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Dispersion Relations for Three-Particle Scattering Amplitudes. II*

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We continue our discussion of the scattering of three nonrelativistic spinless particles interacting via two-body Yukawa potentials. The on-energy-shell T matrix is studied as a function of the total center-of-mass energy E for fixed physical values of the vectors $\mathbf{y}_i=\mathbf{k}_i(2m_iE)^{-1/2}$, $\mathbf{y}_i'=\mathbf{k}_i'(2m_iE)^{-1/2}$, $i=1,2,3$. Here $\mathbf{k}_i$ and $\mathbf{k}_i'$ are the initial and final momenta of the particles, respectively, and m_i are the masses. We show that $T(E)$ can be written as the ratio of two Fredholm series, each of which is uniformly convergent with respect to E for all values of E on the physical sheet including the real axis. Since we have previously seen that each term in these series satisfies a dispersion relation in E with no complex singularities, it follws that the full three-particle amplitude satisfies such a dispersion relation.

I. INTRODUCTION

IN part I of this work[1] we studied the scattering amplitude for three free, nonrelativistic particles interacting via two-body Yukawa potentials. We wrote the on-energy-shell amplitude as a ratio of two Fredholm series and showed that each term in these series satisfied a dispersion relation in the total center-of-mass energy E for fixed physical values of the vectors

$$\mathbf{y}_i=(2m_iE)^{-\frac{1}{2}}\mathbf{k}_i,$$
$$\mathbf{y}_i'=(2m_iE)^{-\frac{1}{2}}\mathbf{k}_i'. \quad i=1,2,3 \tag{1.1}$$

m_i is the mass of the ith particle and $\mathbf{k}_i$ and $\mathbf{k}_i'$ are its initial and final center-of-mass momenta. We then argued that since the Fredholm series are uniformly convergent, the full three-particle scattering amplitude satisfies the same dispersion relation as the individual terms in the series. In the present paper we shall give detailed proof of the convergence of the Fredholm series for the on-energy-shell amplitude for all values of E on the physical sheet including the real axis. We shall always keep the $\mathbf{y}$ vectors fixed and physical.

In order to simplify our proofs we shall only consider the case of simple, two-body Yukawa potentials. However, all of our results would remain valid for a super-position of Yukawa potentials.

Faddeev has previously shown that the on-energy-shell amplitude exists for real energies for a much wider class of potentials than we shall consider.[2] However, in order to complete the proof of our dispersion relation, we need to show that the amplitude exists for complex values of E. In addition, we think that it is worthwhile to present our existence proof for real E since it appears to be much simpler than Faddeev's. The simplicity arises from the fact that we are dealing with a class of potentials that is analytic in the momenta. As a result, we can distort the contours of integration away from the singularities in the Green's functions and thus get simple bounds on the integrals which occur in the problem. This procedure was not available to Faddeev because he did not make any assumptions of analyticity for his potentials.

Since we have been able to show that the Faddeev equations have a Fredholm solutions even for real energies, all of the powerful tools for the numerical and theoretical analysis of Fredholm integral equations can now be applied to the three-body problem.

In Sec. II we shall briefly discuss the two-body problem in order to present our techniques in a familiar setting, and to collect results which will be needed in our discussion of the three-body problem. In Sec. III we show that the on-energy-shell three-body amplitude exists in the entire upper half of the $k=E^{1/2}$ plane, including the real axis, with the possible exception of the imaginary axis for $\mathrm{Im}k\geq(\sqrt{\frac{3}{2}})\mu$, which is the location of the left-hand cut.

Finally, in the Appendices, we give the details of obtaining bounds on the integrals that arise in the text,

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[1] M. Rubin, R. Sugar, and G. Tiktopoulos, Phys. Rev. **146**, 1130 (1966). Hereafter referred to as I.

[2] L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in the Quantum scattering Theory* (Publications of the Stoklov Mathematical Institute No. 69, 1963. (English transl.: Israel Program for Scientific Translations, Jerusalem, Israel, 1965.)

and we present detailed proofs of the existence of the various amplitudes at their threshold energies.

II. BOUNDS OF THE TWO-BODY AMPLITUDE

In this section we shall consider the scattering amplitude for two scalar particles interacting by a Yukawa potential,

$$\langle \mathbf{p}|V|\mathbf{p}'\rangle = V(\mathbf{p},\mathbf{p}') = g[(\mathbf{p}-\mathbf{p}')^2+\mu^2]^{-1}. \quad (2.1)$$

Our starting point is the Lippmann-Schwinger equation for the off-energy-shell scattering amplitude

$$t(\mathbf{p},\mathbf{p}';k^2) = V(\mathbf{p},\mathbf{p}') + \int d^3q\, V(\mathbf{p},\mathbf{q})(q^2-k^2)^{-1} t(\mathbf{q},\mathbf{p}';k^2)$$
$$= V + VG_0 t. \quad (2.2)$$

We are using units in which $\hbar = 2m = 1$ (m is the reduced mass), and we shall consistently neglect numerical factors such as coupling constants and 2π's. We wish to show that the on-energy-shell amplitude, $t(k\hat{n},k\hat{n}';k^2)$ exists in the upper-half k plane, $\mathrm{Im}k \geq 0$, and that the off-energy-shell amplitude is a square-integrable function of either of its momenta in this region.

The solution to Eq. (2.2) can be written formally as

$$t = V + RV = V + VG_0V + VG_0RV, \quad (2.3)$$

where the resolvent R is given by

$$R = VG_0 + VG_0R = VG_0 + RVG_0. \quad (2.4)$$

It is well known that if the kernal VG_0 is square integrable, i.e., if

$$\|G_0V\|^2 = \mathrm{tr}[G_0VVG_0^\dagger]$$
$$= \int d^3q d^3q' [(\mathbf{q}-\mathbf{q}')^2+\mu^2]^{-2} |q^2-k^2|^{-2} < \infty, \quad (2.5)$$

then R will have a Fredholm solution which can be written in the form[3]

$$R = N/D,$$
$$N = \sum_{i=0}^{\infty} N_i, \quad D = \sum_{i=0}^{\infty} D_i. \quad (2.6)$$

The D_i are numbers which depend only on k^2, and the N_i are square-integral operators. Since

$$\|G_0V\|^2 = \pi^4/\mu\mathrm{Im}k, \quad (2.7)$$

R will exist in the half plane $\mathrm{Im}k \geq \epsilon > 0$. In this region the series for D will be uniformly convergent in k. In addition, the series for N will be a relatively, uniformly, absolutely convergent series with respect to k in any part of the upper half plane, and N will be a square integrable operator.[3] This means that if we take matrix elements of each term in the series for N between square integrable functions, the resulting series will be uniformly convergent.

From Eq. (2.3) we see that the on-energy-shell amplitude can be written in the form

$$t(k\hat{n},k\hat{n}';k^2) = \langle k\hat{n}|V|k\hat{n}'\rangle + \langle k\hat{n}|VG_0V|k\hat{n}'\rangle$$
$$+ \sum_{i=0}^{\infty} \langle k\hat{n}|VG_0N_iV|k\hat{n}'\rangle / \sum_{j=0}^{\infty} D_j, \quad (2.8)$$

where $\hat{n}'$ and $\hat{n}$ are unit vectors in the direction of the initial and final momenta, respectively. But[4]

$$\|V|k\hat{n}'\rangle\|^2 = \int d^3q |(\mathbf{q}-k\hat{n}')^2+\mu^2|^{-2}$$
$$\leq \pi^2[\mu^2-(\mathrm{Im}k)^2]^{-1/2}, \quad \mu \geq \mathrm{Im}k \quad (2.9)$$

and

$$\|G_0V|k\hat{n}\rangle\|^2 = \int d^3q |q^2-k^2|^{-2} |(q-k\hat{n})^2+\mu^2|^{-2}$$
$$\leq \pi^2(\mathrm{Im}k)^{-1}[\mu^2-(\mathrm{Im}k)^2]^{-2}. \quad \mu \geq \mathrm{Im}k.$$

As a result, the on-energy-shell amplitude exists and is given by the ratio of two uniformly convergent series in the strip

$$\mu - \epsilon \geq \mathrm{Im}k \geq \epsilon, \quad \epsilon > 0. \quad (2.10)$$

In order to show that the on-energy-shell amplitude exists in the entire upper-half k plane and on the real k axis, it is convenient to introduce an operator which is defined by formally rotating the contours of integration in Eq. (2.2)

$$t_\theta(\mathbf{p},\mathbf{p}';k^2)$$
$$= g[(\mathbf{p}-\mathbf{p}')^2 e^{2i\theta}+\mu^2]^{-1} + \int d^3q [(\mathbf{p}-\mathbf{q})^2 e^{2i\theta}+\mu^2]^{-1}$$
$$\times e^{3i\theta}(q^2e^{2i\theta}-k^2)^{-1} t_\theta(\mathbf{q},\mathbf{p}';k^2)$$
$$= V_\theta + V_\theta G_\theta t_\theta. \quad (2.11)$$

In analogy with Eq. (2.3) we can write the solution to Eq. (2.11) in the form

$$t_\theta = V_\theta + V_\theta G_\theta V_\theta + V_\theta G_\theta R_\theta V_\theta,$$
$$R_\theta + V_\theta G_\theta + V_\theta G_\theta R_\theta = N_\theta / D_\theta. \quad (2.12)$$

Now[4]

$$\|V_\theta G_\theta\|^2$$
$$= \int d^3q d^3q' |(\mathbf{q}-\mathbf{q}')^2 e^{2i\theta}+\mu^2|^{-2} |q^2e^{2i\theta}-k^2|^{-2}$$
$$\leq C_1[\mu \cos^2\theta\ \mathrm{Im}(ke^{-i\theta})]^{-1},$$

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[3] F. Smithies, *Integral Equations* (Cambridge University Press, Cambridge, England, 1958). The explicit form of N_i and D_i for the two-body problem is given in Ref. 1.

[4] The details of obtaining bounds on the integrals that arise in this section are given in Appendix I.

$$\|V_\theta|k\hat{n}\rangle\|^2 = \int d^3q\,|(\mathbf{q}e^{i\theta}-k\hat{n})^2+\mu^2|^{-2} \leq C_2[\mu^2\cos^2\theta-(\mathrm{Im}(ke^{-i\theta}))^2]^{-1}, \tag{2.13}$$

$$\|G_\theta Vk\hat{n}\rangle\|^2 = \int d^3q\,|q^2e^{2i\theta}-k^2|^{-2}|(\mathbf{q}e^{i\theta}-k\hat{n})^2+\mu^2|^{-2} \leq C_3\{\mathrm{Im}(ke^{-i\theta})[\mu^2\cos^2\theta-(\mathrm{Im}(ke^{-i\theta}))^2]^2\}^{-1},$$

where C_1, C_2 and C_3 are constants. It is clear from our previous arguments that the on-energy-shell quantity $t_\theta(ke^{-i\theta}\hat{n},\ ke^{-i\theta}\hat{n}',\ k^2)$ exists and is given by the ratio of two uniformly convergent series in the strip

$$\mu\cos\theta-\epsilon\geq\mathrm{Im}(ke^{-i\theta})\geq\epsilon,\quad \epsilon>0. \tag{2.14}$$

This strip and the one defined by Eq. (2.10) are shown in Fig. 1. Note that for $-\pi/2<\theta<\pi/2$ there is always a region of overlap of the two strips.

