Unitary Restrictions on Scattering Amplitudes from Dynamical Groups*

A. P. BALACHANDRAN, I. GYUK, S. PAKVASA, AND K. RAMAN
Physics Department, Syracuse University, Syracuse, New York
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A method is discussed for deriving restrictions consistent with unitarity on partial-wave amplitudes for two-body reactions, starting with a dynamical group which is a good symmetry only for certain three-point functions. As a simple illustration the application to the quark-antiquark-15 vertex in $U(2,2)$ is worked out in detail. The method is applied to the 364-364*-143 vertex in $U(6,6)$ and sum rules are derived for the partial-wave amplitudes for baryon-antibaryon annihilation into two mesons. The results are seen to be explicitly consistent with unitarity and with the threshold behavior normally expected for partial-wave amplitudes.

I. INTRODUCTION

DURING the past year, several attempts were made to construct a theory which would incorporate the good results of SU_6 [1] in a relativistic framework and enable one to write down S-matrix elements which were invariant under a dynamical group like SU_6 and under the Poincare group. Prominent among these were the $\tilde{U}(12)$ or $U(6,6)$ theory of Salam *et al.*,[2] the $(SU_{12})_{\mathcal{L}}$ scheme of Bég and Pais,[3] and the $M(12)$ theory of Sakita and Wali.[4] It was soon realized, however, that when applied to an arbitrary n-point function, these theories were in conflict with the unitarity of the S matrix.[5–10] Further, for many two-body reaction amplitudes, they predicted results which were at variance with experiment.[7,11]

For three-point functions, however, these theories seemed to make useful predictions; for instance, for the nucleon electromagnetic form factors the predictions of zero-charge form factor for the neutron and the shape equality of the proton electric and magnetic form factors were in fair agreement with the data. It was suggested that perhaps a group like $U(6,6)$ should be regarded as a symmetry only of three-point functions[8,12]; this more restricted assumption would not be in conflict with unitarity.

Given a set of three-point functions—the effective meson-baryon and meson-meson vertex functions—one may immediately write down simple approximations for scattering amplitudes by building these from vertex functions. Such a program was indicated by Salam *et al.*[2]; an N/D calculation of scattering amplitudes starting with vertex functions has been undertaken by Wali *et al.*[13–15]

There is, however, a different method of obtaining information about two-body reaction amplitudes once we are given the three-point functions: The basic idea behind this method is the following. Unitarity states that the absorptive part of a vertex function may be expressed in terms of a set of scattering processes obtained by dispersing the vertex in an appropriate channel (with energy W, say) and inserting a complete set of (physical) intermediate states. For a given value of W, only those intermediate states which have a threshold at an energy below W will contribute to the unitarity relation. Any restrictions on the structure of a vertex function, such as those provided by a symmetry (whether it is an exact symmetry or one that is broken in a specified way) will imply definite restrictions on the scattering processes to which the vertex function is related by unitarity. It is such relations that we shall examine in our present work.

When the unitarity relation can be approximated by a set of two-body intermediate states, these relations become particularly simple. (Note that when the lowest-mass intermediate state is a two-body state, then only this state contributes to the unitarity relation in the energy region between the lowest threshold and the next lowest one, and the two-body unitarity relation is exact.) The unitarity condition now gives a relation between the absorptive part of a vertex function, another set of vertex functions and the partial-wave amplitudes for a set of two-body reactions. If re-

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strictions are imposed on the vertex functions, this gives rise to corresponding restrictions on the partial-wave amplitudes.

The important feature of this method is that the relations thus obtained are automatically consistent with unitarity, as may be expected qualitatively, as they have been obtained from the 2-body unitarity relation for a vertex function.

This method in itself is quite independent of any symmetry group. For instance it may be utilized for obtaining restrictions on two-body amplitudes from empirical information about the form factors. The role of the symmetry group is to provide relations between different three-point functions, which our method translates into relations between partial-wave amplitudes, which are consistent with unitarity.

To illustrate our method we first discuss its application to an internal symmetry group, considering the isospin group for simplicity.

Consider the three-point functions for the πNN and πNN^* vertices. Assuming isospin invariance of the πNN vertex function, one may write each of the vertex functions $\pi^0 pp$ and $\pi^+ np$ as the product of a Clebsch-Gordan coefficient and the same reduced matrix element $\alpha^{(1/2)}(s)$. Approximating the intermediate states in the unitarity relation for these vertex functions by the two-body states (π^0+p) and $(\pi^+ + n)$, we obtain the following equations:

$$-\sqrt{\tfrac{1}{3}}\,\mathrm{Im}\alpha^{(1/2)} = -\sqrt{\tfrac{1}{3}}\alpha^{(1/2)}\langle\pi^0 p|T_{1/2^+}|\pi^0 p\rangle^* + \sqrt{\tfrac{2}{3}}\alpha^{(1/2)}\langle\pi^0 p|T_{1/2^+}|\pi^+ n\rangle^*, \quad (1.1)$$

$$\sqrt{\tfrac{2}{3}}\,\mathrm{Im}\alpha^{(1/2)} = -\sqrt{\tfrac{1}{3}}\alpha^{(1/2)}\langle\pi^+ n|T_{1/2^+}|\pi^0 p\rangle^* + \sqrt{\tfrac{2}{3}}\alpha^{(1/2)}\langle\pi^+ n|T_{1/2^+}|\pi^+ n\rangle^*, \quad (1.2)$$

where the amplitudes $\langle\pi^0 p|T_{1/2^+}|\pi^0 p\rangle$, etc. are the $P_{1/2}$ partial-wave amplitudes (with $l=1$, $J=\frac{1}{2}$).

Using time-reversal invariance and eliminating $\mathrm{Im}\alpha^{(1/2)}/\alpha^{(1/2)}$, we find the relation

$$\langle\pi^0 p|T_{1/2^+}|\pi^0 p\rangle - \langle\pi^+ n|T_{1/2^+}|\pi^+ n\rangle = (1/\sqrt{2})\langle\pi^+ n|T_{1/2^+}|\pi^0 p\rangle. \quad (1.3)$$

By starting with the πNN^* vertex for an N^* with $J=\frac{3}{2}$ and $I=\frac{3}{2}$, the same relation as (1.3) is obtained for the $P_{3/2}$ πN partial-wave amplitudes (with $l=1$, $J=\frac{3}{2}$).

In the above considerations, the N and the N^* could be replaced by other particles with the same internal quantum numbers but different spins; one would then obtain the relation (1.1)–(1.3) for other partial waves. These are the relations for the partial-wave scattering amplitudes which result from the assumptions of SU_2 invariance of the vertex functions and two-body unitarity.

A limitation of this method is immediately apparent: Relations can be obtained only for the amplitudes in those partial waves for which there exist particles with the same quantum numbers, so that a vertex function can be constructed.

If for a given partial wave, particles exist for all the allowed quantum numbers, then the relations obtained (by assuming the invariance of the vertex functions under the purely internal symmetry group) are the same as those resulting from the invariance of the four-point function. This can be shown to be true for an arbitrary internal symmetry group; the reasons are that: (i) The exact symmetry of the vertex function implies that 2-particle states can be written as a sum of orthogonal parts belonging to different irreducible representations of the symmetry group, with the usual Clebsch-Gordan coefficients, which in turn implies the invariance of the scattering amplitude, and (ii) the process of projecting out the partial waves is independent of the internal symmetry group.

Other features of the relations obtained above which are true only for a purely internal symmetry group are the following: (i) These relations are valid at all energies; and (ii) they are independent of the mass of the third particle in the vertex.

When one starts with a dynamical group, the internal symmetry is no longer decoupled from the kinematics, and the relations obtained do not have the properties stated above.

