Two-Dimensional Relativistic Quark Model with Exact Solutions*

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A relativistic two-dimensional quark model is developed in which zero-mass quarks interact through a massive, neutral vector boson. Exact solutions for the quark fields are found, and vector and axial-vector current densities are constructed. An $SU(3)\times SU(3)$ algebra of the time components of these currents is investigated by explicitly calculating the equal-time commutators. It is found that all noncanonical terms (Schwinger terms) are absent.

I. INTRODUCTION

THE quark model of hadrons has been successfully applied to the classification of particles and $SU(6)$ calculations.[1] At the present time quarks have not been detected experimentally. In the current algebra program[2] the quarks describe a model that generates equal-time commutation relations for the fourth component of the current densities. This free-field quark model describes an $SU(3)\times SU(3)$ algebra that leads to sum rules, e.g., the Adler-Weisberger relation for the renormalization of the axial-vector coupling constant in β decay.[3]

The commutation relations are abstracted from the quark model without presupposing that the quarks have a physical reality. But the general consensus is that hadrons are not elementary, but are composed of more fundamental constituents even though there is presently no experimental evidence to support this viewpoint. The results of recent work strongly suggest that these fundamental particles are quarks with fractional charge.

We shall investigate the relativistic quark model in which the quarks are "glued" together by a neutral, massive vector boson.[2] In order to obtain an exact solution to the relativistic problem, we shall extend the two-dimensional Thirring model[4] to an $SU(3)$ formalism based on triplet quarks. It is hoped that with the aid of this model, we may learn how to solve some of the complex problems in the physical system in four dimensions.

II. THE GELL-MANN QUARK MODEL IN TWO DIMENSIONS

We begin with the Lagrangian model of Gell-Mann.[2] There is a triplet[5] ψ_i $(i=1,2,3)$ of two-component fermion fields corresponding to three spin-$\frac{1}{2}$ quarks: the isotopic doublet n and p, with charges $\frac{2}{3}$ and $-\frac{1}{3}$, respectively, and the isotopic spin singlet λ, with charge $-\frac{1}{3}$. The singlet, neutral vector meson is denoted by V_μ. The Lagrangian is

$$\mathcal{L}=-\tfrac{1}{2}:(\partial^\mu V^\nu\partial_\mu V_\nu-m^2V^\nu V_\nu): +i\sum_{i=1}^{3}:\bar{\psi}_i\gamma^\mu\partial_\mu\psi_i:-g::V^\mu:j_\mu:, \quad (1)$$

where m is the boson mass and

$$j_\mu=\sum_{i=1}^{3}\bar{\psi}_i\gamma_\mu\psi_i. \quad (2)$$

The quarks have *zero* mass and the Lagrangian is invariant under $SU(3)$ transformations.

In Eq. (1), we have introduced a "double dot" notation, which is explained in Appendix I.

The equations of motion obtained from the Lagrangian are

$$(\Box+m^2):V_\nu:=g:j_\nu:, \quad (3)$$

$$i:\gamma^\mu\partial_\mu\psi_i:-g::V_\mu:\gamma^\mu\psi_i:=0. \quad (4)$$

The solution for V_μ is

$$:V_\mu(x):=V_\mu^{\text{in}}(x)-g:\int d^2x'\Delta_R(x-x';m) \times:j_\mu(x')::. \quad (5)$$

If we choose $V_\mu^{\text{in}}(x)=0$, then the V boson can be treated as an unstable "elementary" particle. We shall find that this leads to a simple solution of our system of equations. With this in mind, we obtain from Eq. (4):

$$i:\gamma^\mu\partial_\mu\psi_i(x):+g^2:\int d^2x'\Delta_R(x-x';m) \times:j_\mu(x'):\gamma^\mu\psi_i(x):=0. \quad (6)$$

III. EXACT SOLUTIONS OF QUARK FIELD ψ

We shall now solve Eq. (6) for the quark fields $\psi_i(x)$ by using a method that parallels that for the Thirring model given by Glaser.[6]

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[1] R. H. Dalitz, in *Proceedings of the Thirteenth International Conference on High Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, California, 1967).