It was shown in I that if k is held fixed in the region of overlap, then for any term in the series for D or $\langle k\hat{n}|VG_0NV|k\hat{n}'\rangle$, one can simultaneously rotate all of the contours of integration through an angle θ without crossing any singularities of the integrand. As a result, in the region of overlap

$$D_i=D_{\theta i},\quad i=0,1,2,\cdots$$
$$\langle k\hat{n}|VG_0N_iV|k\hat{n}'\rangle=\langle k\hat{n}|V_\theta G_\theta N_{\theta i}V_\theta|k\hat{n}'\rangle. \tag{2.15}$$

Since the series are all uniformly convergent, it follows that $t_\theta(ke^{-i\theta}\hat{n},\ ke^{-i\theta}\hat{n}';\ k^2)$ is the analytic continuation of $t(k\hat{n},k\hat{n}';k^2)$. By rotating θ through the angles

$$-\pi/2+\epsilon\leq\theta\leq\pi/2-\epsilon\quad \epsilon>0, \tag{2.16}$$

we see that the on-energy-shell amplitude exists in the entire upper half k plane including the real axis, with the possible exception of that part of the imaginary axis for which $\mathrm{Im}k\geq\mu$, and the point $k=0$. The limit $k\to0$ is discussed in detail in Appendix II.

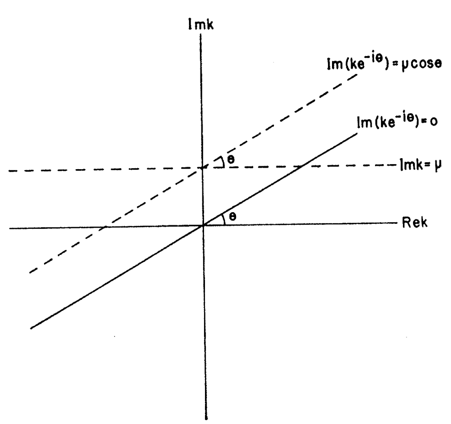

FIG. 1. The regions in which t and t_θ are given by the ratio of two Fredholm series.

We have gone into considerable detail in providing the existence of the on-energy-shell two-body amplitude in order to present the techniques which we will use in the three-body problem in a familiar setting. In the course of our work on the three-body problem we shall need explicit bounds on off-energy-shell, rather than on-energy-shell, two-body amplitudes. We shall collect these results here for future reference. We first note that the off-energy shell amplitude, $t(\mathbf{p},\mathbf{p}',k^2)$, is a square-integrable (L^2) function of $\mathbf{p}$ ($\mathbf{p}'$) in the half plane $\mathrm{Im}k\geq\epsilon>0$. The norm can be bounded independent of $\mathbf{p}'$ ($\mathbf{p}$).

$$\int d^3p\,|t(\mathbf{p},\mathbf{p}';k^2)|^2 = \int d^3p\,|V(\mathbf{p},\mathbf{p}')+RV(\mathbf{p},\mathbf{p}')|^2\leq2\int d^3p\,|V(\mathbf{p},\mathbf{p}')|^2 +2\int d^3p\,|RV(\mathbf{p},\mathbf{p}')|^2\leq\frac{2\pi^2}{\mu}[1+\|R\|^2]. \tag{2.17}$$

In the last line of Eq. (2.17) we have made use of the Schwartz inequality in the form

$$\|RV|\mathbf{p}'\rangle\|^2\leq\|R\|^2\|V|\mathbf{p}'\rangle\|^2. \tag{2.18}$$

In Ref. 3 it is shown that

$$\|N\|^2\leq\|VG_0\|^2\exp[1+\|VG_0\|^2] = (\pi^4/\mu\,\mathrm{Im}k)\exp[1+\pi^4/\mu\,\mathrm{Im}k]. \tag{2.19}$$

In the region in which VG_0 is square integrable the only singularities in $D^{-1}(k^2)$ are poles which occur at solutions of the homogeneous equations,[3] i.e., at bound-state energies. Since $D\to1$ as $k^2\to\infty$, $D^{-1}(k^2)$ can be bounded by a constant if there are no bound states, and by the expression

$$|D^{-1}(k^2)|\leq C_0+\sum_i\left|\frac{C_i}{k^2+B_i}\right|\equiv D_B^{-1}(k^2), \tag{2.20}$$

if there are. The C_i's are constants and the B_i's are the binding energies. As a result

$$\int d^3p\,|t(\mathbf{p},\mathbf{p}';k^2)|^2\leq\frac{2\pi^2}{\mu^2}[1+K_1D_B^{-2}(k^2)],\quad \mathrm{Im}k\geq\epsilon>0. \tag{2.21}$$

K_1 is a constant independent of k or $\mathbf{p}'$. Similarly

$$\int d^3p'\,|t(\mathbf{p},\mathbf{p}';k^2)|^2\leq\frac{2\pi^2}{\mu^2}[1+K_1D_B^{-2}(k^2)]. \tag{2.22}$$

A similar result holds for the half-on-energy-shell amplitude $t(\mathbf{p},k\hat{n};k^2)$ in the strip defined by Eq. (2.10).

$$\int d^3p\,|t(\mathbf{p},k\hat{n};k^2)|^2 \leq 2\|V|k\hat{n}\rangle\|^2[1+\|R\|^2] \leq K_2[1+K_1 D_B^{-2}(k^2)], \quad (2.23)$$

where K_2 is a constant. In general

$$\int d^3p\,|t(\mathbf{p},\mathbf{q},k^2)|^2 \leq K_2[1+K_1 D_B^{-2}(k^2)] \quad (2.24)$$

in the strip $\mu-\epsilon \geq \mathrm{Im}|\mathbf{q}| \geq \epsilon$.

In addition to being an L^2 function of $\mathbf{p}$ or $\mathbf{p}'$, $t(\mathbf{p},\mathbf{p}';k^2)$ can be bounded independent of $\mathbf{p}$ or $\mathbf{p}'$ in the half plane $\mathrm{Im}k \geq \epsilon \geq 0$.

$$\begin{aligned}|t(\mathbf{p},\mathbf{p}';k^2)|^2 &= |V(\mathbf{p},\mathbf{p}')+VG_0V(\mathbf{p},\mathbf{p}')+VG_0RV(\mathbf{p},\mathbf{p}')|^2\\ &\leq 2|VG_0RV|^2+2|V+VG_0V|^2\\ &\leq 2\|R\|^2\|V|\mathbf{p}'\rangle\|^2\|G_0V|\mathbf{p}\rangle\|^2+4|V|^2+4|VG_0V|^2\\ &\leq K_3+K_4D_B^{-2}(k^2). \quad (2.25)\end{aligned}$$

K_3 and K_4 are constants. Similarly, in the strip $\mu-\epsilon \geq \mathrm{Im}k \geq \epsilon$

$$|t(\mathbf{p},k\hat{n};k^2)|^2 \leq K_5+K_5 D_B^{-2}(k^2). \quad (2.26)$$

The same techniques can be used to obtain bounds on the amplitudes on the rotated contours. For example,

$$\int d^3p\,|t_\theta(\mathbf{p},\mathbf{p}';k^2)|^2 \leq \frac{2\pi^2}{\mu\cos\theta}[1+K_{\theta 1}D_B^{-2}(k^2)], \quad \mathrm{Im}(ke^{-i\theta}) \geq \epsilon > 0. \quad (2.27)$$

In Appendix II we extend all of these bounds to include the limit $k \to 0$.

III. THE THREE-BODY AMPLITUDE

In this section we shall consider the scattering amplitude for three free particles interacting via two-body Yukawa potentials. In order to simplify the kinematics we shall only consider the case of equal mass particles, but the generalization of the proof to arbitrary masses will be obvious. We shall always work in the three-body center-of-mass system, so the vectors defined in Eq. (1.1) will satisfy the relations

$$\sum_{i=1}^{3}\mathbf{y}_i = \sum_{i=1}^{3}\mathbf{y}_i' = 0,$$
$$\sum_{i=1}^{3} y_i^2 = \sum_{i=1}^{3} y_i'^2 = 1, \quad (3.1)$$
$$y_i^2,\quad y_i'^2 \leq \tfrac{2}{3},\quad i=1,2,3.$$

We are using units in which $\hbar = 2m = 1$.

Our starting point is the Faddeev equation

$$f = \hat{t} + Kf, \quad (3.2)$$

where

$$f = \begin{pmatrix} f_{12}\\ f_{23}\\ f_{31}\end{pmatrix},\quad \hat{t} = \begin{pmatrix}\hat{t}_{12}\\ \hat{t}_{23}\\ \hat{t}_{31}\end{pmatrix},$$

$$K = \begin{pmatrix} 0 & \hat{t}_{12}G_0 & \hat{t}_{12}G_0\\ \hat{t}_{12}G_0 & 0 & \hat{t}_{23}G_0\\ \hat{t}_{31}G_0 & \hat{t}_{31}G_0 & 0\end{pmatrix}.$$

$\hat{t}_{ij}(k^2)$ are operators in the three-particle Hilbert space which are related to the two-particle off-energy-shell amplitudes by

$$\langle \mathbf{q}_i\mathbf{q}_j\mathbf{q}_k|\,t_{ij}(k^2)\,|\mathbf{q}_i'\mathbf{q}_j'\mathbf{q}_k'\rangle = t_{ij}(\tfrac{1}{2}(\mathbf{q}_i-\mathbf{q}_j),\tfrac{1}{2}(\mathbf{q}_i'-\mathbf{q}_j');k^2-\tfrac{3}{2}q_k^2) \times\delta^3(\mathbf{q}_k-\mathbf{q}_k'). \quad (3.3)$$

G_0 is the three-particle free Green's function. We have factored the momentum-conservation delta function out of the amplitude, so in any intermediate state there are only two independent momenta. Since we are in the center-of-mass system,

$$\sum_{i=1}^{3}\mathbf{q}_i = 0. \quad (3.4)$$

Since all of the elements in K are disconnected, it cannot be a square-integrable kernel. We therefore iterate Eq. (3.1) and write

$$\begin{aligned} f &= (1+K)\hat{t}+K^2 f\\ &= \sum_{n=0}^{7} K^n\hat{t}+K^4\mathcal{R}(K^2+K^3)\hat{t}, \quad (3.5)\end{aligned}$$

where

$$\mathcal{R} = K^2+K^2\mathcal{R} = \mathfrak{N}/\mathfrak{D}. \quad (3.6)$$

Our first task is to find a region in the upper half $k=E^{1/2}$ plane in which K^2 is an L^2 operator, i.e., in which

$$\|K^2\|^2 = \mathrm{tr}[K^2K^{+2}] < \infty. \quad (3.7)$$

A typical term in the trace is[5]

$$J = \mathrm{tr}[\hat{t}_{12}G_0\hat{t}_{23}G_0G_0^\dagger\hat{t}_{23}^\dagger G_0^\dagger\hat{t}_{12}^\dagger]. \quad (3.8)$$

[5] There are also terms of the form $\mathrm{tr}[t_{12}G_0t_{23}G_0G_0^\dagger t_{13}^\dagger G_0^\dagger t_{12}^\dagger]$; however, making use of Schwartz's inequality we see that

$$\begin{aligned}&\mathrm{tr}[t_{12}G_0t_{23}G_0G_0^\dagger t_{13}^\dagger G_0^\dagger t_{12}^\dagger]\\ &= \int d^3q_1d^3q_3d^3q_1'd^3q_3'\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|t_{12}G_0t_{23}G_0|\mathbf{q}_1'\mathbf{q}_2'\mathbf{q}_3'\rangle \times\langle\mathbf{q}_1'\mathbf{q}_2'\mathbf{q}_3'|G_0^\dagger t_{13}^\dagger G_0^\dagger t_{12}^\dagger|\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\rangle\\ &\leq \int d^3q_1d^3q_3\left[\int d^3q_1'd^3q_3'|\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|t_{12}G_0t_{23}G_0|\mathbf{q}_1'\mathbf{q}_2'\mathbf{q}_3'\rangle|^2\right]^{1/2} \times\left[\int d^3q_1''d^3q_3''|\langle\mathbf{q}_1''\mathbf{q}_2''\mathbf{q}_3''|G_0^\dagger t_{13}^\dagger G_0^\dagger t_{12}^\dagger|\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\rangle|^2\right]^{1/2}\\ &\leq \left[\int d^3q_1d^3q_3d^3q_1'd^3q_3'|\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|t_{12}G_0t_{23}G_0|\mathbf{q}_1'\mathbf{q}_2'\mathbf{q}_3'\rangle|^2\right]^{1/2} \times\left[\int d^3q_1d^3q_3d^3q_1''d^3q_3''|\langle\mathbf{q}_1''\mathbf{q}_2''\mathbf{q}_3''|G_0^\dagger t_{13}^\dagger G_0^\dagger t_{12}^\dagger|\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\rangle|^2\right]^{1/2}\\ &= [\mathrm{tr}[t_{12}G_0t_{23}G_0G_0^\dagger t_{23}^\dagger G_0^\dagger t_{12}^\dagger]]^{1/2}\,[\mathrm{tr}[t_{12}G_0t_{13}G_0G_0^\dagger t_{13}^\dagger G_0^\dagger t_{12}^\dagger]]^{1/2}.\end{aligned}$$

So, it is sufficient to consider the trace given in Eq. (3.8).