The relations obtained above are consistent with unitarity. This is seen immediately by writing Eqs. (1.1) and (1.2) in the equivalent form

$$\frac{\mathrm{Im}\alpha^{(1/2)}}{\alpha^{(1/2)}} = \left\langle \frac{1}{\sqrt{3}}\{-|\pi^0 p\rangle + \sqrt{2}|\pi^+ n\rangle\}\,|T_{1/2+}|\; \times\frac{1}{\sqrt{3}}\{-|\pi^0 p\rangle + \sqrt{2}|\pi^+ n\rangle\}\right\rangle; \quad (1.4)$$

$$0 = \left\langle \frac{1}{\sqrt{3}}\{-|\pi^0 p\rangle + \sqrt{2}|\pi^+ n\rangle\}\,|T_{1/2+}|\; \times\frac{1}{\sqrt{3}}\{\sqrt{2}|\pi^0 p\rangle + |\pi^+ n\rangle\}\right\rangle^*. \quad (1.5)$$

As any function f of the form $f=\mathrm{Im}\alpha/\alpha$, with $\alpha\neq 0$, satisfies the partial-wave unitarity relation $\mathrm{Im}f^* = |f|^2$ identically, the partial wave amplitude on the right-hand side of Eq. (1.4) is unitary.

By introducing more 2-body intermediate states, e.g. $(K+\Lambda)$ and $(K+\Sigma)$ in the present case, one obtains equations analogous to (1.1) and (1.2) with more independent functions $\alpha^{(I)}$; these give coupled relations for the partial-wave amplitudes for the reactions $\pi N\to\pi N$, $\pi N\to K\Lambda$ and $\pi N\to K\Sigma$. If, however, one starts with a larger internal symmetry group, e.g., SU_3, such that K and π are in the same multiplet (and similarly for N, Λ and Σ), the number of independent functions is again reduced.

To apply our method to obtain relations from a dynamical group like $U(6,6)$ we first give a precise statement of our assumptions.[16]

The conjecture that $U(6,6)$ may be a good symmetry only for three-point functions needs qualification. Firstly, it is only particles at rest that can be assigned to irreducible representations of $U(6,6)$. Particles with nonzero momenta, subject to the equations of motion, cannot be assigned to irreducible representations of the group, so that the symmetry is immediately broken. It is because the symmetry of the vertex is broken right at the outset that it will be possible to obtain in a consistent way scattering amplitudes which are noninvariant (as they must be if they are to be compatible with unitarity). Even though the vertex symmetry is broken, our assumption that particles at rest can be classified according to representations of the dynamical group gives rise to relations between vertex functions with different spin states.

Secondly, the assumption of $U(6,6)$ invariance of the 3-point function (subject to the intrinsic symmetry breaking through the equations of motion) in itself gives exact relations between different 3-point functions only at one value of the energy variable. To obtain relations which are a function of this variable, additional assumptions are required. Thus the results for the nucleon electromagnetic form factors[2,4] were obtained by making the additional assumption that these form factors were dominated by vector meson poles. Results for the baryon-baryon-vector meson vertex may be obtained if one assumes that the vector meson wave function obeys the mass-shell conditions (i.e., the equations of motion), which are strictly true only at one value of the energy. Thus the prescription for writing 3-point functions starting with $U(6,6)$ can give only approximate relations for vertex functions considered as functions of the energy variable.

Thirdly, it is known that the intrinsically broken $U(6,6)$ is good only for certain vertex functions; specifically it forbids most vertex functions involving an orbital angular momentum higher than one.[17] Among the forbidden vertices are, for instance, those corresponding to the experimentally observed decays $f^0 \to 2\pi$, $A_2 \to \rho\pi$, $N^*(\frac{3}{2}-) \to N\pi$, $Y_0^*(\frac{3}{2}-) \to Y\pi$, etc., when we assume the simplest assignments for the π, ρ, N, and Y to representations of $U(6,6)$.[17] Thus it is not possible to assume as a general principle that dynamical groups like $U(6,6)$ are good for all 3-point functions. However, for certain 3-point functions, in particular the meson-baryon-baryon vertex, good results have been obtained from $U(6,6)$ and for these vertices we may use our method to obtain relations for the partial-wave amplitudes involved.

[16] For convenience, most of our statements about dynamical groups will be made in terms of $U(6,6)$; they may be modified so as to apply to other groups as well.

[17] H. Harari, Phys. Rev. Letters **14**, 1100 (1965).

We now formulate our basic assumptions:

(1) Single-particle states can be usefully classified by the finite-dimensional irreducible representations of $U(6,6)$.[18,19] States at rest are assigned to such representations of $U(6,6)$; for states with a nonzero momentum the wave functions are obtained by applying the Bargmann-Wigner equations to the wave functions in the usual way.[2]

(2) The structure of the simplest three-point functions which involve an orbital angular momentum less than or equal to unity, in particular the B-B-P, B-B-V, B-B^*-P, B-B^*-V, P-P-V, P-V-V, and V-V-V vertex functions, is correctly prescribed by writing a formally $U(6,6)$-invariant vertex and putting in for the particle wave functions the forms obtained by imposing the Bargmann-Wigner equations. (Here B is a $\frac{1}{2}^+$ octet baryon, B^* a $\frac{3}{2}^+$ decuplet baryon, P a singlet or octet pseudoscalar meson, and V a singlet or octet vector meson.)

Note that we do not classify 2-particle states by representations of $U(6,6)$. Doing this would immediately result in all the "bad" predictions of $U(6,6)$ invariance for four-point functions. That classifying 2-particle states into representations of $U(6,6)$ would lead to difficulties is evident when we note that the direct product of 2 finite-dimensional representations of $U(6,6)$ contains only states with a finite number of angular momenta, while a two-particle state in which each particle has a definite momentum is a superposition of states with an infinite number of angular momenta.

The contents of this paper are briefly as follows: In Sec. II we apply our method to the quark-antiquark-15 vertex in $U(2,2)$ and obtain the restrictions on the quark-antiquark $^1S_0$, $^3S_1$, and $^3D_1$ elastic partial-wave amplitudes; these show in a simple way the nature of the results obtained for a dynamical group. In Sec. III we obtain some relations for baryon-antibaryon annihilation into two mesons by applying our method to the **364-364*-143** vertex in $U(6,6)$. In Sec. IV we summarize our conclusions. In the Appendix we collect some expressions referred to in the text.

In subsequent papers we shall obtain detailed results for baryon-antibaryon annihilation into 2 mesons and for meson-baryon scattering; we shall also apply our method to groups like $SU(6)_W$.[20,14]

[18] This application of a dynamical group for purposes of classification of particles rather than as a conventional invariance group is akin to the use of these groups as "noninvariance" groups, or "spectrum-generating" groups. For a discussion of these, see Ref. 19.

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II. THE QUARK-ANTIQUARK-15 VERTEX IN $U(2,2)$

We shall now apply the ideas of the preceding section to the dynamical group $U(2,2)$ and derive the restrictions on the quark-antiquark (Q-$\bar{Q}$) elastic partial-wave amplitudes with $J=1^-$ and 0^- that follow from the assumptions of

(i) $U(2,2)$ invariance of the Q-$\bar{Q}$-15 vertex function, where this invariance is understood to be subject to an intrinsic symmetry breaking through the application of the equations of motion, and; (ii) elastic two-particle unitarity, where the Q-$\bar{Q}$-15 vertex function is assumed to be dominated by the elastic $(Q+\bar{Q})$ intermediate state.

Denoting the quark and antiquark operators by ψ_i and $\bar{\psi}^j$ and the meson operator by $\varphi_j{}^i$, $U(2,2)$ invariance of the Q-$\bar{Q}$-15 vertex implies that the vertex operator is of the form $\bar{\psi}^j\psi_i\varphi_j{}^i$ and that the effective vertex function is of the following form in the $\bar{Q}$-Q center-of-momentum (c.m.) frame:

$$\mathfrak{F}(s)=[\bar{v}^j(\mathbf{p})u_i(-\mathbf{p})\bar{\Phi}_j{}^i]\alpha(s)\,, \tag{2.1}$$

where $s=(p_1+p_2)^2$, p_1 and p_2 are the four-momenta of the antiquark and quark, respectively, and $\alpha(s)$ is an undetermined function of s.