[2] M. Gell-Mann, Physics 1 63 (1964). For a review of this subject see: J. W. Moffat, in *Proceedings of the Istanbul Advanced Study Institute on Symmetry Principles and Fundamental Particles*, edited by B. Kurşunoğlu and A. Perlmutter (W. H. Freeman and Company, San Francisco, 1967).

[3] S. L. Adler, Phys. Rev. Letters **14**, 1015 (1965); W. I. Weisberger, *ibid.* **14**, 1047 (1965).

[4] W. Thirring, Ann. Phys. **9**, 91 (1958). F. Schwabel, W. Thirring, and J. Wess, Report No. IC/66/96, 1966 (unpublished).

[5] The model may be worked out for any number of fermion fields. One has only to replace $\sum_{i=1}^{3}$ by $\sum_{i=1}^{N}$.

[6] V. Glaser, Nuovo Cimento **9**, 990 (1958).

The metric we use is of the form

$$x^1=x,\quad x^2=t,\quad g^{11}=-1,\quad g^{22}=+1. \tag{7}$$

We shall employ the γ-matrix representation

$$\gamma^1=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \gamma^2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{8}$$

and the notation

$$\psi_i=\begin{pmatrix}\psi_{1,i}\\ \psi_{2,i}\end{pmatrix},\quad (i=1,2,3)$$

$$u_1=-x+t,\quad u_2=x+t. \tag{9}$$

With the help of a little algebra, we get

$$:\frac{\partial}{\partial u_2}\psi_{1,i}(x):-ig^2:\int d^2x'\Delta_R(x-x';m)\times\sum_{j=1}^{3}:\psi_{2,j}^\dagger(x')\psi_{2,j}(x'):\psi_{1,i}(x):=0 \tag{10}$$

and a similar equation obtained by making the substitution $(1\leftrightarrow 2)$. The solutions will be expressed in terms of the incoming free-quark fields $\phi_{r,i}(x)$ $(r=1,2;\ i=1,2,3)$ which satisfy

$$\frac{\partial}{\partial u_2}\phi_{1,i}(x)=\frac{\partial}{\partial u_1}\phi_{2,i}(x)=0. \tag{11}$$

Thus the incoming fields have the special x,t dependence $\phi_{r,i}(u_r)\,(r=1,2)$

The Hilbert space satisfies the conditions

$$\langle 0|0\rangle=1, \tag{12}$$

and

$$a_i(p)|0\rangle=0,\quad b_i(p)|0\rangle=0, \tag{13}$$

where a_i and b_i denote the quark and antiquark annihilation operators, respectively.

The solution of the equations of motion (10) is of the form

$$\psi_{1,i}(x)=\phi_{1,i}(u_1)\exp\left[ig^2\int_{-\infty}^{u_2}du_2'\times\int d^2x''\Delta_R(x'-x'';m)\rho_2(u_2'')\right], \tag{14}$$

The solution for $\psi_{2,i}(x)$ is obtained by making the substitution $(1\leftrightarrow 2)$ in (14) and

$$\rho_r(u_r)=\sum_{j=1}^{3}:\phi_{r,j}^\dagger(u_r)\phi_{r,j}(u_r): \tag{15}$$

We observe that the :: notation corresponds to the removal of all disconnected graphs from the solution. The exponential operators in Eq. (14) may be written as a product of a field renormalization constant $Z^{1/2}$ and a normal-ordered exponential operator.[7]

By using the canonical equal-time commutation relations of the free fields, and the special space-time dependence of these fields, one can easily verify that

$$\{\phi_{r,i}(u_r),\phi_{s,j}(u_s')\}=0,$$
$$\{\phi_{r,i}^\dagger(u_r),\phi_{s,j}(u_s')\}=\delta_{rs}\delta_{ij}\delta(u_r-u_s'). \tag{16}$$