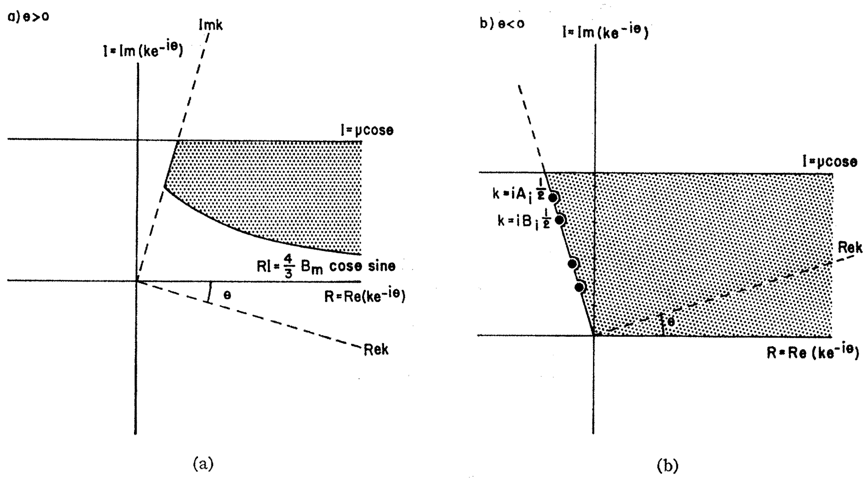

FIG. 2. The regions in which f_θ is given by the ratio of two Fredholm series.

In Appendix III we show that J is bounded by

$$J \leq C\{I\mu[R^2+(I+2^{1/2}\mu)^2]^{1/2}\}^{-1}, \tag{3.9}$$

where C is a constant and $k=R+iI$. The bound holds in the entire upper half k plane, $I\geq\epsilon>0$, except for the segment of the imaginary axis for which $B_m\geq I\geq 0$, where B_m is the largest binding energy in the problem. We note that $\|K^2\|^2\to 0$ as $|k|\to\infty$, so it will be easy to prove the convergence of the Born series at high energies.

In order to complete the proof of the existence of the on-energy-shell amplitude, we must show that the vectors $K^2t|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, $K^3t|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, and $\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K^4$ have finite norms. In Appendix III we show that the norm of each of these vectors can be bounded by a constant in the strip

$$(\sqrt{\tfrac{3}{2}})\mu-\epsilon\geq I\geq\epsilon>0$$
$$R\geq\epsilon>0. \tag{3.10}$$

The bound also holds on the imaginary axis for $(\sqrt{\frac{3}{2}})\mu>I>(\frac{4}{3}B_m)^{1/2}$. It follows from our discussion in Sec. II that the on-energy-shell three-body amplitude exists and is given by the ratio of two uniformly convergent series in this strip.

$$f(k)=\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|f|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle$$
$$=\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|\sum_{n=0}^{7}K^n\hat{t}|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle$$
$$+\sum_{i=0}^{\infty}\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K^4N_i(K_2+K_3)\hat{t}|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle\Big/\sum_{j=0}^{\infty}\mathfrak{D}_j. \tag{3.11}$$

It was shown in I that the individual terms in the series for $\mathfrak{D}$ and $\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K^4\mathfrak{N}(K^2+K^3)\hat{t}|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle$ are analytic functions of k for fixed physical values of the vectors $\mathbf{y}_i$ and $\mathbf{y}_i'$, and k in the strip defined by Eq. (3.10). Since

$$\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|\sum_{n=0}^{7}K^n\hat{t}|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle$$

is also an analytic function of k in this region, it follows that $f(k)$ is.

In order to show that the amplitude exists in the rest of the first quadrant of the k plane, we shall use the rotation of contour argument introduced in Sec. II. In analogy with Eqs. (2.11) and (3.2) we write

$$f_\theta=\hat{t}_\theta+K_\theta f_\theta$$
$$=\sum_{n=0}^{7}K_\theta^n t_\theta+K_\theta^4\mathfrak{R}_\theta(K_\theta^2+K_\theta^3)t_\theta, \tag{3.12}$$

where

$$\langle\mathbf{q}_i\mathbf{q}_j\mathbf{q}_k|\hat{t}_{\theta ij}(k^2)|\mathbf{q}_i'\mathbf{q}_j'\mathbf{q}_k'\rangle$$
$$=t_{\theta ij}(\tfrac{1}{2}(\mathbf{q}_i-\mathbf{q}_j),\tfrac{1}{2}(\mathbf{q}_i'-\mathbf{q}_j');k^2-\tfrac{3}{2}q_k^2e^{2i\theta})\times\delta^3(\mathbf{q}_k-\mathbf{q}_k'), \tag{3.13}$$

and

$$G_\theta=e^{3i\theta}[\mathbf{q}_1^2e^{2i\theta}+\mathbf{q}_3^2e^{2i\theta}+(\mathbf{q}_1+\mathbf{q}_3)^2e^{2i\theta}-k^2]^{-1}.$$

We shall be interested in both positive and negative values of θ. However we cannot rotate contours of integration through an arbitrary negative angle since we will eventually encounter the resonance poles of the two-particle t matrices. However, since all resonances have a finite width (zero-energy resonances will not prevent us from rotating contours), we can always rotate through a finite negative angle $-\theta_0$ before encountering the first resonance pole. We shall only consider values of θ greater than $-\theta_0$.

In Appendix III we show that

$$\|K_\theta\|^2\leq C\{I\mu\cos\theta\,[(I+\sqrt{2}\mu\cos\theta)^2+(|R|-\sqrt{2}\mu|\sin\theta|)^2]^{1/2}\}^{-1}, \tag{3.14}$$

where $ke^{-i\theta}=R+iI$. For $\pi/2-\epsilon\geq\theta\geq 0$, the bound holds in the first quadrant of the $ke^{-i\theta}$ plane above the hyperbola $RI=B_m \sin\theta \cos\theta$. B_m is the largest binding energy in the problem. For $0>\theta>-\theta_0$, the bound holds in the region of the $ke^{-i\theta}$ plane bounded by the lines $\mathrm{Re}k=0$, $I=\epsilon>0$, provided we exclude small semicircles about the points $k=iB_i^{1/2}$. B_i are the two-particle binding energies. We note that $\|K_\theta^2\|^2 \to 0$ as $k\to\infty$.

In Appendix III we also obtain bounds on the norms of the vectors $K_\theta^2\hat{t}_\theta|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, $K_\theta^3\hat{t}_\theta|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, and $\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3| \times K_\theta^4$. For $\pi/2>\theta\geq 0$ we find that the norms can be bounded by a constant in the region bounded by the curves (see Fig. 2a).

$$\begin{aligned} RI&=\tfrac{4}{3}B_m \sin\theta\cos\theta+\epsilon,\\ I&=(\surd\tfrac{3}{2})\mu\cos\theta-\epsilon, \qquad (3.15)\\ \mathrm{Re}k&=0. \end{aligned}$$

For $0>\theta>-\theta_0$ the norms can be bounded by a constant in the region bounded by the lines

$$\begin{aligned} I&=(\surd\tfrac{3}{2})\mu\cos\theta-\epsilon,\\ I&=\epsilon, \qquad (3.16)\\ \mathrm{Re}k&=0, \end{aligned}$$

provided we exclude small semicircles about the normal thresholds $k=iB_i^{1/2}$ and the anomalous thresholds whose positions are given in Eq. (3.20). (See Fig. 2b).

It follows from our previous discussion that the on-energy-shell quantity

$$f_\theta(k)=\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|f_\theta|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle \qquad (3.17)$$

exists and is given by the ratio of two uniformly convergent series in the regions defined by Eqs. (3.15) and (3.16).

$$f_\theta(k)=\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|\sum_{n=0}^{7}K_\theta^n\hat{t}_\theta|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle + \sum_{i=0}^{\infty}\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K_\theta^4\mathfrak{N}_{\theta i}(K_\theta^2+K_\theta^3)\hat{t}_\theta|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle \Big/ \sum_{j=0}^{\infty}\mathfrak{D}_{\theta j}. \qquad (3.18)$$

It was shown in I that the individual terms in the series for $f_\theta(k)$ are analytic functions of k for fixed, physical values of the vectors $\mathbf{y}_i$ and $\mathbf{y}_i'$, and k in the regions given by Eqs. (3.15) and (3.16). As a result $f_\theta(k)$ is an analytic function of k in these regions.

It will be noted that for $\pi/2>\theta>-\theta_0$ there is always an overlap between the regions in which $f(k)$ and $f_\theta(k)$ exist. It was shown in I that if k was held fixed in the region of overlap, then for any term in the series for $f(k)$ one could simultaneously rotate all contours of integration through an angle θ without crossing a singularity of the integrand. As a result, in the region of overlap

$$\begin{aligned} \mathfrak{D}_i&=\mathfrak{D}_{\theta i},\\ \langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K^4\mathfrak{N}_i(K^2+K^3)\hat{t}|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle &=\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K_\theta^4\mathfrak{N}_{\theta i}(K_\theta^2+K_\theta^3)\hat{t}_\theta|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle,\\ \langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|\sum_{n=0}^{7}K^n\hat{t}|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle &=\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|\sum_{n=0}^{7}K_\theta^n\hat{t}_\theta|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle. \end{aligned} \qquad (3.19)$$

So $f_\theta(k)$ is the analytic continuation of $f(k)$.

By rotating contours through the angles $\pi/2>\theta>-\theta_0$ we see that the on-energy-shell amplitude exists in the entire first quadrant of the k plane including the positive real axis, with the possible exception of the normal thresholds, $k=0$, $k=iB_i^{1/2}$, the anomalous thresholds, and the line $\mathrm{Re}k=0$, $\mathrm{Im}k\geq(\surd\tfrac{3}{2})\mu$. In Appendix V we show that the amplitude does in fact exist at the normal thresholds, $k=0$ and $k=iB_i^{1/2}$.

There is a slight delicacy associated with the anomalous thresholds. It was shown in I that in general each term in the numerator series will have anomalous thresholds. If, for example, we are considering a term in which particles 1 and 2 interact last then there will be an anomalous threshold at

$$k^2=-A_i=-B_i\{1-\tfrac{3}{8}[y_3^2-(2-3y_3^2)^{1/2}]^2\}^{-1};\quad y_3^2\geq\tfrac{1}{2}. \qquad (3.20)$$

B_i is the binding energy of a bound state of either the (2,3) or (1,3) systems. The anomalous threshold enters the physical sheet through the normal threshold at $k^2=-B_i$ when $y_3^2=\frac{1}{2}$. It moves down the negative k^2 axis as y_3^2 increases reaching the point $k^2=-\frac{4}{3}B_i$ when $y_3^2=\frac{2}{3}$. There will of course be anomalous thresholds which depend on each of the $\mathbf{y}_i$ and $\mathbf{y}_i'$.

We have already seen that the kernel is square-integrable at the anomalous thresholds. However, in Appendix 3 we show that the vectors $\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|K^2\hat{t} \times|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, $\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|K^3\hat{t}|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, and $\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K^4|\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\rangle$ all blow up logarithmically at the anomalous thresholds. But in each case, the coefficient of the logarithm remains a square integrable function of the $\mathbf{q}_i$. As a result, the full amplitude will blow up logarithmically at the anomalous thresholds, but the coefficient of the logarithm will still be given by the ratio of two convergent series.