The meson wave function $\bar{\Phi}_j{}^i(q)$ in $U(2,2)$, after application of the Bargmann-Wigner equations, is of the form

$$\bar{\Phi}_j{}^i(q)=\left[\frac{(\gamma\cdot q+\mu)}{\mu}\gamma_\lambda\right]_j^i\bar{\varphi}_\lambda+\left[\frac{(\gamma\cdot q+\mu)}{\mu}\gamma_5\right]_j^i\bar{\varphi}_5\,, \tag{2.2}$$

where $\bar{\varphi}_\lambda$ and $\bar{\varphi}_5$ are the wave functions of the vector and pseudoscalar mesons, respectively, contained in the **15** representation of $U(2,2)$, each meson having a mass μ. We shall work throughout in the $\bar{Q}$-Q c.m. frame, where the final meson is at rest.

Since the application of the Bargmann-Wigner equations made above is, strictly, valid only when the energy W of the $\bar{Q}Q$ state is equal to the meson mass μ,[21] an extrapolation is required in order that results be obtained which would hold over a range of energies. A prescription for making this extrapolation will be suggested later in this section.

We quantize the initial $\bar{Q}$-Q state along the z axis and the final meson wave function along the direction (θ,φ). Wave functions with definite helicity for the antiquark, quark and final meson are given in the Appendix.

The vertex functions for the couplings of the quark-antiquark state to a vector meson and a pseudoscalar meson, with the initial and final particles quantized as specified above, must have the following form:

$$\mathfrak{F}_{\rho;\lambda_1\lambda_2}{}^V(s)=\alpha(s)K_{\rho;\lambda_1\lambda_2}{}^V(s)D_{\lambda\rho}{}^{1*}(\varphi,\theta,-\varphi)\,; \tag{2.3a}$$

$$\mathfrak{F}_{0;\lambda_1\lambda_2}{}^P(s)=\alpha(s)K_{0;\lambda_1-\lambda_1}{}^P(s)\delta_{\lambda_2,-\lambda_1}\,. \tag{2.3b}$$

In (2.3), $\mathfrak{F}^V$ describes the coupling of the $\bar{Q}Q$ state to a vector meson (V) and $\mathfrak{F}^P$ the coupling to a pseudoscalar meson (P). λ_1, λ_2, ρ are the helicities of the antiquark, quark and meson, respectively, and $\lambda=(\lambda_1-\lambda_2)$.

The functions $\alpha(s)K_{\rho;\lambda_1\lambda_2}{}^V(s)$ and $\alpha(s)K_{0;\lambda_1\lambda_2}{}^P(s)$, which we shall term the partial-wave vertex amplitudes, are the matrix elements of the vertex operator between the final meson state and an initial $\bar{Q}$-Q state with definite total angular momentum J and definite helicities for the Q and $\bar{Q}$:

$$\alpha(s)K_{\rho;\lambda_1\lambda_2}{}^V(s)=\langle J=1;\lambda_1\lambda_2|\mathfrak{F}|J=1;\rho\rangle\,; \tag{2.4a}$$

$$\alpha(s)K_{0;\lambda_1\lambda_2}{}^P(s)=\langle J=0;\lambda_1\lambda_2|\mathfrak{F}|J=0;0\rangle\,. \tag{2.4b}$$

The form of the dependence on (θ,φ) of the right-hand sides of (2.3a) and (2.3b) follows from rotational invariance. The additional restriction imposed by $U(2,2)$ invariance of the three-point function shows itself in the appearance of the same function $\alpha(s)$ in (2.3a) and (2.3b).

Substituting Eqs. (A.1)–(A.5) into (2.1) with $\bar{\Phi}_j{}^i$ replaced by (2.2), we obtain the following expressions for the functions K^V and K^P defined by (2.3):

$$K_{\rho;++}{}^V=K_{\rho;--}{}^V=\mathfrak{N}\left[-2\beta m(m+E)+\frac{2p^2}{\mu}(W+2m)\right]; \tag{2.5a}$$

$$K_{\rho;+-}{}^V=K_{\rho;-+}{}^V=-\frac{\beta E}{m\sqrt{\mu}}\,; \tag{2.5b}$$

$$K_{0;++}{}^P=-K_{0;--}{}^P=-\frac{i\beta E}{m\sqrt{\mu}}\,; \tag{2.5c}$$

$$K_{0;+-}{}^P=-K_{0;-+}{}^P=0\,; \tag{2.5d}$$

where

$$\beta=\left[1+\frac{2m}{\mu}\right], \tag{2.6}$$

and

$$\mathfrak{N}=[2\mu]^{-1/2}[2m(m+E)]^{-1}. \tag{2.7}$$

In Eqs. (2.5)–(2.7), m is the quark mass, and E and p are its energy and momentum respectively in the c.m. frame. The first equality in each of the Eqs. (2.5) follows from parity conservation.

Our assumption (ii) of elastic 2-particle unitarity for the vertex function may be expressed in terms of the partial-wave vertex amplitudes as follows:

$$\mathrm{Im}[\alpha(s)K_{\rho;\lambda_1\lambda_2}{}^{V,P}(s)]=\sum_{\lambda_1',\lambda_2'}T^*{}_{\lambda_1'\lambda_2';\lambda_1\lambda_2}{}^{V,P}(s)\,\alpha(s)\,K_{\rho;\lambda_1'\lambda_2'}{}^{V,P}(s)\,, \tag{2.8}$$

[21] Note that for a stable meson, this value of W lies below the $\bar{Q}Q$ threshold (and the absorptive part of the vertex function vanishes at this energy); to obtain significant results involving scattering amplitudes at an energy above threshold, an extrapolation is necessary.

where $\rho=0$ in $K_{\rho;\lambda_1\lambda_2}{}^P$, and $T_{\rho;\lambda_1'\lambda_2'}{}^{V,P}(s)$ are the partial-wave helicity amplitudes for elastic $\bar{Q}$-Q scattering in the $J=1^-(V)$ and $J=0^-(P)$ partial waves. $\mathrm{Im}G(s)$ for a function $G(s)$ denotes

$$\mathrm{Im}G(s)\equiv\frac{1}{2i}[G(s+i0)-G(s-i0)]. \tag{2.9}$$

In studying the threshold behavior of the partial-wave amplitudes, it is convenient to work in an L-S representation. States of the initial $\bar{Q}$-Q pair with definite total spin S, orbital angular momentum L, total angular momentum J and its projection M, which are relevant to our problem are the ${}^1S_0$ state with spin-parity 0^- and the ${}^3S_1$ and ${}^3D_1$ states with spin-parity 1^-. The relations between these states and states with definite J, M, λ_1, λ_2 are given in Eqs. (A.6)–(A.8) in the Appendix.

Using (2.4), (2.5) and (A6)–(A8), partial-wave vertex amplitudes in the J, M, L, S representation may be written as follows:

$$\alpha(s)\mathcal{K}^P(s)=\langle P|\mathfrak{F}|{}^1S_0\rangle=-\alpha(s)\frac{i\beta E}{m\sqrt{\mu}}, \tag{2.10a}$$

$$\alpha(s)\mathcal{K}_S{}^V(s)=\langle V|\mathfrak{F}|{}^3S_1\rangle=\alpha(s)\frac{\mathfrak{N}}{\sqrt{3}}\left\{-\beta[p^2+3(E+m)^2]+2p^2\frac{(W+2m)}{\mu}\right\}, \tag{2.10b}$$

$$\alpha(s)\mathcal{K}_D{}^V(s)=\langle V|\mathfrak{F}|{}^3D_1\rangle=\alpha(s)\frac{2\sqrt{2}}{\sqrt{3}}\mathfrak{N}p^2\times\left[\beta+\frac{(W+2m)}{\mu}\right]. \tag{2.10c}$$

These equations define $\mathcal{K}^P$, $\mathcal{K}_S{}^V$, and $\mathcal{K}_D{}^V$.

Here, $|V\rangle$ and $|P\rangle$ denote a vector meson and a pseudoscalar meson, respectively, and the superscripts V and P denote 1^- and 0^- partial-wave amplitudes.