Moreover, it follows that

$$:\psi_{r,j}^\dagger\psi_{r,j}:=:\phi_{r,j}^\dagger\phi_{r,j}:=\rho_{r,j}, \tag{17}$$

and

$$[\rho_{r,i}(u_r),\rho_{s,j}(u_s')]=0 \tag{18}$$

for all r,s and i,j. Since we can now differentiate the exponential operators as if they were c numbers, it follows that

$$:\frac{\partial}{\partial u_2}\psi_{1,i}(x):=ig^2:\int d^2x'\Delta_R(x-x';m)\rho_2(u_2')\psi_{1,i}(x): =ig^2:\int d^2x'\Delta_R(x-x';m)\times\sum_{j=1}^{3}:\psi_{2,j}^\dagger(x')\psi_{2,j}(x'):\psi_{1,i}(x):. \tag{19}$$

and similar equations for $\psi_{2,i}(x)$. This verifies that the equations of motion for the quark fields are satisfied.

Here we note that if $V_\mu^{\text{in}}(x)\neq 0$, then the equation of motion for the quarks is

$$:\frac{\partial}{\partial u_2}\psi_{1,i}(x):+ig:\left(V_+^{\text{in}}(x)-g\int d^2x'\Delta_R(x-x';m)\times\sum_{j=1}^{3}:\psi_{2,j}^\dagger(x')\psi_{2,j}(x'):\right)\psi_{1,i}(x):=0. \tag{20}$$

The equation of motion for $\psi_{2,i}(x)$ is given by making the substitutions $[1\leftrightarrow 2;\ V_+^{\text{in}}(x)\to V_-^{\text{in}}(x)]$ where

$$V_\pm^{\text{in}}(x)=\pm V_1^{\text{in}}(x)+V_2^{\text{in}}(x). \tag{21}$$

One cannot simply multiply the above solution Eq. (21) by

$$\exp\left[-ig\int_{-\infty}^{u_2}du_2'V_+^{\text{in}}(u_1,u_2')\right], \tag{22}$$

since

$$[V_+^{\text{in}}(u_1,u_2),\ V_+^{\text{in}}(u_1,u_2')]\neq 0 \tag{23}$$

and the exponential cannot be differentiated as if it were a c number.

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[7] W. Thirring, Nuovo Cimento 9, 1007 (1958).

IV. CANONICAL COMMUTATION RELATIONS FOR INTERACTING QUARK FIELDS ψ

The following identity can be derived[6]:

$$\exp[-ig^2 Q_{r,i}(u_r)]\phi_{r,i}(u_r')\exp[ig^2 Q_{r,i}(u_r)] = \exp\left[ig^2\int d^2x''\Delta_R(x'';m)\theta(u_r-u_r'-u_r'')\right] \times\phi_{r,i}(u_r'), \quad (24)$$

where

$$Q_{r,i}(u_r)=\int_{-\infty}^{u_r} du_r'\int d^2x''\Delta_R(x'-x'';m)\rho_{r,i}(u_r''). \quad (25)$$

We can establish this identity by differentiating each side with respect to g^2 and showing that each side satisfies the same first-order differential equation in g^2. For $g^2=0$ each side is equal to $\phi_{r,i}(u_r')$. With the help of the identity (24), we can obtain the commutation relations for the interacting quark fields.

Let us put

$$F(u_r-u_r')=\exp\left[ig^2\int d^2x''\Delta_R(x'';m) \times\theta(u_r-u_r'-u_r'')\right], \quad (26)$$

and

$$Q_r(u_r)=\sum_{i=1}^{3} Q_{r,i}(u_r).$$

The solution may be written

$$\psi_{1,i}(x)=\phi_{1,i}(u_1)\exp\left[ig^2\sum_{j=1}^{3} Q_{2,j}(u_2)\right], \quad (27)$$

with a similar equation for $\psi_{2,i}(x)$. We now find that

$$\{\psi_{1,i}(x),\psi_{1,j}(y)\}=\{\phi_{1,i}(u_1),\phi_{1,j}(v_1)\}\exp\{ig^2[Q_2(u_2)+Q_2(v_2)]\}=0. \quad (28)$$