Since the on-energy-shell amplitude exists in the first quadrant of the k plane, it must exist in the second quadrant because it is given in this quadrant by

$$f(k)=f^*(-k^*). \qquad (3.21)$$

We have thus shown that the on-energy-shell amplitude exists in the entire physical sheet of the $E=k^2$ plane with the possible exception of the line $\mathrm{Im}E=0$, $\mathrm{Re}E\leq-\frac{3}{2}\mu^2$. This line lies on the left-hand cut. This completes our proof that the on-energy-shell amplitude satisfies a dispersion relation in E. Since the amplitude is given by the ratio of two uniformly convergent series, it can only have the singularities of the individual terms in these series. They were discussed in I.

We recall that $\mathrm{tr}(K^2K^{+2})\to 0$ as $E\to\infty$ in any direction. As a result, the Born series will converge at large energies and the amplitude will go to the first Born approximation. In this limit the three-body Fredholm determinant, $\mathfrak{D}$, goes to 1. Although our proof of the existence of the three-body amplitude breaks down along a portion of the negative real axis, our proof of the existence of $\mathfrak{D}$ does not, since it depends only on the existence of $\mathrm{tr}(K^2K^{+2})$. As a result, $\mathfrak{D}$ can have no essential singularities. Since $\mathfrak{D}$ goes to 1 at infinity, it must have a finite number of zeros, so there will be a finite number of three-particle bound states.[6]

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APPENDIX I

In this Appendix we shall give the details of obtaining bounds on some of the integrals which appeared in our study of the two-body amplitude in Sec. II. We first note that

$$\int d^3q\,|(\mathbf{q}-\mathbf{q}')^2e^{2i\theta}+\mu^2|^{-2}=\int d^3q''\,|q''^2+\mu^2e^{-2i\theta}|^{-2}$$

$$=\int d^3q''[(q''^2+\mu^2\cos^2\theta-\mu^2\sin^2\theta)^2+4\mu^4\sin^2\theta\cos^2\theta]^{-1}$$

$$\leq\int d^3q''[(q''^2-\mu^2\sin^2\theta)^2+\mu^4\cos^4\theta]^{-1}$$

$$=2^{1/2}\pi^2[(\sin^4\theta+\cos^4\theta)^2+\sin^2\theta]^{1/2}/\mu\cos^2\theta\,, \tag{A1.1}$$

so

$$\int d^3q\,|(\mathbf{q}-\mathbf{q}')^2e^{2i\theta}+\mu^2|^{-2}\leq C/\mu\cos^2\theta\,. \tag{A1.2}$$

In addition

$$\int d^3q\,|(\mathbf{q}-\hat{n}ke^{-i\theta})^2+\mu^2e^{-2i\theta}|^2=\int d^3q'\,|(\mathbf{q}'-iI\hat{n})^2+\mu^2e^{-2i\theta}|^2=\int d^3q'[(q'^2-I^2+\mu^2\cos2\theta)^2+(\mu^2\sin2\theta+2qIz)^2]^{-1}$$

$$\leq\int d^3q'[(q'^2-\mu^2\sin^2\theta)^2+(\mu^2\cos^2\theta+I^2)^2]^{-1}\leq 2^{1/2}\pi^2(\mu^2\cos^2\theta-I^2)^{-1}\,, \tag{A1.3}$$

where $ke^{-i\theta}=R+iI$, $I\leq\mu\cos\theta$.

The bounds on the remaining integrals can be obtained by the same techniques.

APPENDIX II

In this Appendix we shall show that the on-energy-shell two-body amplitude exists in the limit $k\to 0$ and that the off-energy-shell amplitude remains an L^2 function of either of its momenta in this limit.

We start by considering the operator P, which, acting on a state of definite momentum, $|\mathbf{p}\rangle$, gives

$$P|\mathbf{p}\rangle=|\mathbf{p}|\,|\mathbf{p}\rangle. \tag{A2.1}$$

We then define

$$A=P(P+\mu)^{-1}. \tag{A2.2}$$

Now consider the amplitude $A^{-1}t$.

$$A^{-1}t=A^{-1}V+A^{-1}VG_0A(A^{-1}t)$$
$$=A^{-1}V+A^{-1}VG_0A(A^{-1}V)$$
$$+A^{-1}VG_0AR'A^{-1}V\,, \tag{A2.3}$$

where

$$R'=A^{-1}VG_0A+A^{-1}VG_0AR'. \tag{A2.4}$$

Now

$$\|A^{-1}VG_0A\|^2=\int d^3qd^3q'(1+\mu/q)^2[(\mathbf{q}-\mathbf{q}')^2+\mu^2]^{-2}$$
$$\times|q'^2-k^2|^{-2}q'^2(q'+\mu)^{-2}. \tag{A2.5}$$

Now

$$\int d^3q(1+\mu/q)^2[(\mathbf{q}-\mathbf{q}')^2+\mu^2]^{-2}[\theta(q^2-\mu^2)+\theta(\mu^2-q^2)]$$

$$\leq 4\int d^3q[(\mathbf{q}-\mathbf{q}')^2+\mu^2]^{-2}\theta(q^2-\mu^2)$$

$$+\frac{1}{\mu^4}\int d^3q(1+\mu/q)^2\theta(\mu^2-q^2)$$

$$\leq 4\left[\frac{\pi^2}{\mu}+\frac{4\pi}{\mu}\right]=C. \tag{A2.6}$$

[6] It is shown in Ref. 2 that there is a one-to-one correspondence between bound states and solutions of the homogeneous Faddeev equations. D can have zeros only at solutions of the homogeneous equation (Ref. 3).

So

$$\|A^{-1}VG_0A\|^2 \leq C\int d^3q'\,|q'^2-k^2|^{-2}q'^2(q'+\mu)^{-2}. \quad \text{(A2.7)}$$

Since $|q'^2-k^2|^2 \geq [I^2/(R^2+I^2)]q'^4$, we have

$$\|A^{-1}VG_0A\|^2 \leq C\frac{R^2+I^2}{I^2}4\pi\int_0^\infty dq'\,(q'+\mu)^{-2}$$

$$= C'\frac{R^2+I^2}{I^2}. \quad k=R+iI. \quad \text{(A2.8)}$$

We note that $\|A^{-1}VG_0A\|^2$ exists in the limit $k\to 0$ if we approach the origin in any direction except along the real axis.

By the same argument that gave (A2.6) we see that

$$\|A^{-1}V|k\hat{n}\rangle\|^2 = \int d^3q(1+\mu/q)^2|(\mathbf{q}-k\hat{n})^2+\mu^2|^{-2}$$

$$\leq C_1[\mu^2-I^2]^{-1/2}+C_2\mu^3(\mu^2-I^2)^{-2}. \quad \text{(A2.9)}$$

and that

$$\|AG_0V|k\hat{n}\rangle\|^2$$

$$= \int d^3q\, q^2(\mu+q)^{-2}|q^2-k^2|^{-2}|(\mathbf{q}-k\hat{n})^2+\mu^2|^{-2}$$

$$\leq C\frac{(R^2+I^2)}{I^2}(\mu^2-I^2)^{-2}. \quad \text{(A2.10)}$$

As a result, the on-energy-shell amplitude,

$$\langle k\hat{n}|t|k\hat{n}\rangle = \langle k\hat{n}|[V+VG_0+VG_0AR'A^{-1}V]|k\hat{n}'\rangle, \quad \text{(A2.11)}$$

exists in the limit $k\to 0$. From our rotation of contour arguments it is clear that this limit exists even if we approach the origin along the real axis.

It is now easy to see that $t(\mathbf{p},\mathbf{p}';k^2)$ is an L^2 function of $\mathbf{p}$ in the limit $k\to 0$. Using the arguments of Eqs. (2.17)–(2.24) we see that $\langle\mathbf{p}|A^{-1}t|\mathbf{p}'\rangle$ is an L^2 function of $\mathbf{p}$ whose norm is independent of $\mathbf{p}'$. But

$$C[1+\|R'\|^2] \geq \int d^3p\,|\langle\mathbf{p}|A^{-1}t|\mathbf{p}'\rangle|^2$$

$$= \int d^3p(1+\mu/p)^2|t(\mathbf{p},\mathbf{p}';k^2)|^2$$

$$\geq \int d^3p\,|t(\mathbf{p},\mathbf{p}';k^2)|^2. \quad \text{(A2.12)}$$

Similarly, using the arguments which led to Eq. (2.25) we see that

$$|t(\mathbf{p},\mathbf{p}';k^2)|^2 \leq C'+C''\|R'\|^2. \quad \text{(A2.13)}$$

Similar bounds can, of course, be obtained for t_θ in the usual way.

APPENDIX III

In this Appendix we shall obtain the bounds on the norms of the kernels and vectors which were quoted in Sec. III.

Let us start with the trace given in Eq. (3.8). We have

$$J=\mathrm{tr}[\hat{t}_{12}G_0\hat{t}_{23}G_0G_0^\dagger t_{23}^\dagger G_0^\dagger t_{12}^\dagger] = \int d^3q_1d^3q_3d^3q_1'd^3q_3'\,|t_{12}(\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_3,\, \mathbf{q}_1'+\tfrac{1}{2}\mathbf{q}_3;\, k^2-\tfrac{3}{2}q_3^2)|^2$$

$$\times|t_{23}(\mathbf{q}_3+\tfrac{1}{2}\mathbf{q}_1',\, \mathbf{q}_3'+\tfrac{1}{2}\mathbf{q}_1';\, k^2-\tfrac{3}{2}q_1'^2)|^2\;|\mathbf{q}_1'^2+\mathbf{q}_3^2+(\mathbf{q}_1'+\mathbf{q}_3)^2-k^2|^{-2}\;|\mathbf{q}_1'^2+\mathbf{q}_3'^2+(\mathbf{q}_1'+\mathbf{q}_3')^2-k^2|^{-2}. \quad \text{(A3.1)}$$

Our choice of variables is shown in Fig. 3. Since the integrand depends on $\mathbf{q}_1$ only through one of the arguments of t_{12}, we can do the $\mathbf{q}_1$ integration immediately by making use of Eq. (2.21). We note that in the first quadrant of the k plane, R, $I\geq\epsilon>0$ $(k=R+iI)$, $D_B^{-1}(k^2-\frac{3}{2}q_3^2)$ can be bounded by a constant; so in this region

$$J\leq C\int d^3q_1'd^3q_3d^3q_3'\,|t_{23}(\mathbf{q}_3+\tfrac{1}{2}\mathbf{q}_1',\mathbf{q}_3'+\tfrac{1}{2}\mathbf{q}_1';\,k^2-\tfrac{3}{2}q_1'^2)|^2|\mathbf{q}_1'^2+\mathbf{q}_3^2+(\mathbf{q}_1'+\mathbf{q}_3)^2-k^2|^{-2}|\mathbf{q}_1'^2+\mathbf{q}_3'^2+(\mathbf{q}_1'+\mathbf{q}_3')-k^2|^{-2}$$

$$=C\int d^3q_1'd^3pd^3p'\,|t_{23}(\mathbf{p},\mathbf{p}';\,k^2-\tfrac{3}{2}q_1'^2)|^2|2p^2-(k^2-\tfrac{3}{2}q_1'^2)|^{-2}|2p'^2-(k^2-\tfrac{3}{2}q_1'^2)|^{-2}. \quad \text{(A3.2)}$$

