_0=L_0-2$, $L_0-4, \cdots, 0$, and we find for the ratio of the widths:

$$R\equiv\frac{\Gamma_\kappa}{\Gamma_0}=\frac{\begin{pmatrix}2n+1\\ \kappa\end{pmatrix}\left(\dfrac{16q_0^2}{u_0-2(M^2+\mu^2)}\right)^{-\kappa}}{\left[\begin{pmatrix}n\\ \kappa\end{pmatrix}\right]^2} \times C_\kappa^{n-\kappa+1}\left(\frac{M^2-\mu^2}{[u_0(2(M^2+\mu^2)-u_0)]^{1/2}}\right). \qquad (7.3)$$

Here q_0, u_0, $L_0=n$ are the c.m. momentum, mass squared, and orbital momentum of the parent resonance. Let us recognize that the first factor is proportional to the ratio of the barrier penetration factors.

As an illustration of the predictions made by the four-dimensional symmetry,[23] let us ask the question: Does the N_α trajectory produce observable daughter particles? The internal quantum numbers are: $P=+1$, $I=\frac{1}{2}$, (three-dimensional) signature $=+1$. The usual

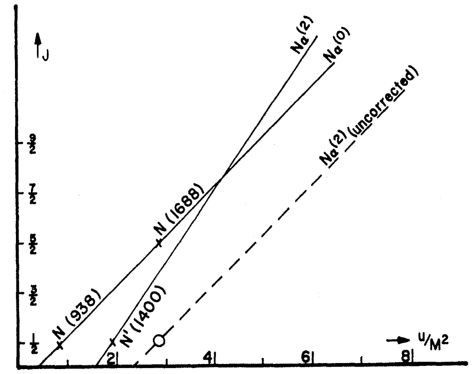

FIG. 2. The $N_\alpha^{(0)}$-trajectory and its first even daughter $N_\alpha^{(2)}$. The superscript indicates the value of κ. The expected position of $N'(1400)$ is indicated by a circle, while the observed one by a cross. The dashed line for $N_\alpha^{(2)}$ shows the trajectory with $\epsilon=0$, while the $N_\alpha^{(2)}$ trajectory corrected for the symmetry breaking is drawn with a full line.

straight-line fit to N_α (Chew-Frautschi plot) is shown in Fig. 2, together with the expected first even daughter trajectory (dashed line) ($\kappa=2$) in the symmetry limit [$\epsilon=0$ in Eq. (6.4)]. We write the value of κ as a superscript to the symbol of each trajectory. A resonant state on $N_\alpha^{(0)}$ is the $N_\alpha(1688)$ with $j=\frac{5}{2}$; in the spectroscopic notation $(L_{2I,2J}{}^P)$ it is $F_{15}{}^+$. Thus (if $\epsilon=0$) we expect a $P_{11}{}^+$ resonance lying on $N_\alpha^{(2)}$ at the same mass and internal quantum numbers. In fact a resonance with these quantum numbers has been observed,[24] with a mass around 1400 MeV. Can we associate $N'(1400)$ with $N_\alpha^{(2)}$? According to its quantum numbers, it cannot lie on other known trajectories (N_γ,Δ_δ), except $N_\alpha^{(0)}$ itself, but then it is hard to understand the "accidental" doubling of the nucleon. If we associate $N'(1400)$ with the lowest physical state along $N_\alpha^{(2)}$, knowing the parameters of the parent trajectory ($\alpha_0\approx-0.37$, $\gamma\approx1.02$ in units of the nucleon mass squared), and using the known mass (1400 MeV) we find from (6.4) that the symmetry-breaking parameter is indeed small: $\epsilon\approx-0.05$ in the same units, i.e., about 5% of the slope in the symmetry limit. (The trajectory $N_\alpha^{(2)}$ corrected for the symmetry breaking is drawn in Fig. 2 with a full line.) Thus it is not unreasonable to assign $N'(1400)$ to $N_\alpha^{(2)}$. If we do so, Eq. (7.3) relates its width in the πN channel to the width of its "parent" $N(1688)$. Numerically we find from Eq. (7.3):

$$R\approx1.1.$$

This is to be compared with the same ratio deduced from the (rather inaccurate) experimental data,[24] giving

$$\left(\frac{M\Gamma_{el}(N'(1400))}{M\Gamma_{el}(N(1688))}\right)_{\text{exp}}\approx1.5.$$

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[22] There is no such trouble in the equal-mass case, because the residues of the odd-κ poles vanish.