A further calculation gives

$$\begin{aligned}\{\psi_{1,i}(x),\psi_{2,j}(y)\} &=\phi_{1,i}(u_1)\exp[ig^2Q_2(u_2)]\phi_{2,j}(v_2)\exp[ig^2Q_1(v_1)]+\phi_{2,j}(v_2)\exp[ig^2Q_1(v_1)]\phi_{1,i}(u_1)\exp[ig^2Q_2(u_2)],\\ &=\{\phi_{1,i}(u_1)\exp[ig^2Q_{2,j}(u_2)]\phi_{2,j}(v_2)\exp[-ig^2Q_{2,j}(u_2)]+\phi_{2,j}(v_2)\exp[ig^2Q_{1,i}(v_1)]\phi_{1,i}(u_1)\exp[-ig^2Q_{1,i}(v_1)]\}\\ &\qquad\times\exp[ig^2(Q_1(v_1)+Q_2(u_2))],\\ &=[F^*(u_2-v_2)-F^*(v_1-u_1)]\phi_{1,i}(u_1)\phi_{2,j}(v_2)\exp[ig^2(Q_1(v_1)+Q_2(v_2))]. \end{aligned} \quad (29)$$

At equal times $x_2=y_2$, we have

$$F^*(u_2-v_2)=F^*(v_1-u_1) \quad (30)$$

and thus at equal times $x_2=y_2$, we have

$$\{\psi_{r,i}(x),\psi_{s,j}(y)\}=0. \quad (31)$$

Proceeding in the same way, we get

$$\begin{aligned}\{\psi_{1,i}(x),\psi_{1,j}^\dagger(y)\} &=\{\phi_{1,i}(u_1),\phi_{1,j}^\dagger(v_1)\}\exp\{ig^2[Q_2(u_2)-Q_2(v_2)]\},\\ &=\delta_{ij}\delta(u_1-v_1).\end{aligned} \quad (32)$$

For equal times $x_2=y_2$ this gives

$$\{\psi_{r,i}(x),\psi_{r,j}^\dagger(y)\}=\delta_{ij}\delta(x^1-y^1)\quad(r=1,2). \quad (33)$$

Similarly,

$$\begin{aligned}\{\psi_{1,i}(x),\psi_{2,j}^\dagger(y)\} &=[F(u_2-v_2)-F(v_1-u_1)]\phi_{1,i}(u_1)\phi_{2,j}^\dagger(v_2)\\ &\qquad\times\exp\{ig^2[Q_2(u_2)-Q_1(v_1)]\}\end{aligned} \quad (34)$$

and for equal times $x_2=y_2$, we have

$$\{\psi_{r,i}(x),\psi_{s,j}^\dagger(y)\}=\delta_{rs}\delta_{ij}\delta(x^1-y^1). \quad (35)$$

From these results, we have established that the interacting quark fields satisfy the canonical commutation relations at equal times

$$\begin{aligned}\{\psi_{r,i}(x,t),\psi_{s,j}(x',t)\}&=0,\\ \{\psi_{r,i}(x,t),\psi_{s,j}^\dagger(x',t)\}&=\delta_{rs}\delta_{ij}\delta(x-x').\end{aligned} \quad (36)$$

V. THE $SU(3)\times SU(3)$ CURRENT ALGEBRA

We can now construct octets of vector and axial-vector current densities in terms of our quark fields, and with the aid of our exact solutions for the ψ fields explicitly calculate the equal-time commutation relations of the time component of the densities in this model.

The octet of vector currents is

$$J_{\mu(\alpha)}(x,t)=\sum_{i,j=1}^{3}\bar\psi_i(x,t)\gamma_\mu(\lambda\alpha/2)_{ij}\psi_j(x,t), \quad (37)$$

where $i,j=1,2,3$ and the internal-symmetry subscript $(\alpha)=1,2,\ldots,8$; the $\lambda\alpha$ are Gell-Mann's matrices. The space-time suffix takes on the values $\mu=1,2$. We adjoin to Eq. (37) the octet of axial-vector current densities

$$J_{\mu(\alpha)}{}^5(x,t)=\sum_{i,j=1}^{3}\bar{\psi}_i(x,t)\gamma_\mu\gamma_5(\lambda\alpha/2)_{ij}\psi_j(x,t)\,, \quad (38)$$

where $\gamma_5=\gamma_1\gamma_2$.