In the L-S representation, the $\bar{Q}$-Q elastic partial-wave amplitudes relevant to our problem are the following:

$$\begin{aligned} f(s)&:\ {}^1S_0\rightarrow{}^1S_0,\\ g_{SS}(s)&:\ {}^3S_1\rightarrow{}^3S_1,\\ g_{DD}(s)&:\ {}^3D_1\rightarrow{}^3D_1,\\ g_{SD}(s)=g_{DS}(s)&:\ {}^3S_1\rightleftarrows{}^3D_1. \end{aligned} \tag{2.11}$$

The amplitudes $g_{SD}(s)$ and $g_{DS}(s)$ are equal by time-reversal invariance.

The unitarity relations in terms of the amplitudes in the L-S representation give the following equations:

$$0^-\ \textit{Vertex:}\ \frac{\mathrm{Im}\alpha^*(s)}{\alpha^*(s)}=f(s), \tag{2.12}$$

$$1^-\ \textit{Vertex:}\ \frac{\mathrm{Im}\alpha^*(s)}{\alpha^*(s)}\begin{bmatrix}\mathcal{K}_S{}^V(s)\\ \mathcal{K}_D{}^V(s)\end{bmatrix}=\begin{bmatrix}g_{SS}(s) & g_{SD}(s)\\ g_{SD}(s) & g_{DD}(s)\end{bmatrix}\begin{bmatrix}\mathcal{K}_S{}^V(s)\\ \mathcal{K}_D{}^V(s)\end{bmatrix}. \tag{2.13}$$

Eliminating $\mathrm{Im}\alpha^*(s)/\alpha^*(s)$ from (2.12) and (2.13), we obtain the equations

$$\begin{bmatrix}g_{SS}(s) & g_{SD}(s)\\ g_{SD}(s) & g_{DD}(s)\end{bmatrix}\begin{bmatrix}\mathcal{K}_S{}^V(s)\\ \mathcal{K}_D{}^V(s)\end{bmatrix}=f(s)\begin{bmatrix}\mathcal{K}_S{}^V(s)\\ \mathcal{K}_D{}^V(s)\end{bmatrix}. \tag{2.14}$$

As observed earlier, these results obtained from the $\bar{Q}$-Q-15 vertex would strictly be true only at $W=\mu$.[21] To obtain relations for W different from μ and above the $\bar{Q}Q$ threshold, we shall assume that to a good approximation, the restrictions imposed by the Bargmann-Wigner equations may be extended to an off-mass-shell meson with mass $\sqrt{(q^2)}=W$ by replacing μ by W in the above equations.[22] *A priori*, it is not clear how good an approximation this would be for values of W considerably larger than μ; this must be judged by comparing the results with experiment in a realistic problem. We expect it will be a good approximation at low energies.

On making the above extrapolation we obtain from Eqs. (2.14) the following energy-dependent sum rules:

$$\left\{-\beta[p^2+3(E+m)^2]+2p^2\left(1+\frac{2m}{W}\right)\right\}\times[g_{SS}(s)-f(s)]+2\sqrt{2}p^2\left[\beta+1+\frac{2m}{W}\right]\times g_{SD}(s)=0; \tag{2.15a}$$

$$2\sqrt{2}p^2\left[\beta+1+\frac{2m}{W}\right][g_{DD}(s)-f(s)]+\left\{-\beta[p^2+3(E+m)^2]+2p^2\left(1+\frac{2m}{W}\right)\right\}\times g_{SD}(s)=0. \tag{2.15b}$$

These are the restrictions on the 0^- and 1^- $\bar{Q}Q$ elastic partial-wave amplitudes which follow from the assumptions (i) and (ii) about the $\bar{Q}$-Q-15 3-point function.

We now examine briefly the nature of the restrictions we have obtained on the partial-wave amplitudes.

A. Consistency with Unitarity

It is easy to see explicitly that Eqs. (2.12) and (2.13) are consistent with unitarity. From (2.12), the 0^- partial-wave amplitude $f(s)$ satisfies the unitarity relation

$$\mathrm{Im}f^*(s)=|f(s)|^2$$

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[22] When the meson is off the mass shell, it has additional components (which become redundant on the mass shell). We assume that these components are small when W is not very different from μ.

identically because of its representation as $\text{Im}\alpha^*(s)/\alpha^*(s)$.

Equation (2.13) states that

$$\begin{bmatrix} \mathcal{K}_S{}^V(s) \\ \mathcal{K}_D{}^V(s) \end{bmatrix}$$

is an eigenvector of the matrix $[g]$ occurring on the right-hand side of Eq. (2.13), belonging to the eigenvalue $\text{Im}\alpha^*/\alpha^* = f(s)$. Equation (2.13) [or (2.14)] specifies the action of $[g]$ only on this eigenvector. Further, (2.13) implies that the relation $\text{Im}g^* = g^\dagger g = gg^\dagger$ holds identically when operating on this eigenvector. Hence (2.13) is consistent with partial-wave unitarity.

B. Threshold Behavior

Consider the equations (2.15a) and (2.15b), divided by factors p and p^3 respectively. Noting that the partial-wave amplitudes $f(s)$, $g_{SS}(s)$, etc., are the amplitudes with the behavior of $[\exp i\delta_l]\sin\delta_l$, we assume the threshold behavior

$$\begin{aligned} f(s) &\sim p, \quad \text{as} \quad p^2 \to 0; \\ g_{SS}(s) &\sim p, \quad \text{as} \quad p^2 \to 0; \\ g_{SD}(s) &\sim p^3, \quad \text{as} \quad p^2 \to 0; \\ g_{DD}(s) &\sim p^5, \quad \text{as} \quad p^2 \to 0. \end{aligned} \tag{2.16}$$

The equations (2.15) when evaluated at threshold (after division by p and p^3) then give the following relations:

$$A(^1S_0) = A(^3S_1) = -\frac{3\sqrt{2}}{2} m^2 A_{SD}, \tag{2.17}$$

where $A(^1S_0)$ and $A(^3S_1)$ are the singlet and the triplet S-wave scattering lengths, and A_{SD} is the triplet $S \to D$ scattering length in $\bar{Q}Q$ scattering:

$$\begin{aligned} A(^1S_0) &= \left[\frac{f(s)}{p}\right]_{p^2=0}; \\ A(^3S_1) &= \left[\frac{g_{SS}(s)}{p}\right]_{p^2=0}; \\ A_{SD} &= \left[\frac{g_{SD}(s)}{p^3}\right]_{p^2=0}. \end{aligned} \tag{2.18}$$

Our approach applied to $\bar{Q}$-Q scattering thus gives relations which are consistent with unitarity and with a nonvanishing scattering at threshold.

We have dealt with the problem of the $\bar{Q}$-Q-15 vertex in some detail, as it shows in a simple manner the qualitative features of the predictions of our approach, which are preserved in more realistic and more complicated cases. Such a realistic example we shall discuss in the next section.

III. THE 364-364*-143 VERTEX IN $U(6,6)$ AND BARYON-ANTIBARYON ANNIHILATION INTO TWO MESONS

In this section, we shall apply our considerations to obtain restrictions on physically measurable reaction amplitudes, namely those for baryon-antibaryon annihilation into two mesons. The baryons and antibaryons we consider will be components of spin-$\frac{1}{2}$ SU_3 octets belonging to the **364** and **364*** representations, respectively, of $U(6,6)$, and the two mesons will be two vector mesons, or two pseudoscalar mesons, or a vector meson and a pseudoscalar meson, where in each case the two mesons belong to the same **143** representation of $U(6,6)$.

We first note that the representations common to the decompositions of **364**×**364*** and **143**×**143** (with identical **143**'s) are just a **143** and a **5940** (besides the trivial 1-dimensional representation).

We consider the **364-364*-143** and the **364-364*-5940** vertex functions and assume that the unitarity relation for each of these is dominated by the contribution of two-meson states, where we include all two-meson states consisting of vector and/or pseudoscalar mesons belonging to the **143** representation.

This is admittedly only a rough approximation, as 3-meson annihilation is known to be strong in nucleon-antinucleon collisions. However, we may expect that the results obtained with this approximation will not be bad at low energies.