[23] A more detailed analysis is given in G. Domokos, Phys. Letters **24B**, 293 (1967).

[24] A. H. Rosenfeld, A. Barbaro-Galtieri, W. J. Podolsky, L. R. Price, M. Roos, P. Soding, W. J. Willis, and C. G. Wohl, University of California Radiation Laboratory Report No. UCRL-8030 (Rev.), 1967 (unpublished).

Thus the agreement between the prediction and the experimental result is fair (28%). [In Eq. (7.3) we inserted the mass of the parent particle, consistently with our assumption that the symmetry breaking is small.] It is to be emphasized that this ratio cannot be obtained from other "accepted" symmetry schemes.

It is worth remarking that the $N_\alpha^{(2)}$ trajectory, corrected for the symmetry breaking, intersects its parent at about $u \approx 4.5M^2$. This should give rise to a characteristic anomaly (corresponding to a double pole) in the $j=\frac{9}{2}^+$ pion-nucleon phase shift. We just mention that in the meson system the A_2 trajectory has the best chance to produce an observable daughter state. It should be a 0^+ octet, with a mass ≈ 1300 MeV. There seem to be "bumps" around this mass value[24] and probably they have spin zero, but at present a detailed analysis is hardly feasible.

8. CONCLUSION

Summarizing, we have shown that:

(a) A four-dimensional symmetry is a meaningful concept for scattering amplitudes. In the scattering of particles of unequal mass the symmetry is intrinsically broken by the mass differences, even at $u=0$, but still has a meaning for the spectrum.

(b) The most important prediction of the symmetry is that Regge trajectories occur in families; each one of the known top-ranking trajectories is accompanied by an infinite series of "daughters"; the daughters follow their parents at integer intervals at $u=0$. The members of each family can be labeled by a quantum number following from the existence of the higher symmetry.

(c) On the basis of the group-theoretical formalism we deduced a one-parameter formula describing the deviation of the slopes of Regge trajectories from the value required by the symmetry. Experimentally, the symmetry breaking in the πN system seems to be reasonably small, so the daughter trajectories should give rise to observable particles with specified quantum numbers and masses comparable to the parent particles. The symmetry predicts the coupling strength of a daughter in terms of its parent's vertex function.

(d) The higher symmetry resolves the long-standing problem of the asymptotic behavior of the unequal-mass scattering; thus we have verified the conjecture of Freedman and Wang.[4]

Let us finally reemphasize two points.

(1) As we already mentioned in the Introduction, the symmetry treated in this paper is entirely different in nature from other known symmetries. It is neither an internal symmetry nor a purely kinematical one, but rather seems to be the result of a delicate interplay between the (*exact*) kinematical symmetry and the analytic properties of a scattering amplitude.

(2) The second remark concerns dynamics. The usual approximate S-matrix dynamical equations fail to produce the four-dimensional symmetry. It is reasonable to expect that by improving the approximation (e.g., by including many-particle states, etc.) one will be able to produce the symmetry in the formalism of the analytic S-matrix theory as well.[25] In fact, loosely speaking, the continuation of a scattering amplitude off the mass shell probes its multiparticle structure (as is by now evident from the example of *ep* scattering). The Bethe-Salpeter equation does not seem to offer a satisfactory solution to the problem, because (apart from computational problems) it fails to account for the fact that the hadrons are "Reggeized," i.e., going "off the mass shell" their spin changes along with the mass. Nevertheless, from the point of view of experimenting with models, the Bethe-Salpeter equation seems to be better than a nonrelativistic Schrödinger equation in that the former has at least the same symmetry group as a full relativistic theory.

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[25] This has been particularly emphasized by Chew [G. F. Chew (private communication)].