In Sec. II we used units in which $m=1$. In our present units, $m=\frac{1}{2}$, the two-particle free Green's functions for an intermediate state with center-of-mass momentum p and energy E is $(2p^2-E)^{-1}$. Using Eq. (2.3) and the Schwartz inequality, we see that

$$\int d^3pd^3p'\langle\mathbf{p}|G_0t_{23}G_0|\mathbf{p}'\rangle|^2 = \int d^3pd^3p'\langle\mathbf{p}|G_0V_{23}G_0+G_0V_{23}G_0R|\mathbf{p}'\rangle|^2 \leq 2\int d^3pd^3p'[|\langle\mathbf{p}|G_0V_{23}G_0|\mathbf{p}'\rangle|^2$$

$$+|\langle\mathbf{p}|G_0V_{23}G_0R|\mathbf{p}'\rangle|^2] \leq 2\int d^3pd^3p'\,|\langle\mathbf{p}|G_0V_{23}G_0|\mathbf{p}'\rangle|^2[1+\|R(k^2-\tfrac{3}{2}q_1'^2)\|^2]. \quad \text{(A3.3)}$$

In the quadrant R, $I\geq\epsilon>0$, $\|R(k^2-\frac{3}{2}q_1'^2)\|^2$ is bounded by a constant, so

$$J\leq C'\int d^3q_1'd^3pd^3p'\,|2p^2+\tfrac{3}{2}q_1'^2-k^2|^{-2}|2p'^2+\tfrac{3}{2}q_1'^2-k^2|^{-2}[(\mathbf{p}-\mathbf{p}')^2+\mu^2]^{-2}. \tag{A3.4}$$

In Appendix IV we obtain a bound on this integral and show that

$$J\leq C''|I\mu[R^2+(I+2^{1/2}\mu)^2]^{1/2}|^{-1},\quad k=R+iI,\quad R,I\geq\epsilon>0. \tag{A3.5}$$

In order to complete the proof of the existence of the on-energy-shell amplitude, we must show that the "vectors" $K^2\hat{t}|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, $K^3\hat{t}|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, and $\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K^4$ have finite norms. A typical term in the norm of $K^2\hat{t}|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$ is

$$\begin{aligned}J_1&=\|\hat{t}_{12}G_0\hat{t}_{23}G_0\hat{t}_{12}|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle\|^2\\
&=\int d^3q_1d^3q_3d^3qd^3q'\,t_{12}(\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_3,\tfrac{1}{2}(\mathbf{q}+\mathbf{q}_3-\mathbf{k}_3);k^2-\tfrac{3}{2}q_3^2)t_{12}^*(\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_3,\tfrac{1}{2}(\mathbf{q}'+\mathbf{q}_3-\mathbf{k}_3);k^2-\tfrac{3}{2}q_3^2)\\
&\quad\times[\mathbf{q}_3^2+\tfrac{1}{4}(\mathbf{q}-\mathbf{k}_3)^2+(\mathbf{q}_3+\tfrac{1}{2}(\mathbf{q}-\mathbf{k}_3))^2-k^2]^{-1}[\mathbf{q}_3^2+\tfrac{1}{4}(\mathbf{q}'-\mathbf{k}_3)^2+(\mathbf{q}_3+\tfrac{1}{4}(\mathbf{q}'-\mathbf{k}_3))^2-k^2]^{*-1}\\
&\quad\times t_{23}(\mathbf{q}_3+\tfrac{1}{4}(\mathbf{q}-\mathbf{k}_3),\tfrac{3}{4}\mathbf{k}_3+\tfrac{1}{4}\mathbf{q}';k^2-\tfrac{3}{8}(\mathbf{q}-\mathbf{k}_3)^2)t_{23}^*(\mathbf{q}_3+\tfrac{1}{4}(\mathbf{q}'-\mathbf{k}_3),\tfrac{3}{4}\mathbf{k}_3+\tfrac{1}{4}\mathbf{q}';k^2-\tfrac{3}{8}(\mathbf{q}'-\mathbf{k}_3)^2)\\
&\quad\times(\tfrac{1}{2}q^2-\tfrac{1}{2}k_{12}^2)^{-1}(\tfrac{1}{2}q'^2-\tfrac{1}{2}k_{12}^2)^{*-1}t_{12}(\mathbf{q},\mathbf{k}_{12},\tfrac{1}{2}k_{12}^2)t_{12}^*(\mathbf{q}',\mathbf{k}_{12},\tfrac{1}{2}k_{12}^2),\end{aligned} \tag{A3.6}$$

where $\frac{1}{2}\mathbf{k}_{12}^2=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)^2=k^2-\frac{3}{2}k_3^2$. Our choice of variables is shown in Fig. 4.

For simplicity we start by assuming that there are no bound states in the (2,3) system, so there will be no difficulty with anomalous thresholds.

Since the only terms in the integrand which depend on $\mathbf{q}_1$ are t_{12} and t_{12}^*, one can do the $\mathbf{q}_1$ integration immediately using the Schwartz inequality and (2.24)

$$\begin{aligned}&\left|\int d^3q_1t_{12}(\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_3,\tfrac{1}{2}(\mathbf{q}+\mathbf{q}_3-\mathbf{k}_3);q^2-\tfrac{3}{2}q_3^2)t_{12}^*(\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_3,\tfrac{1}{2}(\mathbf{q}'+\mathbf{q}_3-\mathbf{k}_3);k^2-\tfrac{3}{2}q_3^2)\right|\\
&\leq\left|\int d^3q_1|t_{12}(\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_3,\tfrac{1}{2}(\mathbf{q}+\mathbf{q}_3-\mathbf{k}_3);k^2-\tfrac{3}{2}q_3^2)|^2\right|^{1/2}\left|\int d^3q_1|t_{12}(\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_3,\tfrac{1}{2}(\mathbf{q}'+\mathbf{q}_3-\mathbf{k}_3);k^2-\tfrac{3}{2}q_3^2)|^2\right|^{1/2}\\
&\leq C.\end{aligned} \tag{A3.7}$$

For $\mu>\frac{1}{2}\,\mathrm{Im}k_3>0$. Since $\mathbf{k}_3=\mathbf{y}_3k$ and $y_3^2\leq\frac{2}{3}$, the bound holds in the strip

$$R\geq\epsilon>0,\quad(\sqrt{6})\mu-\epsilon\geq I>\epsilon. \tag{A3.8}$$

Before doing the $\mathbf{q}_3$ integration it is convenient to bound the Green's functions that depend on $\mathbf{q}_3$.

$$\begin{aligned}|G_0^{-1}|^2&=|\mathbf{q}_3^2+\tfrac{1}{4}(\mathbf{q}-\mathbf{k}_3)^2+(\mathbf{q}_3+\tfrac{1}{2}\mathbf{q}-\tfrac{1}{2}\mathbf{k}_3)^2-k^2|^2\\
&=|\tfrac{3}{2}\mathbf{q}_3^2+\tfrac{1}{2}(\mathbf{q}_3+\mathbf{q})^2-\mathbf{k}\mathbf{y}_3\cdot(\mathbf{q}_3+\mathbf{q})-k^2(1-\tfrac{1}{2}y_3^2)|^2\\
&=[\tfrac{3}{2}\mathbf{q}_3^2+\tfrac{1}{2}(\mathbf{q}_3+\mathbf{q})^2-Ry_3|\mathbf{q}_3+\mathbf{q}|z-(R^2-I^2)(1-\tfrac{1}{2}y_3^2)]^2+[Iy_3|\mathbf{q}_3+\mathbf{q}|z+2RIz(1-\tfrac{1}{2}y_3^2)]^2,\end{aligned} \tag{A3.9}$$

where $z=\hat{y}_3.\ (\mathbf{q}_3+\mathbf{q})/|\mathbf{q}_3+\mathbf{q}|$. Minimizing $|G_0^{-1}|^2$ with respect to z without regard to the restriction $-1\leq z\leq1$ gives

$$\begin{aligned}|G_0^{-1}|^2&\geq\frac{I^2}{R^2+I^2}[\tfrac{3}{2}\mathbf{q}_3^2+\tfrac{1}{2}(\mathbf{q}_3+\mathbf{q})^2+(R^2+I^2)(1-\tfrac{1}{2}y_3^2)]^2\\
&\geq I^2(R^2+I^2)(1-\tfrac{1}{2}y_3^2).\end{aligned} \tag{A3.10}$$

Since $y_3^2\leq\frac{2}{3}$, we have

$$|G_0|\leq(\tfrac{3}{2})^{1/2}/I(I^2+R^2)^{1/2}. \tag{A3.11}$$

Using this bound on G_0, the integrand will depend on $\mathbf{q}_3$ only through t_{23} and t_{23}^*, so we can do the $\mathbf{q}_3$ integration just as we did the $\mathbf{q}_1$ integration. The Fredholm determinant of t_{23}, $D_{23}(k^2-\frac{3}{8}(\mathbf{q}-\mathbf{k}_3)^2)$ can be bounded by a constant since we have assumed that there are no

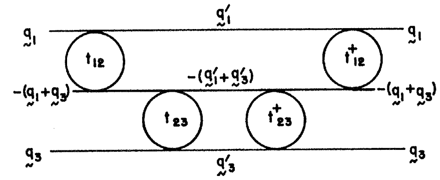

FIG. 3. The diagram corresponding to $\mathrm{tr}[t_{12}G_0t_{23}G_0G_0^\dagger t_{23}^\dagger G_0^\dagger t_{12}^\dagger]$.

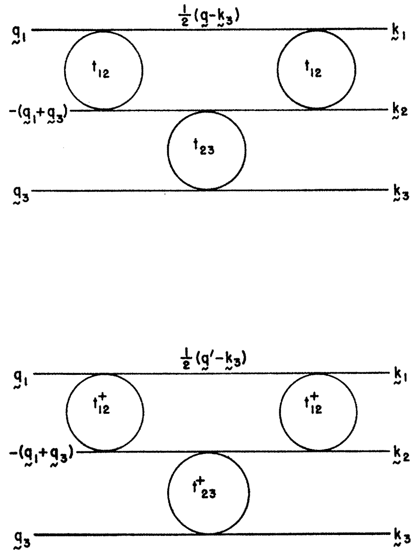

FIG. 4. The diagram corresponding to $\|t_{12}G_0t_{23}G_0t_{12}|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle\|$.

bound states in the (2,3) system, and

$$\mathrm{Im}[k^2-\tfrac{3}{8}(\mathbf{q}-\mathbf{k}_3)^2]^{1/2}>0$$

for $\mathrm{Im}k>0$. We then have

$$J_1\leq\frac{C}{I^2(R^2+I^2)}\int d^3q|\tfrac{1}{2}q^2-\tfrac{1}{2}k_{12}^2|^{-1}|t_{12}(\mathbf{q},\mathbf{k}_{12};\tfrac{1}{2}k_{12}^2)|$$
$$\times\int d^3q'|\tfrac{1}{2}q'^2-\tfrac{1}{2}k_{12}^2|^{-1}|t_{12}(\mathbf{q}',\mathbf{k}_{12};\tfrac{1}{2}k_{12}^2)|\,. \quad \text{(A3.12)}$$

If we try to bound the integrals in (3.20) by direct application of the Schwartz inequality, the bound will blow up in the limit $y_{12}=k_{12}/k\rightarrow 0$ (In the physical region $0\leq y_{12}^2\leq 2$.) so we must be more careful. We first note that

$$|q^2-k_{12}^2|^2$$
$$=[q^2-(\mathrm{Re}k_{12})^2+(\mathrm{Im}k_{12})^2]^2+4[\mathrm{Re}k_{12}\,\mathrm{Im}k_{12}]^2$$
$$\geq\frac{(\mathrm{Im}k_{12})^2}{(\mathrm{Re}k_{12})^2+(\mathrm{Im}k_{12})^2}q^4=\frac{I^2}{R^2+I^2}q^4. \quad \text{(A3.13)}$$

As a result

$$J_1'=\int d^3q|q^2-k_{12}^2|^{-1}|t_{12}(\mathbf{q},\mathbf{k}_{12};\tfrac{1}{2}k_{12}^2)|$$
$$\leq\frac{(R^2+I^2)^{1/2}}{I}\int d^3q\,q^{-2}|t_{12}(\mathbf{q},\mathbf{k}_{12};\tfrac{1}{2}k_{12}^2)|$$
$$\times[\theta(\mu^2-q^2)+\theta(q^2-\mu^2)]. \quad \text{(A3.14)}$$

In Appendix II we show that the bounds on the t matrix given in Eqs. (2.23) and (2.26) hold in the limit $k\rightarrow 0$ provided we do not approach the origin along the real axis. In the present case $\mathrm{Im}k_{12}/\mathrm{Re}k_{12}=I/R>0$ since $I\geq\epsilon>0$. We then have

$$J_1'\leq\frac{(R^2+I^2)^{1/2}}{I}\Bigg[C_1\int d^3q\,q^{-2}\theta(\mu^2-q^2)$$
$$+\bigg[\int d^3q\,q^{-4}\theta(q^2-\mu^2)\bigg]^{1/2}$$
$$\times\bigg[\int d^3q'|t_{12}(\mathbf{q}',\mathbf{k}_{12};\tfrac{1}{2}k_{12}^2)|^2\theta(q^2-\mu^2)\bigg]^{1/2}\Bigg]. \quad \text{(A3.15)}$$

We have used Eq. (2.26) to bound $|t_{12}|$ in the first integral. Using (2.23) we see that

$$J_1'\leq\frac{(R^2+I^2)^{1/2}}{I}C_1' \quad \text{(A3.16)}$$

so

$$J_1\leq C'/I^4\leqslant C''; \quad \sqrt{2}\mu-\epsilon\geq I\geq\epsilon; \quad R\geq\epsilon. \quad \text{(A3.17)}$$

The factor $\sqrt{2}$ arises because $0\leq y_{12}^2\leq 2$.