With this dynamical assumption, our method when applied to the **364-364*-143** and the **364-364*-5940** vertex functions will predict a set of relations among the partial-wave amplitudes for baryon-antibaryon ($B\bar{B}$) annihilation into 2 mesons.

Noting that the parities of $U(6,6)$ multiplets are assigned uniquely by the assumption that they transform as

$$\Phi_B{}^A \sim \bar{\psi}^A \otimes \psi_B, \tag{3.1}$$

etc., and that the basic quark field is a Dirac field, the **143** multiplet will, on application of the Bargmann-Wigner equations, describe 0^- and 1^- SU_3 singlets and octets, while the **5940** will describe states with spin-parity 0^+ and 2^+ belonging to the **1**, **8** and **27** representations of SU_3, and 1^+ states belonging to the **1**, **8**, **10**, **10*** and **27** representations of SU_3.

Therefore our method when applied to the **364-364*-143** vertex will give relations involving the 0^- and 1^- partial waves, and when applied to the **364-364*-5940** vertex will give relations involving the 0^+, 1^+ and 2^+ partial waves.

A complete treatment of this problem will be given in a subsequent paper. Here we shall outline the application of our method to the **364-364*-143** vertex and discuss some of the results obtained.

The procedure is analogous to that for the Q-$\bar{Q}$-15 vertex in the last section. $U(6,6)$ invariance of the

364-364*-143 vertex implies that the effective vertex function is of the form

$$\alpha(s)\cdot\bar{\Psi}^{(ABC)}\Psi_{(ABC)}\varphi_C{}^D\,, \tag{3.2}$$

where the completely symmetric $\Psi_{(ABC)}$ is the wave function for the **364** and the traceless tensor $\varphi_C{}^D$ is the wave function for the **143**. $\alpha(s)$ is an undetermined function of $s=W^2=(p_1+p_2)^2$, in a notation analogous to that of Sec. II.

Similarly, $U(6,6)$ invariance of the **143-143-143** vertex (which is coupled to the **364-364*-143** vertex through the unitarity relation) implies that the effective vertex function is of the form

$$\beta(s)\Phi_B{}^C\Phi_D{}^B\varphi_C{}^D\,, \tag{3.3}$$

where the first two **143** wave functions refer to the same **143** and $\beta(s)$ is an undetermined function of s.

Writing the $SU_3\times U(2,2)$ decomposition of the wave functions $\Psi_{(ABC)}$, $\Phi_B{}^A$ and $\varphi_B{}^A$, and applying the Bargmann-Wigner equations in a well-known way, we can separate the part of the coupling (3.2) involving the spin-$\frac{1}{2}$ baryon and antibaryon octets and write it as a sum of couplings of these baryon and antibaryons to the 0^- and 1^- SU_3 singlets and octets in the **143** tensor $\varphi_C{}^D$. The assumption of $U(6,6)$ invariance thus enables us to express each of these couplings as a known factor $\mathcal{K}$ times the same function $\alpha(s)$.

A similar procedure applied to the coupling (3.3) enables us to express it as a sum of couplings of (V_r+V_s), (V_r+P_s) and (P_r+P_s) states to the 0^- and 1^- SU_3 singlets and octets in $\varphi_C{}^D$. Here V_r and P_s denote vector (V) and pseudoscalar (P) mesons belonging to the r and s representations of SU_3, where r, s may each be an octet or a singlet. Again, each of these couplings is given by a known factor $\mathcal{G}$ times the function $\beta(s)$.

With the assumption of SU_3 invariance for the partial-wave reaction amplitudes occurring in the unitarity relation for the vertex functions, the 2-meson intermediate states that can contribute to the unitarity relation for the different vertex functions are the following[23]:

(*a*) *The* $(0^-,1)$ *vertex:*

$$(V_1+V_1);\quad (V_8+V_8);\quad (V_1+P_1);\quad (V_8+P_8).$$

(*b*) *The* $(0^-,8)$ *vertex:*

$$(V_1+V_8);\quad (V_8+V_8)_d;\quad (V_1+P_8);\quad (V_8+P_1);\quad (V_8+P_8)_f.$$

(*c*) *The* $(1^-,1)$ *vertex:*

$$(V_1+V_1);\quad (V_8+V_8);\quad (V_1+P_1);\quad (V_8+P_8);\quad (P_1+P_1);\quad (P_8+P_8).$$

[23] Here we characterize the vertex functions by (J^P,R) where J^P gives the spin-parity and R the SU_3 representation of the part of the tensor $\varphi_D{}^C$ which occurs in each of these vertices. The subscripts d and f denote the symmetric and antisymmetric 8 representations occurring in the direct product of the two 8 representations (when both the mesons are SU_3 octets.)

(*d*) *The* $(1^-,8)$ *vertex:*

$$(V_1+V_8);\quad (V_8+V_8)_f;\quad (V_1+P_8);\quad (V_8+P_1);\quad (V_8+P_8)_d;\quad (P_1+P_8);\quad (P_8+P_8)_f.$$

In obtaining these, we have taken into account the requirements of SU_3 invariance, parity conservation, and charge-conjugation invariance. Further restrictions are imposed when the two mesons are identical.

The different partial-wave reaction amplitudes that enter the unitarity relation are the following:

(*i*) $\bar{B}B\to V+V$ *in the* 0^- *state:*

$$^1S_0\to{}^3p_0.$$

(*ii*) $\bar{B}B\to V+V$ *in the* 1^- *state:*

$$^3S_1\to{}^1p_1;\quad {}^3S_1\to{}^3p_1;\quad {}^3S_1\to{}^5p_1;\quad {}^3S_1\to{}^5f_1;$$
$$^3D_1\to{}^1p_1;\quad {}^3D_1\to{}^3p_1;\quad {}^3D_1\to{}^5p_1;\quad {}^3D_1\to{}^5f_1.$$

(*iii*) $\bar{B}B\to V+P$ *in the* 0^- *state:*

$$^1S_0\to{}^3p_0.$$

(*iv*) $\bar{B}B\to V+P$ *in the* 1^- *state:*

$$^3S_1\to{}^3p_1;\quad {}^3D_1\to{}^3p_1.$$

(*v*) $\bar{B}B\to P+P$ *in the* 1^- *state:*

$$^3S_1\to{}^1p_1;\quad {}^3D_1\to{}^1p_1.$$

Here we have used the well-known spectroscopic notation

$$^{2S+1}L_J \tag{3.4}$$

to denote the spin S, orbital angular momentum L, and total angular momentum J of a two-particle state. For the initial $\bar{B}B$ state, $L=0, 1, 2, 3, \cdots$ are denoted by the letters $S, P, D, F, \cdots$, whereas for the final two-meson state they are denoted by the letters $s, p, d, f, \cdots$.

The unitarity relations for the $(0^-,1)$, $(0^-,8)$, $(1^-,1)$, and $(1^-,8)$ vertex functions may now be written down as follows:

(*1*) $(0^-,1)$ *vertex:*

$$\frac{\mathrm{Im}\alpha(s)}{\beta(s)}\mathcal{K}^{P(1)}=[\mathcal{T}_{11}{}^{P(1)*}\mathcal{G}_{11}{}^{P(1)}+\mathcal{T}_{88}{}^{P(1)*}\mathcal{G}_{88}{}^{P(1)}+\mathcal{T}_{11}{}'^{P(1)*}\mathcal{G}_{11}{}'^{P(1)}+\mathcal{T}_{88}{}'^{P(1)*}\mathcal{G}_{88}{}'^{P(1)}]. \tag{3.5}$$

(*2*) $(1^-,1)$ *vertex:*

$$\frac{\mathrm{Im}\alpha(s)}{\beta(s)}\mathcal{K}^{V(1)}(i)=[\sum_f\{\mathcal{T}_{11}{}^{V(1)*}(i\to f)\mathcal{G}_{11}{}^{V(1)}(f)+\mathcal{T}_{88}{}^{V(1)*}(i\to f)\mathcal{G}_{88}{}^{V(1)}(f)\}+\mathcal{T}_{11}{}'^{V(1)*}(i)\mathcal{G}_{11}{}'^{V(1)}+\mathcal{T}_{88}{}'^{V(1)*}(i)\mathcal{G}_{88}{}'^{V(1)}+\mathcal{T}_{11}{}''^{V(1)*}(i)\mathcal{G}_{11}{}''^{V(1)}+\mathcal{T}_{88}{}''^{V(1)*}(i)\mathcal{G}_{88}{}''^{V(1)}]. \tag{3.6}$$

Similar equations may be written for the $(0^-,8)$ and $(1^-,8)$ vertices. The notation used in Eqs. (3.5) and (3.6) is the following:

In Eq. (3.6), $\mathcal{K}^{V(1)}(i)$ is the kinematic factor [obtained from (3.2)] in the vertex function connecting a $\bar{B}B$ system in a state i (where $i={}^3S_1$ or ${}^3D_1$) to a vector (V) particle in the SU_3 representation 1 [denoted by the superscript (1)].

Similarly, $\mathcal{G}_{11}{}^{V(1)}(f)$ is the kinematic factor [obtained from (3.3)] in the vertex function connecting a (V_1+V_1) system in the state f (where $f={}^1p_1$, ${}^3p_1$, ${}^5p_1$ or ${}^5f_1$) to a vector (V) particle in the SU_3 representation 1. The subscripts **11** denote that both the vector particles in this vertex function are SU_3 singlets. (Similarly, the subscripts in $\mathcal{G}_{88}{}^{V(1)}$ indicate that both the vector particles in this vertex function are SU_3 octets.)

$\mathcal{T}_{11}{}^{V(1)}(i\to f)$ is the 1^- partial-wave amplitude (V) for the reaction $\bar{B}+B\to V_1+V_1$ connecting a $\bar{B}B$ state i to a (V_1+V_1) state f, both states being combinations of particle states corresponding to the SU_3 representation **1**. Thus, $\mathcal{T}_{11}{}^{V(1)}(i\to f)$ is an SU_3 reduced matrix element for the 1^- partial-wave amplitude. Again $i={}^3S_1$ or ${}^3D_1$, while f runs over ${}^1p_1$, ${}^3p_1$, ${}^5p_1$, and ${}^5f_1$.

The functions $\mathcal{G}_{11}{}'^{V(1)}$ and $\mathcal{T}_{11}{}'^{V(1)}$ refer to similar quantities when the two-meson system is (V_1+P_1), while $\mathcal{G}_{11}{}''^{V(1)}$ and $\mathcal{T}_{11}{}''^{V(1)}(i)$ refer to functions corresponding to a two-meson system (P_1+P_1). The quantities in Eq. (3.5) are defined similarly; the superscript P (for pseudoscalar) denotes that here the spin-parity of the partial-wave amplitudes and the vertex amplitudes (considered in the appropriate channel) are 0^-.

Eliminating $\mathrm{Im}\alpha(s)/\beta(s)$ from Eq. (3.5) and the two Eqs. (3.9) (for $i={}^3S_1$ and ${}^3D_1$) gives 2 equations. These equations relate the SU_3 reduced matrix elements (corresponding to the representation **1**) for the 0^- and 1^- partial-wave amplitudes occurring in Eqs. (3.5) and (3.6).

A similar set of two equations may be obtained for the SU_3 reduced matrix elements corresponding to the representation **8** by eliminating $\mathrm{Im}\alpha(s)/\beta(s)$ between the unitarity relations for the $(0^-,8)$ and the $(1^-,8)$ vertex functions.

Finally, by eliminating $\mathrm{Im}\alpha(s)/\beta(s)$ between the unitarity relations for the $(0^-,1)$ and $(0^-,8)$ vertex functions, we obtain a relation between the singlet and octet reduced matrix elements for the same (0^-) partial-wave amplitude. (A similar relation may be obtained between the singlet and octet reduced matrix elements for the 1^- partial-wave amplitude.)

These are the restrictions on the baryon-antibaryon annihilation amplitudes that result from our assumptions of $U(6,6)$ invariance for the vertex functions and 2-particle unitarity.

Before discussing the details of some of these results, we note some general properties of them:

(1) The relations obtained here will involve the mass a of the meson occurring in the **143** representation in the vertex function; this enters through the application of the Bargmann-Wigner equations, which are strictly valid only at $W=a$. To obtain relations for $W\neq a$, we shall assume, as in Sec. II, that replacing a by W gives relations that are approximately valid. This will be done in the relations written below.

(2) Our results give relations involving the different *partial-wave* amplitudes. Although this requires detailed measurements to be carried out for these reactions, it has the advantage that when the required experimental information becomes available, it will provide a reliable test of our basic assumptions because a fortuitous agreement of experiment with such detailed predictions would be unlikely.

(3) Our results are automatically consistent with unitarity, for the same reasons as discussed in Sec. II.

(4) The restrictions we derive for the $\bar{B}B$ annihilation amplitudes do not require them to vanish at the $\bar{B}B$ threshold. We recall that assuming $U(6,6)$ invariance directly for the four-point functions and restricting the couplings to the simplest (regular) type require the annihilation amplitudes to vanish at threshold,[11] which is at variance with experiment.

We now list and briefly discuss some of the simpler relations obtained by eliminating $\mathrm{Im}\alpha/\beta$ from (3.5), (3.6), and the other two unitarity equations. We shall express these relations in terms of the amplitudes f^P and f^V obtained by dividing the amplitudes $\mathcal{T}^P$ and $\mathcal{T}^V$ by the momentum factor

$$(k_i)^{l_i+1/2}(k_f)^{l_f+1/2}, \tag{3.7}$$

so as to factor out the threshold zeros in $\mathcal{T}^P$ and $\mathcal{T}^V$.

Here, k_i and l_i are the c.m. momentum and the orbital angular momentum respectively for the initial $\bar{B}B$ state, and k_f, l_f are the corresponding quantities for the final 2-meson state.

For convenience in writing the relations below, we make the approximation of taking equal masses among the vector mesons, and among the pseudoscalar mesons, but keep the vector meson and pseudoscalar meson masses different; we also assume that the mass of each meson is less than the baryon mass.

The simplest relations are those for the 2-meson annihilation amplitudes at the baryon-antibaryon ($\bar{B}B$) threshold. The equations obtained by eliminating $\mathrm{Im}\alpha/\beta$ from (3.5) and (3.6) give the following relation for the $(0^-,1)$ amplitudes at the $\bar{B}B$ threshold:

$$i[f_{88}{}^{P(1)}+\tfrac{1}{3}f_{11}{}^{P(1)}]-H_0f_{88}{}'^{P(1)}=0. \tag{3.8}$$

The notation here is as explained after Eq. (3.6) and before (3.7). H_0 is a function of the meson and baryon masses; it is the value at threshold of the function

$$H=-\frac{\mu_V\mu_0}{\mu_P}\frac{k'}{kW}\left[-\frac{1}{\omega_P}\left\{W-\frac{\Delta}{\mu_0}(\omega_V-\omega_P)\right\}+\frac{2\omega_P\Delta}{\mu_P\mu_V}\right], \quad (3.9)$$

where W is the total c.m. energy, k and k' are the meson c.m. momenta for the reactions $\bar{B}B\to VV$ and $\bar{B}B\to VP$; ω_V and μ_V are the energy and mass of the final vector meson in the process $\bar{B}B\to VP$ and ω_P and μ_P the corresponding quantities for the pseudoscalar meson; and

$$\mu_0=\tfrac{1}{2}(\mu_V+\mu_P);\quad \Delta=\tfrac{1}{2}(\mu_V-\mu_P);\quad m_0=\tfrac{1}{2}(m_1+m_2), \quad (3.10)$$

where m_1 and m_2 are the anti-baryon and baryon masses respectively. In the limit $\mu_V\approx\mu_P\approx\mu$, H_0 has the value μ/m_0.