We must now consider the case in which there are bound states in the (2,3) system. We can then write the Fredholm denominator of t_{23} in the form

$$D_{23}^{-1}(k^2-\tfrac{3}{8}(\mathbf{q}-\mathbf{k}_3)^2)=1+\int_0^\infty\frac{\rho(x)dx}{x-k^2+\frac{3}{8}(q-k_3)^2}$$
$$+\sum_i\frac{C_i}{\frac{3}{8}(\mathbf{q}-\mathbf{k}_3)^2-k^2-B_i}. \quad \text{(A3.18)}$$

The C_i are constants and the B_i are the two particle binding energies. The dispersion integral can again be bounded by a constant for $\mathrm{Im}k\geq 0$.

We now substitute Eq. (A3.18) and the corresponding expression for $D_{23}^{*-1}[k^2-\frac{3}{8}(\mathbf{q}'-\mathbf{k}_3)^2]$ into Eq. (A3.6). In each term which contains a bound-state pole we apply the Feynman identity to the bound-state denominator and the free Green's function $(\frac{1}{2}q^2-\frac{1}{2}k_{12}^2)^{-1}$. We have

$$[q^2-k_{12}^2]^{-1}[(\mathbf{q}-\mathbf{k}_3)^2-(8/3)(k^2+B_i)]^{-1}$$
$$=\int_0^1 dx[(\mathbf{q}-x\mathbf{k}_3)^2-k^2(x^2y_3^2+x(\tfrac{2}{3}+2y_3^2)$$
$$+2-3y_3^2)-(8/3)xB_i]^{-2}. \quad \text{(A3.19)}$$

We now make the change of variables $\mathbf{q}\rightarrow\mathbf{q}+x\mathbf{k}_3$, $\mathbf{q}'\rightarrow\mathbf{q}'+x'\mathbf{k}_3$. We can now do the $\mathbf{q}_1$ and $\mathbf{q}_3$ integrations as before. We will be left with integrals of the form (A3.14) and terms of the form

$$J_1''=C\int_0^1 dx\int d^3q|t_{12}(\mathbf{q}+x\mathbf{k}_3,\mathbf{k}_{12};\tfrac{1}{2}k_{12}^2)|$$
$$\times[q^2-k^2(x^2y_3^2+x(\tfrac{2}{3}+2y_3^2)+2-3y_3^2)$$
$$-(8/3)xB_i]^{-2}. \quad \text{(A3.20)}$$

We can bound $|t_{12}|$ by a constant and do the $\mathbf{q}$ integration. We then have

$$J_1''\leq C\int_0^1 dx[x^2k^2y_3^2+x((8/3)B_i+k^2(\tfrac{2}{3}+2y_3^2)) +2-3y_3^2]^{-1/2}. \quad (A3.21)$$

This integral can be done explicitly; however, for our purposes it is sufficient to note that it exists for all complex values of k and goes to zero as $|k|\to\infty$. We thus see that J_1 is bounded by a constant in the region given by Eq. (3.10).

For future reference we note that the integral can only blow up if the argument of the square root has a double zero as a function of x. Such a zero corresponds to a pinch between the bound-state pole and the free Green's function. We have seen in I that this pinch gives rise to an anomalous threshold at the point $k^2=-A_i$ [see Eq. (3.20)]. It is clear that the matrix element $\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|\hat{t}_{12}G_0\hat{t}_{23}G_0\hat{t}_{12}|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$ goes like $\ln(k^2+A_i)$ as $k^2\to -A_i$ for all values of the $\mathbf{q}_i$. Since it is a square integrable function of the $\mathbf{q}_i$ for all values of $k^2\neq -A_i$, the coefficient of the logarithm must be an L^2 function of the $\mathbf{q}_i$.

The procedure just outlined can also be used to bound the norms of the vectors $K^3\hat{t}|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$ and $\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K^4$. The results are identical to those just obtained.

We now turn to the problem of obtaining bounds on the norms of K_θ^2, $K_\theta^2\hat{t}_\theta|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, $\cdots$. These bounds will go through exactly as the bounds on the norm of K^2, $K^2\hat{t}|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, $\cdots$ except that we must be somewhat careful in bounding the two particle Fredholm denominators. The bound on $D^{-1}(k^2)$ given in Eq. (2.10) is only good in the upper-half k plane since $D^{-1}(k^2)$ will have additional poles associated with resonances in the lower-half k plane.

We are interested in bounding $D^{-1}(k^2-\frac{3}{2}q^2e^{2i\theta})$. (We have dropped the subscript on D_θ since we have seen that it is the analytic continuation of D.) Let us first consider the case $\pi/2>\theta\geq 0$. It will be sufficient to restrict ourselves to the first quadrant of the $ke^{-i\theta}$ plane. In this case the imaginary part of $[k^2-\frac{3}{2}q^2e^{2i\theta}]^{1/2}$ is positive, so the bound on D^{-1} given in Eq. (2.20) holds. A typical denominator in $D_B^{-1}(k^2-\frac{3}{2}q^2e^{2i\theta})$ is

$$d_i=B_i+k^2-\tfrac{3}{2}q^2e^{2i\theta} =e^{2i\theta}[B_ie^{-2i\theta}+(ke^{-i\theta})^2-\tfrac{3}{2}q^2]. \quad (A3.22)$$

If we write $ke^{-i\theta}=R+iI$, we see that d_i can vanish along the hyperbola

$$RI=B_i\sin\theta\cos\theta, \quad (A3.23)$$

provided

$$R\geq B_i^{1/2}\sin\theta,\quad I\leq B_i^{1/2}\cos\theta. \quad (A3.24)$$

If we wish to bound $D_B^{-1}(k^2-\frac{3}{2}q^2e^{2i\theta})$ by a constant, it is clearly sufficient to restrict ourselves to that part of the first quadrant of the $ke^{-i\theta}$ plane above the hyperbola

$$RI=B_m\cos\theta\sin\theta. \quad (A3.25)$$

If we now consider rotating contours through a negative angle, $[k^2-\frac{3}{2}q^2e^{2i\theta}]^{1/2}$ can have a negative imaginary part. However, if we restrict ourselves to the half plane $I\geq 0$, then

$$\text{phase }[k^2-\tfrac{3}{2}q^2e^{2i\theta}]^{1/2}\geq -|\theta|. \quad (A3.26)$$

Since resonances cannot lie on the real axis (zero energy resonances clearly cause no trouble here) we can always rotate through a finite angle $-\theta_0$ before encountering the first resonance. Then for $\theta\geq-\theta_0$, the bound on D^{-1} given in Eq. (2.20) still holds. In this case, d_i can vanish along the hyperbola

$$RI=-B_i|\sin\theta\cos\theta|, \quad (A3.27)$$

provided

$$|R|\geq B_i^{1/2}\cos\theta,\quad I\geq B_i^{1/2}|\sin\theta|. \quad (A3.28)$$

We see that $B_B^{-1}(k^2-\frac{3}{2}q^2e^{2i\theta})$ can be bounded by a constant in the entire first quadrant of the k plane except for semicircles of radius ϵ about the points $k=iB_i^{1/2}$.

If we restrict ourselves to the regions of the k plane in which we can bound the two-particle Fredholm denominator, we can bound $\|K_\theta^2\|^2$ in the same way as we did $\|K^2\|^2$. Using exactly the same procedure with which we went from Eq. (A3.1) to Eq. (A3.4) we find

$$J_\theta\equiv \mathrm{tr}[\hat{t}_{\theta12}G_\theta\hat{t}_{\theta23}G_\theta G_\theta^\dagger\hat{t}_{\theta23}^\dagger G_\theta^\dagger\hat{t}_{\theta12}^\dagger]$$

$$\leq C'\int d^3q',d^3pd^3p'|2p^2+\tfrac{3}{2}q_1'^2-(ke^{-i\theta})^2|^{-2} \times|2p'^2+\tfrac{3}{2}q_1'^2-(ke^{-i\theta})^2|^{-2} \times|(\mathbf{p}-\mathbf{p}')^2+\mu^2e^{-2i\theta}|^{-2}. \quad (A3.29)$$

In Appendix IV we obtain a bound on this integral and show that

$$J_\theta\leq C''\{I\mu\cos\theta[(I+2^{1/2}\mu\cos\theta)^2 +(|R|-2^{1/2}\mu|\sin\theta|)^2]^{1/2}\}^{-1}. \quad (A3.30)$$

We can obtain bounds on the norms of the vectors $K_\theta^2\hat{t}_\theta|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, $K_\theta^3\hat{t}_\theta|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, and $\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K_\theta^4$ by the same arguments that we used on the unrotated contours. The norms exist and are bounded by a constant in the regions given in Eqs. (3.15) and (3.16). We note that even after rotating the contours of integration we will always be left with the integral of Eq. (A3.21) after all of the momentum integrations have been performed. As a result, the vectors will all blow up logarithmically at the anomalous thresholds.

APPENDIX IV

In this Appendix we shall give the details of obtaining bounds on the integrals which appear in Eqs. (A3.4) and (A3.29). Let us start with Eq. (A3.4) and write

$$J'=\int d^3qd^3pd^3p'|p^2+q^2-\tfrac{1}{2}k^2|^{-2}|p'^2+q^2-\tfrac{1}{2}k^2|^{-2} \times[(\mathbf{p}-\mathbf{p}')^2+\mu^2]^{-2}, \quad (A4.1)$$

where we have made the change of variables $\mathbf{q}=(\sqrt{\tfrac{3}{4}})\mathbf{q}_1'$.

Applying the Feynman identity to the denominators in Eq. (A4.1) allows us to do the **p**, **p**′, and **q** integrations. We have

$$J'=\int d^3qd^3pd^3p'\prod_{i=1}^{5}dx_i\delta(1-\sum_{i=1}^{5}x_i)x_5D^{-6}$$
$$=K\int\prod_{i=1}^{5}dx_i\delta(1-\sum_{i=1}^{5}x_i)x_5d^{-3/2}C^{-3/2},\qquad\text{(A4.2)}$$

where

$$D=x_1[p^2+q^2-\tfrac{1}{2}k^2]+x_2[p^2+q^2-\tfrac{1}{2}k^{*2}]+x_3[p'^2+q^2-\tfrac{1}{2}k^2]+x_4[p'^2+q^2-\tfrac{1}{2}k^{*2}]+x_5[(\mathbf{p}-\mathbf{p}')^2+\mu^2],$$

$$d=\begin{vmatrix}1-x_5 & 0 & 0\\ 0 & x_1+x_2+x_5 & -x_5\\ 0 & -x_5 & x_3+x_4+x_5\end{vmatrix}\qquad\text{(A4.3)}$$
$$=(1-x_5)[x_5(1-x_5)+(x_1+x_2)(x_3+x_4)],$$
$$C=\mu^2x_5-\tfrac{1}{2}(R^2-I^2)(1-x_5)-iRI(x_1+x_3-x_2-x_4),$$

and

$$k=R+iI.$$

We now perform the x_5 integration with the aid of the δ function, and make the change of variables

$$x=x_1+x_2+x_3+x_4,\quad a=x_1-x_3,$$
$$y=x_1+x_3-x_2-x_4.\quad b=x_2-x_4.$$

The a and b integration can be done immediately to give

$$J'=8K\int_0^1 dx\,x^{-5/2}(1-x)(1-\tfrac{3}{4}x)^{-1}\int_{-x}^{x}dyC^{-3/2}\times[[x-\tfrac{3}{4}x^2-\tfrac{1}{4}y^2]^{1/2}-[x(1-x)]^{1/2}].\qquad\text{(A4.4)}$$

Since $|y|\leq x$ and $x\leq 1$,

$$[x-\tfrac{3}{4}x^2-\tfrac{1}{4}y^2]^{1/2}-[x(1-x)]^{1/2}\leq\tfrac{1}{8}x^{3/2}.\qquad\text{(A4.5)}$$

As a result

$$J'\leq K\int_0^1 dx\,x^{-1}\int_{-x}^{x}dy[\mu^2-(\mu^2+\tfrac{1}{2}R^2-\tfrac{1}{2}I^2)x-iRIy]^{-3/2}$$
$$=\frac{4K}{RI\mu}\sin^{-1}\left[\frac{R}{R^2+(I+\sqrt{2}\mu)^2}\right].\qquad\text{(A4.6)}$$

Since $\pi/2x\geq\sin^{-1}x$ for $0\leq x\leq 1$, we have the result quoted in Eq. (A3.5).

$$J'\leq K'/I\mu[R^2+(I+\sqrt{2}\mu)^2],$$

or (A4.7)

$$J\leq C''/I\mu[R^2+(I+\sqrt{2}\mu)^2].$$

The integral given in Eq. (A3.24) can be bounded in a similar manner. We have

$$J_\theta'=\int d^3qd^3pd^3p'|p^2+q^2-\tfrac{1}{2}k^2e^{-2i\theta}|^{-2}\times|p'^2+q^2-\tfrac{1}{2}k^2e^{-2i\theta}|^{-2}|(\mathbf{p}-\mathbf{p}')^2+\mu^2e^{-2i\theta}|^{-2}$$
$$=\int d^3qd^3pd^3p'\prod_{i=1}^{6}dx_i\delta(1-\sum_{i=1}^{6}x_i)D_\theta^{-6}$$
$$=\int\prod_{i=1}^{6}dx_i\delta(1-\sum_{i=1}^{6}x_i)d_\theta^{-3/2}C_\theta^{-3/2},\qquad\text{(A4.8)}$$

where

$$D_\theta=x_1[p^2+q^2-\tfrac{1}{2}k^2e^{-2i\theta}]+x_2[p^2+q^2-\tfrac{1}{2}k^{*2}e^{2i\theta}]+x_3[p'^2+q^2-\tfrac{1}{2}k^2e^{-2i\theta}]+x_4[p'^2+q^2-\tfrac{1}{2}k^{*2}e^{2i\theta}]+x_5[(\mathbf{p}-\mathbf{p}')^2+\mu^2e^{-2i\theta}]+x_6[(\mathbf{p}-\mathbf{p}')^2+\mu^2e^{2i\theta}],$$
$$d_\theta=(1-x_5-x_6)[(x_5+x_6)(1-x_5-x_6)+(x_1+x_2)(x_3+x_4)],\qquad\text{(A4.9)}$$
$$C_\theta=(x_5+x_6)\mu^2\cos 2\theta-i(x_5-x_6)\mu^2\sin 2\theta-\tfrac{1}{2}(1-x_5-x_6)(R^2-I^2)-i(x_1+x_3-x_2-x_4)RI,$$

and $ke^{-i\theta}=R+iI$. We now perform the x_6 integration with the aid of the δ function and make the change of variables

$$x=x_1+x_2+x_3+x_4,\quad a=x_5-x_6,$$
$$y=x_1+x_3-x_2-x_4,\quad b=x_2-x_3,\qquad\text{(A4.10)}$$
$$c=x_2-x_4.$$

The b and c integrations can be done immediately. Making use of Eq. (A4.5) we find

$$J_\theta'\leq K\int_0^1 dx\,x^{-1}\int_{-x}^{x}dy\int_{-(1-x)}^{(1-x)}da\,C_\theta^{-3/2}.\qquad\text{(A4.11)}$$

The remaining integrals are straightforward. We find

$$J_\theta'\leq\frac{8K}{RI\mu\sin 2\theta}\left[e^{-i\theta}\ln\left(\frac{\sqrt{2}\mu e^{-i\theta}+I+iR}{\sqrt{2}\mu e^{-i\theta}+I-iR}\right)+e^{i\theta}\ln\left(\frac{\sqrt{2}\mu e^{i\theta}+I-iR}{\sqrt{2}\mu e^{i\theta}+I+iR}\right)\right].\qquad\text{(A4.12)}$$

Using the fact that $\pi/2x\geq\sin^{-1}x$ for $0\leq x\leq 1$, we have

$$J_\theta'\leq K'/I\mu\cos\theta[(I+\sqrt{2}\mu\cos\theta)^2+(|R|-\sqrt{2}\mu|\sin\theta|)^2]^{1/2}.\qquad\text{(A4.13)}$$

This completes the proof of Eq. (A3.30). We note that both J' and J_θ' are finite in the limit $k\to 0$ as long as we do not approach the origin along the I axis.

APPENDIX V

In this Appendix we shall show that the three-body amplitude exists at the three-body threshold and at the thresholds for the scattering of a particle off a two-body bound state.

Let us start with the limit $k\rightarrow 0$. We shall consider $f_\theta(k)$ for negative values of θ so that we can bound the two-body Fredholm denominators by constants. First we consider $\|K_\theta^2\|^2$ for small values of k. We start from the expression for J_θ given in Eq. (A3.29) and again do the $\mathbf{q}_1$ integration making use of Eq. (A2.12).

$$J_\theta\leq C[1+\|R_\theta'\|^2]\int d^3q_1'd^3pd^3p'\,|2p^2+\tfrac{3}{2}q_1'^2-k^2e^{-2i\theta}|^{-2}|2p'^2+\tfrac{3}{2}q_1'^2-k^2e^{-2i\theta}|^{-2}|t_\theta(\mathbf{p},\mathbf{p}';k^2-\tfrac{3}{2}q_1'^2e^{2i\theta})|^2. \quad (A5.1)$$

We recall that $\|R_\theta'\|^2$ is a function of $R^2+I^2/I^2(ke^{-i\theta}=R+iI)$ and can blow up only when this quantity does. From Fig. 2b we see that $\|R_\theta'\|^2$ remains finite as we approach the origin from any direction in the first quadrant of the k plane.

Making use of Eq. (A3.13) we can bound the two Green's functions by

$$|2p^2+\tfrac{3}{2}q_1'^2-k^2e^{-2i\theta}|^{-2}\leq\frac{R^2+I^2}{I^2}[2p^2+\tfrac{3}{2}q_1'^2]^{-2},\quad |2p'^2+\tfrac{3}{2}q_1'^2-k^2e^{-2i\theta}|^{-2}\leq\frac{R^2+I^2}{I^2}[2p'^2+\tfrac{3}{2}q_1'^2]^{-2}. \quad (A5.2)$$

So

$$J_\theta=C\left[\frac{R^2+I^2}{I^2}\right]^2[1+\|R_\theta'|^2]\int d^3pd^3q_1'd^3p'[2p'^2+\tfrac{3}{2}q_1'^2]^{-2}$$
$$\times[2p^2+\tfrac{3}{2}q_1'^2]^{-2}|t_\theta(\mathbf{p},\mathbf{p}';k^2-\tfrac{3}{2}q_1'^2e^{2i\theta})|^2[\theta(p^2-\mu^2)+\theta(\mu^2-p^2)]. \quad (A5.3)$$

Now

$$J_\theta^+=\int d^3pd^3p'd^3q_1'[2p'^2+\tfrac{3}{2}q_1'^2]^{-2}[2p^2+\tfrac{3}{2}q_1'^2]^{-2}|t_\theta|^2\theta(p^2-\mu^2)$$
$$\leq\int d^3pd^3p'd^3q_1'[2p'^2+\tfrac{3}{2}q_1'^2]^{-2}[2\mu^2+\tfrac{3}{2}q_1'^2]^{-2}|t_\theta(\mathbf{p},\mathbf{p}';k^2-\tfrac{3}{2}q_1'^2)|^2$$
$$\leq C(1+\|R_\cdot'\|^2)\int d^3q_1'd^3p'[2p'^2+\tfrac{3}{2}q_1'^2]^{-2}[2\mu^2+\tfrac{3}{2}q_1'^2]^{-2}$$
$$=C'[1+\|R_\theta'\|^2]. \quad (A5.4)$$

We have made use of Eq. (A2.12) in doing the $\mathbf{p}$ integration.

On the other hand

$$J_\theta^-=\int d^3pd^3p'd^3q_1'[2p'^2+q_1'^2]^{-2}[2p^2+\tfrac{3}{2}q_1'^2]^{-2}|t_\theta|^2\theta(\mu^2-p^2)$$
$$\leq[C'+C''\|R_\theta'\|^2]\int d^3pd^3p'd^3q_1'[2p'^2+\tfrac{3}{2}q_1']^{-2}[2p^2+\tfrac{3}{2}q_1'^2]^{-2}\theta(\mu^2-p^2)$$
$$=C_1[C_2+\|R_\theta'\|^2]. \quad (A5.5)$$

We have made use of Eq. (A2.13) in bounding $|t_\theta|^2$.

As a result

$$J_\theta\leq C\frac{(R^2+I^2)^2}{I^2}[1+\|R_\theta'\|^2][C_2+\|R_\theta'\|^2]. \quad (A5.6)$$

We can also obtain bounds on the norms of $K_\theta^2\hat{t}_\theta|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, $K_\theta^3\hat{t}_\theta|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, and $\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K_\theta^4$ that are finite in the limit $k\rightarrow 0$. Let us first consider

$$J_{\theta1}=\|\hat{t}_{\theta12}G_\theta\hat{t}_{\theta23}G_\theta\hat{t}_{\theta12}|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle\|^2. \quad (A5.7)$$

The choice of variables is the same as for Eq. (A3.6). We first do the $\mathbf{q}_1$ integration as in Eq. (A3.7) making use of the bound given in Eq. (A2.12). We then use Eq. (A3.10) to bound the Green's function which depend on $\mathbf{q}_3$.

$$|\mathbf{q}_3^2+\tfrac{1}{4}(\mathbf{q}-\mathbf{k}_3e^{-i\theta})^2+(\mathbf{q}_3+\tfrac{1}{2}\mathbf{q}+\tfrac{1}{2}\mathbf{k}_3e^{-i\theta})^2-k^2e^{-2i\theta}|^{-2}\leq\frac{R^2+I^2}{I^2}[\tfrac{3}{2}\mathbf{q}_3^2+\tfrac{1}{2}(\mathbf{q}_3+\mathbf{q})^2]^{-2}\leq 4\frac{(R^2+I^2)}{I^2}(\mathbf{q}_3+\mathbf{q})^{-4}. \quad (A5.8)$$

Using Eq. (A2.13) to bound $t_{\theta 23}$ we have

$$J_\theta \le C\int d^3q_3 d^3q d^3q' |\mathbf{q}_3+\mathbf{q}|^{-2}|\mathbf{q}_3+\mathbf{q}'|^{-2}|q^2-k_{12}{}^2e^{-2i\theta}|^{-1}|q'^2-k_{12}{}^2e^{-2i\theta}|^{-1}|t_{\theta 12}(\mathbf{q},\mathbf{k}_{12};\tfrac{1}{2}k_{12}{}^2e^{-2i\theta})||t_{\theta 12}(\mathbf{q}',\mathbf{k}_{12};\tfrac{1}{2}k_{12}{}^2e^{-2i\theta})|$$

$$=C'\int d^3q d^3q' |\mathbf{q}-\mathbf{q}'|^{-1}|q^2-k_{12}{}^2e^{-2i\theta}|^{-1}|q'^2-k_{12}{}^2e^{-2i\theta}|^{-1}|t_{\theta 12}(\mathbf{q},\mathbf{k}_{12})||t_{\theta 12}(\mathbf{q}',\mathbf{k}_{12})|\,, \quad \text{(A5.9)}$$

where C and C' are polynomials in $\|R_\theta'\|^2$.