The relation (3.8) is obtained by noting that for $i={}^3D_1$ the right-hand side of Eq. (3.6) vanishes at threshold faster than $\mathcal{K}^{V(1)}$ on the left-hand side, which gives $\mathrm{Im}\alpha/\beta=0$ at threshold. Equation (3.5) then gives the relation (3.8); while Eq. (3.6) for $i={}^3S_1$ gives the following relation (where the initial $\bar{B}B$ system is in a ${}^3S_1$ state):

$$\sum_r a_0(r)[f_{88}{}^{V(1)}(r)+\tfrac{1}{3}f_{11}{}^{V(1)}(r)]+b_0[f_{88}{}'^{V(1)}+\tfrac{1}{3}f_{11}{}'^{V(1)}]+c_0[f_{88}{}''^{V(1)}+\tfrac{1}{3}f_{11}{}''^{V(1)}]=0. \quad (3.11)$$

Here r runs over the different possible angular momentum configurations for the $(V+V)$ final state in the process $\bar{B}+B\to V+V$: $r={}^1p_1$, ${}^3p_1$, ${}^5p_1$ and ${}^5f_1$. The quantities a_0, b_0 and c_0 are the following functions of the masses:

$$a_0({}^1p_1)=\frac{1}{\sqrt{3}}\left[-\left(1+\frac{m_0}{\mu_V}\right)\frac{8k_0}{\mu_V}\left(2+\frac{k_0^2}{m_0^2}+\frac{m_0^2}{\mu_V^2}\right)+\frac{4m_0}{\mu_V}\left(1+\frac{\mu_V}{m_0}-\frac{k_0^2}{2\mu_V m_0}\right)\right]; \quad (3.12a)$$

$$a_0({}^3p_1)=-8\sqrt{2}\frac{k_0}{\mu_V}\left(1+\frac{2\mu_V}{a}\right); \quad (3.12b)$$

$$a_0({}^5p_1)=\frac{16}{\sqrt{15}}\frac{k_0}{\mu_V}\left[\left(1+\frac{m_0}{\mu_V}\right)\left(1-\frac{k_0^2}{m_0^2}-\frac{m_0^2}{\mu_V^2}\right)+\frac{4m_0}{\mu_V}\left(1+\frac{\mu_V}{m_0}-\frac{k_0^2}{2\mu_V m_0}\right)\right] \quad (3.12c)$$

$$=-(\sqrt{\tfrac{2}{3}})a_0({}^5f_1); \quad (3.12d)$$

$$b_0=\frac{8ik_0'm_0}{\mu_V\mu_P}; \quad (3.12e)$$

$$c_0=\frac{8k_0}{\mu_P}\left(1+\frac{m_0}{\mu_P}\right). \quad (3.12f)$$

Here k_0 and k_0' are the values of k and k' at the $\bar{B}B$ threshold.

We obtain relations for the $(0^-,8)$ and $(1^-,8)$ amplitudes at the $\bar{B}B$ threshold in a similar manner. For the $(0^-,8)$ amplitudes, the relation is

$$i[f_{88}{}^{P(8)}+(\sqrt{\tfrac{2}{3}})f_{81}{}^{P(8)}]-H_0f_{88}{}'^{P(8)}=0, \quad (3.13)$$

where H_0 is again given by (3.9) evaluated at threshold. For the $(1^-,8)$ amplitudes at threshold (with the initial $\bar{B}B$ pair in a ${}^3S_1$ state), the relation is

$$\sum_r a_0(r)[f_{88}{}^{V(8)}(r)+(\sqrt{\tfrac{2}{3}})f_{81}{}^{V(1)}(r)]+b_0[f_{88}{}'^{V(8)}+(\sqrt{\tfrac{2}{3}})f_{81}{}'^{V(8)}+\sqrt{\tfrac{2}{3}}f_{18}{}'^{V(8)}]+c_0f_{88}{}''^{V(8)}=0, \quad (3.14)$$

where $a_0(r)$, b_0 and c_0 are given by Eqs. (3.12), and the notation r is as in (3.11).

Finally, from the unitarity equations for the $(0^-,1)$ and $(0^-,8)$ states, we obtain the following momentum-dependent sum rule:

$$i[f_{88}{}^{P(1)}+\tfrac{1}{3}f_{11}{}^{P(1)}]-Hf_{88}{}'^{P(1)}=-6\sqrt{3}\{i[f_{88}{}^{P(8)}+\sqrt{\tfrac{2}{3}}f_{81}{}^{P(8)}]-Hf_{88}{}'^{P(8)}\}, \quad (3.15)$$

where H is given by (3.19).

All the relations given above are for partial-wave amplitudes which are reduced matrix elements with respect to SU_3. To obtain relations for amplitudes expressed in terms of particle states, it is best to express the unitarity relation directly in terms of such amplitudes. We give below two such sum rules, for $\bar{p}p$ annihilation in the ${}^1S_0$ state at threshold. In obtaining these, we have used the restrictions imposed by the symmetry character of the final state (in addition to those from parity conservation and charge conjugation invariance); also, we have taken into account φ-ω mixing and have omitted the final states $(\varphi^0+\varphi^0)$, (φ^0+X^0), and (X^0+X^0), which are forbidden at threshold when one assigns the particles their physical masses. The relations for $\bar{p}p$ annihilation amplitudes are the following:

$$(8/9)[f^P(\rho^0\rho^0)+\sqrt{2}f^P(\rho^+\rho^-)]+(2/3)f^P(\omega\omega)+(5/9)f^P(\omega\varphi)+(19\sqrt{2}/18)[f^P(K^{*+}K^{*-})+f^P(K^{*0}\bar{K}^{*0})]$$
$$=H_0\{f'^P(\bar{K}^{*-}K^+)-f'^P(K^{*+}K^-)+f'^P(\bar{K}^{*0}K^0)-f'^P(K^{*0}\bar{K}^0)+(\sqrt{2}/9\sqrt{3})[f'^P(\omega X^0)-\sqrt{2}f'^P(\omega\eta)+f'^P(\varphi\eta)]\}; \quad (3.16)$$

$$f^P(\rho^0\rho^0)+\sqrt{2}f^P(\rho^+\rho^-)+(5/9)f^P(\omega\omega)+(4/9)f^P(\omega\varphi)-(\sqrt{2}/5)f^P(\omega\rho)$$
$$-(4/15)f^P(\varphi\varphi)+(9\sqrt{2}/10)f^P(K^{*-}K^{*+})+(11\sqrt{2}/10)f^P(K^{*0}\bar{K}^{*0})$$
$$=-H_0(\sqrt{2}/5)\{(1/\sqrt{2})f'^P(\rho^-\pi^+)-(1/\sqrt{2})f'^P(\rho^+\pi^-)+(1/\sqrt{3})f'^P(\rho^0X^0)-(\sqrt{2}/3)f'^P(\omega\pi^0)+(1/3)f'^P(\varphi\pi^0)$$
$$+f'^P(K^{*-}K^+)-f'^P(K^{*+}\bar{K}^-)-f'^P(K^{*0}\bar{K}^0)+f'^P(\bar{K}^{*0}K^0)\}. \quad (3.17)$$

We have for simplicity written the relations here with degenerate meson masses (among the vector mesons and among the pseudoscalar mesons); they may be written with the physical meson masses by substituting for H_0, a_0, etc. more complicated expressions.

The sum rules (3.16) and (3.17) give inequalities relating the partial cross sections for $\bar{p}p$ annihilation in the $^1S_0$ state into the different allowed two-meson final states. The available data on $\bar{p}p$ annihilation[24] give only the total cross sections. We may make a rough comparison by assuming that the partial cross sections for $\bar{p}p$ annihilation are roughly in the same ratio as the total cross sections; the inequalities are then found to be satisfied. However, more detailed measurements are required before an adequate comparison of our predictions with experiment can be made.

We have in this section shown the different types of relations that may be derived by our method by starting with $U(6,6)$ invariance of the meson-baryon vertex function. A detailed treatment of baryon-antibaryon annihilation will be given in a subsequent paper.

IV. CONCLUSIONS

In this paper we have discussed a method of obtaining predictions for scattering amplitudes from dynamical groups which are at best invariance groups only of certain three-point functions. The method yields sum rules which are consistent with unitarity and, at least for $U(6,6)$, also with the threshold behavior normally expected for partial-wave amplitudes. In Sec. II we have illustrated the procedure in detail for the $\bar{Q}$-Q-15 vertex of $U(2,2)$, and in Sec. III we have obtained sum rules relating the different amplitudes for baryon-antibaryon annihilation into two mesons.