We now make use of the fact that the integrand is symmetric in $\mathbf{q}$ and $\mathbf{q}'$ to write

$$J_{\theta 1}\le 2C'\int_0^\infty q^2dqd\Omega q|q^2-k_{12}{}^2e^{-2i\theta}|^{-1}|t_{\theta 12}(\mathbf{q},\mathbf{k}_{12})|\int_0^q q'^2dq'd\Omega' q'|q'^2-k_{12}{}^2e^{-2i\theta}|^{-1}\,|t_{\theta 12}(\mathbf{q}',\mathbf{k}_{12})|\,|\mathbf{q}-\mathbf{q}'|^{-1}$$

$$=4\pi C'\int_0^\infty qdqd\Omega q|q^2-k_{12}{}^2e^{-2i\theta}|^{-1}|t_{\theta 12}(\mathbf{q},\mathbf{k}_{12})|\int_0^q q'^2dq'|q'^2-k_{12}{}^2e^{-2i\theta}|^{-1}|t_{\theta 12}(\mathbf{q}',\mathbf{k}_{12})|\,. \quad \text{(A5.10)}$$

Using Eqs. (A3.13) and (A2.13) to bound the Green's function and t matrix which depend on $\mathbf{q}'$, we have

$$J_{\theta 1}\le C''\int d^3q|q^2-k_{12}{}^2e^{-2i\theta}|^{-1}|t_{\theta 12}(\mathbf{q},\mathbf{k}_{12})|$$

$$\le C'''. \quad \text{(A5.11)}$$

The last step follows from the same reasoning which led to Eq. (A3.16). C''' is a polynomial in $\|R_\theta'\|^2$ and $(R^2+I^2)/I^2$ so $\|K_\theta{}^2\hat{t}_\theta|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle\|^2$ exists in the limit $k\to 0$ if we approach the origin from any direction in the first quadrant of the k plane. Similar bounds on $\|K_\theta{}^3\hat{t}_\theta|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle\|^2$ and $\|K_\theta{}^{+4}|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle\|^2$ follow from the arguments just presented so the three-body amplitude exists in the limit $k\to 0$.

Let us now consider the threshold for the scattering of a particle off a two-body bound state. The problem is similar to the one discussed in Appendix II. In the two-body problem the free Green's function is not square integrable at $k=0$. In the present case, the two-body Fredholm denominator, $D^{-1}(k^2-\frac{3}{2}q^2)$ is not an L^2 function of $\mathbf{q}$ at the threshold for the scattering of a particle off a two-body bound state, since $D^{-1}(k^2-\frac{3}{2}q^2)$ is proportional to $(k^2+B-\frac{3}{2}q^2)^{-1}$. B is the two-body binding energy.

Our procedure is the same as in Appendix II. We first define the operators P_i by

$$P_i|\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\rangle=|\mathbf{q}_i||\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\rangle,\quad i=1,2,3. \quad \text{(A5.12)}$$

Suppose the 12 system has a bound state of binding energy B. We define the operator a_{12} by

$$a_{12}=(\tfrac{3}{2}P_3{}^2-B-k^2)^{1/2}/(P_3+\mu) \quad \text{(A5.13)}$$

and write

$$A=\begin{bmatrix} a_{12} & 0 & 0\\ 0 & a_{12}{}^{-1} & 0\\ 0 & 0 & a_{12}{}^{-1}\end{bmatrix}. \quad \text{(A5.14)}$$

In analogy with Eq. (A2.3) we consider the amplitude Af.

$$Af=A(1+K)\hat{t}+(AK^2A^{-1})Af. \quad \text{(A5.15)}$$

The kernel of Eq. (A5.15) is

$$AK^2A^{-1}$$

$$=\begin{bmatrix} a_{12}\hat{t}_{12}G_0\hat{t}_{23}G_0a_{12}{}^{-1}+a_{12}\hat{t}_{12}G_0\hat{t}_{31}G_0a_{12}{}^{-1} & a_{12}\hat{t}_{12}G_0\hat{t}_{31}G_0a_{12}{}^{-1} & a_{12}\hat{t}_{12}G_0\hat{t}_{23}G_0a_{12}{}^{-1}\\ a_{12}{}^{-1}\hat{t}_{23}G_0\hat{t}_{13}G_0a_{12} & a_{12}{}^{-1}\hat{t}_{23}G_0\hat{t}_{12}G_0a_{12}+a_{12}{}^{-1}\hat{t}_{23}G_0\hat{t}_{31}G_0a_{12} & a_{12}{}^{-1}\hat{t}_{23}G_0\hat{t}_{12}G_0a_{12}\\ a_{12}{}^{-1}\hat{t}_{31}G_0\hat{t}_{23}G_0a_{12} & a_{12}{}^{-1}\hat{t}_{31}G_0\hat{t}_{12}G_0a_{12} & a_{12}{}^{-1}\hat{t}_{31}G_0\hat{t}_{23}G_0a_{12}+a_{12}{}^{-1}\hat{t}_{31}G_0\hat{t}_{12}G_0a_{12}\end{bmatrix}. \quad \text{(A5.16)}$$

We now consider Eq. (A5.15) on a contour which has been rotated through a *negative* angle, θ. A typical term in

$\|A_\theta K_\theta^2 A_\theta^{-1}\|^2$ is

$$J_\theta=\operatorname{tr}[a_{\theta 12}\hat{t}_{\theta 12}G_\theta\hat{t}_{\theta 23}G_\theta a_{\theta 12}{}^{-1}a_{\theta 12}{}^{\dagger\,-1}G_\theta{}^\dagger\hat{t}_{\theta 23}{}^\dagger G_\theta{}^\dagger\hat{t}_{\theta 12}{}^\dagger a_{\theta 12}{}^\dagger],$$

$$=\int d^3q_1d^3q_3d^3q_1'd^3q_3'\frac{|\tfrac{3}{2}q_3{}^2e^{2i\theta}-k^2-B|}{|q_3e^{i\theta}+\mu|^2}|t_{\theta 12}(\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_3,\ \mathbf{q}_1'+\tfrac{1}{2}\mathbf{q}_3;\ k^2-\tfrac{3}{2}q_3{}^2e^{2i\theta})|^2|q_1'+q_3{}^2+(q_1'+q_3)^2-k^2e^{-2i\theta}|^{-2}$$

$$\times|q_1'^2+q_3'^2+(q_1'+q_3')^2-k^2e^{-2i\theta}|^{-2}|t_{\theta 23}(\mathbf{q}_3+\tfrac{1}{2}\mathbf{q}_1',\ \mathbf{q}_3'+\tfrac{1}{2}\mathbf{q}_1';\ k^2-\tfrac{3}{2}q_1e^{2i\theta})^2|\frac{|q_3'e^{i\theta}+\mu|^2}{|\tfrac{3}{2}q_3'^2e^{2i\theta}-k^2-B|}. \quad (A5.17)$$

This integral is the same as the one considered in Eq. (A3.29) except for the factors of $|a_{\theta 12}(q_3)|^2$ and $|a_{\theta 12}(q_3')|^{-2}$. They clearly do not effect the convergence of the integral at ∞. $|a_{\theta 12}(q_3)|^2$ can vanish in the first quadrant of the k plane only at the point $k^2=-B$ (see Fig. 4b). In that case $|a_{\theta 12}(q_3')|^{-2}\sim q_3'^{-2}$ so the $\mathbf{q}_3'$ integral will still exist. Similarly, $|a_{\theta 12}(q_3)t_{\theta 12}(\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_3,\ \mathbf{q}_1'+\tfrac{1}{2}\mathbf{q}_3,\ k^2-\tfrac{3}{2}q_3{}^2e^{2i\theta})|^2\sim q_3{}^{-2}$ at $k^2=-B$, $q_3{}^2\to 0$. As a result, the $\mathbf{q}_3$ integral is now convergent, which would not have been the case if we had not introduced A. It is clear that J_θ' exists in the limit $k^2\to -B$, and an explicit bound can be obtained by the usual procedure.

Similarly the vectors $\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|A_\theta K_\theta{}^2\hat{t}_\theta|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, $\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|A_\theta K_\theta{}^3\hat{t}_\theta|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$, and $\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K_\theta{}^4A_\theta{}^{-1}|\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\rangle$ will now have finite norms at threshold. From Eqs. (A5.14), (A5.16), and (A3.6) we see that each $t_{\theta 12}$ which depends on $\mathbf{q}_3$ will be multiplied by $a_{\theta 12}$, so the $\mathbf{q}_3$ integral will converge. On the other hand, if $t_{\theta 12}$ appears in an internal loop then $D_{12}{}^{-1}$ and $D_{12}{}^{\dagger-1}$ will depend on different momenta, so there will be no convergence problems. Again explicit bounds on the norms of these vectors can be obtained in the usual way.

The on-energy-shell amplitude can now be written in the form

$$f_\theta(k)=\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|\sum_{i=0}^{7}K_\theta{}^i\hat{t}_\theta|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle+\langle\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|K_\theta{}^4A_\theta{}^{-1}\mathcal{R}_\theta{}^AA_\theta(K_\theta{}^2+K_\theta{}^3)\hat{t}_\theta|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle \quad (A5.18)$$

where

$$\mathcal{R}_\theta{}^A=AK_\theta{}^2A^{-1}+AK_\theta{}^2A^{-1}\mathcal{R}_\theta{}^A.$$

It follows from our previous arguments that $f_\theta(k)$ and consequently $f(k)$ exists at the threshold for the scattering of particle 3 off the bound state, B.

The argument can be repeated at each threshold. If two systems, say 12 and 23, have a bound state of the same energy, we write

$$a_{12}=(\tfrac{3}{2}P_3{}^2-k^2-B)^{1/2}/(P_3+\mu),\quad a_{23}=(\tfrac{3}{2}P_1{}^2-k^2-B)^{1/2}/(P_1+\mu), \quad (A5.19)$$

and then define A to be

$$A=\begin{bmatrix} a_{12}a_{23}{}^{-1} & 0 & 0\\ 0 & a_{23}a_{12}{}^{-1} & 0\\ 0 & 0 & a_{12}{}^{-1}a_{23}{}^{-1}\end{bmatrix}. \quad (A5.20)$$

Our argument now goes through unchanged. Finally if all three two-body systems have a bound state of the same energy we take

$$A=\begin{bmatrix} (a_{12})^{3/2-\epsilon}a_{23}{}^{-1}a_{31}{}^{-1} & 0 & 0\\ 0 & (a_{23})^{3/2-\epsilon}a_{12}{}^{-1}a_{31}{}^{-1} & 0\\ 0 & 0 & (a_{31})^{3/2-\epsilon}a_{12}{}^{-1}a_{23}{}^{-1}\end{bmatrix},\quad \tfrac{1}{2}>\epsilon>0 \quad (A5.21)$$

and our argument again goes through. It then follows that the on-energy-shell three-body amplitude exists at each of its thresholds.