Although dynamical groups like $U(6,6)$ have met with serious objections, they have also made a few successful predictions. The present work is an attempt to combine the good results of such groups with dynamical assumptions so as to obtain useful results which avoid the traditional objections to these groups, particularly the conflict with unitarity.

The relations predicted by our method for baryon-antibaryon ($\bar{B}B$) annihilation can in principle be tested experimentally. In subsequent papers we shall discuss in detail these and other predictions for $\bar{B}B$ annihilation as well as predictions for meson-baryon scattering. We shall also discuss how by making further assumptions, more restrictive relations may be obtained which may be easier to test. We shall also apply our method to groups like $SU(6)_W$.

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APPENDIX

A quark wave function with definite helicity, quantized along the $-z$ direction, and an antiquark wave function with definite helicity, quantized along the $+z$ direction are given by the following:

$$u_{(\pm)}(-\mathbf{p})=[2m(E+m)]^{-1/2}\begin{bmatrix}(E+m)\chi_{(\pm)}\\ -(\boldsymbol{\sigma}\cdot\mathbf{p})\chi_{(\pm)}\end{bmatrix}, \quad (A1)$$

where

$$\chi_{(+)}=\begin{bmatrix}0\\ -1\end{bmatrix};\quad \chi_{(-)}=\begin{bmatrix}1\\ 0\end{bmatrix}; \quad (A2)$$

and

$$v_{(\pm)}(\mathbf{p})=[2m(E+m)]^{-1/2}\begin{bmatrix}(\boldsymbol{\sigma}\cdot\mathbf{p})\xi_{(\pm)}\\ (E+m)\xi_{(\pm)}\end{bmatrix}, \quad (A3)$$

where

$$\xi_{(+)}=\begin{bmatrix}0\\ 1\end{bmatrix};\quad \xi_{(-)}=\begin{bmatrix}1\\ 0\end{bmatrix}. \quad (A4)$$

For the final vector meson at rest, quantized along the direction (θ,φ), a wave function $\bar{\varphi}_\rho$ with a definite helicity ρ is given by

$$\bar{\varphi}_\rho=\frac{(-1)^\rho}{[2\mu]^{1/2}}\begin{pmatrix}D_{1\rho}{}^1(\varphi,\theta,-\varphi)\\ D_{0\rho}{}^1(\varphi,\theta,-\varphi)\\ D_{-1\rho}{}^1(\varphi,\theta,-\varphi)\end{pmatrix}, \quad (A5)$$

where μ is the mass of the vector meson and the $D_{\lambda\mu}{}^J(\varphi,\theta,-\varphi)$ are the rotation matrices.

The relations between the $^1S_0$, $^3S_1$, and $^3D_1$ states for the $\bar{Q}$-Q system and states $|JM;\lambda_1\lambda_2\rangle$ with a total angular momentum J and definite helicities λ_1 and λ_2 for the antiquark ($\bar{Q}$) and the quark (Q) are given by the following:

$$|^1S_0\rangle=\frac{1}{\sqrt{2}}\{|00;++\rangle-|00;--\rangle\}; \quad (A6)$$

[24] We are indebted to Dr. T. Kalogeropoulos for a discussion of the experimental data on $\bar{N}N$ annihilation into 2 mesons.

$$|{}^3S_1\rangle=\frac{1}{\sqrt{6}}\{|11;++\rangle+|11;--\rangle\}$$
$$+\frac{1}{\sqrt{3}}\{|11;+-\rangle+|11;-+\rangle\};\quad (A.7)$$

$$|{}^3D_1\rangle=-\frac{1}{\sqrt{3}}\{|11;++\rangle+|11;--\rangle\}$$
$$+\frac{1}{\sqrt{6}}\{|11;+-\rangle+|11;-+\rangle\}.\quad (A.8)$$

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Weak-Electromagnetic Decays of Hyperons in a Broken $SU(3)\otimes SU(3)$ Model*

D. S. CARLSTONE AND WALTER A. SIMMONS
Department of Physics, Purdue University, Lafayette, Indiana
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The weak (strangeness-changing)–electromagnetic decays of the hyperons of the type $B\rightarrow B\gamma$ are considered in a broken $SU(3)\otimes SU(3)$ model. Various sets of assumptions regarding the transformation properties of the particles, the breaking, and the Hamiltonian are examined and for each a set of sum rules for the decay amplitudes is obtained.

I. INTRODUCTION

SINCE the assumption that the weak hadron currents generate the algebra of $SU(3)\otimes SU(3)$ has recently found wide application in weak-interaction physics, several authors have investigated the possibility that the $SU(3)\otimes SU(3)$ may be an approximate invariance group of the weak Hamiltonian. Iizuka and Miyamoto[1] investigate the possibility that $SU(3)\otimes SU(3)$ may be an exact invariance group of H_w and found that the nonleptonic decays cannot be correctly described in such a scheme. Later, Schechter and Ueda[2] found that broken chiral $SU(3)\otimes SU(3)$ can be used to obtain a new sum rule for the hyperon nonleptonic decays which is in rough agreement with experiment.

The purpose of this paper is to investigate the application of the group structure of broken chiral $SU(3)\otimes SU(3)$ to the weak-electromagnetic processes which have recently been examined with a current algebra approach by Graham and Pakvasa.[3]

II. THE $SU(3)\otimes SU(3)$ MODEL

The space-time structure of the weak-electromagnetic (WE) Hamiltonian is assumed to be given by[4,5]

$$H_{\rm WE}=\bar{N}(A+B\gamma_5)\sigma_{\mu\nu}NF_{\mu\nu},$$

where N is a baryon 4-spinor. Before examining the $SU(3)\otimes SU(3)$ structure of this Hamiltonian, let us review the model.

According to the model of Marshak, Mukunda, and Okubo[6] the $SU(3)\otimes SU(3)$ algebra is generated by two sets of $SU(3)$ matrices, $A_\nu{}^\mu$ and $B_{\nu'}{}^{\mu'}$, which satisfy the commutation relations

$$[A_\nu{}^\mu,A_\beta{}^\alpha]=\delta_\nu{}^\alpha A_\beta{}^\mu-\delta_\beta{}^\mu A_\nu{}^\alpha,$$
$$[B_{\nu'}{}^{\mu'},B_{\beta'}{}^{\alpha'}]=\delta_{\nu'}{}^{\alpha'}B_{\beta'}{}^{\mu'}-\delta_{\beta'}{}^{\mu'}B_{\nu'}{}^{\alpha'},$$
$$[A_\nu{}^\mu,B_{\beta'}{}^{\alpha'}]=0.$$

For every $SU(3)\otimes SU(3)$ tensor, primes are used throughout this paper to indicate those indices transformed by $B_{\nu'}{}^{\mu'}$ and the unprimed indices are transformed by $A_\nu{}^\mu$. Under parity P and charge conjugation C, the generators are transformed:

$$P:\quad A_\nu{}^\mu\rightarrow B_{\nu'}{}^{\mu'},\qquad B_{\nu'}{}^{\mu'}\rightarrow A_\nu{}^\mu,$$
$$C:\quad A_\nu{}^\mu\rightarrow -B_{\mu'}{}^{\nu'},\quad B_{\nu'}{}^{\mu'}\rightarrow -A_{\mu'}{}^{\nu'}.$$

An irreducible representation of $SU(3)\otimes SU(3)$, (μ_1,μ_2) is transformed according to

$$P:\quad (\mu_1,\mu_2)\rightarrow(\mu_2,\mu_1),$$
$$C:\quad (\mu_1,\mu_2)\rightarrow(\mu_2^*,\mu_1^*).$$

In the notation of Schechter and Ueda,[7] a four-component baryon spinor $N_b{}^a$ transforms under $SU(3)\otimes SU(3)$ as

$$N_b{}^a=\begin{pmatrix} f_b{}^{a'} \\ -i\sigma_2 g^a{}_{b'}\end{pmatrix},$$

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