With the aid of the equal-time canonical commutation relations for the interacting quark fields given by Eq. (36), we can easily show that at equal times

$$[J_{2(\alpha)}(x,t),\,J_{2(\beta)}(x',t)]=if_{\alpha\beta\gamma}J_{2(\gamma)}(x,t)\delta(x-x')\,,$$
$$[J_{2(\alpha)}(x,t),\,J_{2(\beta)}{}^5(x',t)]=if_{\alpha\beta\gamma}J_{2(\gamma)}{}^5(x,t)\delta(x-x')\,,$$
$$[J_{2(\alpha)}{}^5(x,t),\,J_{2(\beta)}{}^5(x',t)]=if_{\alpha\beta\gamma}J_{2(\gamma)}(x,t)\delta(x-x')\,. \quad (39)$$

Thus, the $SU(3)\times SU(3)$ algebra of the current densities that we have derived from the exact solutions of the quark fields does not lead to any gradients or higher derivatives of δ functions (so-called Schwinger terms).[2] This result also can be shown to hold for the commutators of the spatial components of the currents. The absence of such additional terms is due to our removal of all disconnected graphs proportional to c numbers of the type $\langle 0|\psi^2|0\rangle$ by means of our "double dot" procedure. We view this as a strong indication that all noncanonical terms in the current algebras are of an unphysical nature, since they are generated by unobservable processes.[8]

VI. CONCLUDING REMARKS

The S matrix in this model is trivial and there is no creation or scattering of quarks.[6] However, the interaction between the quarks mediated by the boson V is nonzero, and a derivation of the equal-time commutation relations between the time component of the current densities has shown that they are of the canonical form, as originally assumed by Gell-Mann.[2]

It would be interesting to study the solutions of the quark equations of motion when quarks with nonzero masses are introduced.[9] This would provide us with a solvable model of broken $SU(3)\times SU(3)$ based on field theory, and might shed some light on the nature of the symmetry violations in the physical theory.

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[8] J. S. Bell, Nuovo Cimento 47, 616 (1967).
[9] C. M. Sommerfield, Ann. Phys. (N. Y.) 26, 1 (1963).

APPENDIX I

Let us now study the properties of the "double dot" procedure introduced in Sec. II. We have for a given local, boson field operator $f(x)$:

$$:f^2(x):\equiv f^2(x)+Af(x)+B\,, \quad (A1)$$

where A and B are constants. Moreover, the variational derivative of the functional $:f^2(x):$ is defined by

$$[\delta/\delta f(x)]:f^2(x):=2f(x)+A\,. \quad (A2)$$

The constants A and B are determined by the conditions

$$\langle 0|:f^2(x):|0\rangle=0 \quad (A3)$$

and

$$\langle 0|[\delta/\delta f(x)]:f^2(x):|0\rangle=0\,. \quad (A4)$$

It follows that

$$A=-2\langle 0|f(x)|0\rangle\,,$$
$$B=-\langle 0|f^2(x)|0\rangle+2[\langle 0|f(x)|0\rangle]^2\,. \quad (A5)$$

Now consider the case $f(x)=\chi(x)$ where $\chi(x)$ represents a free field. Then, we have

$$\chi(x)=\chi^+(x)+\chi^-(x)\,, \quad (A6)$$

where $\chi^\pm$ denotes the positive and negative frequency parts of χ. This gives

$$\chi^2(x)=\chi^+\chi^++\chi^+\chi^-+\chi^-\chi^++\chi^-\chi^-\,. \quad (A7)$$

In terms of the usual Wick ordering

$$:\chi^2(x):=\chi^+\chi^++2\chi^-\chi^++\chi^-\chi^-\,. \quad (A8)$$

By virtue of the commutation relations satisfied by the χ operators, we have

$$:\chi^2(x):-\chi^2(x)=\chi^-\chi^+-\chi^+\chi^-=c\text{ number}$$
$$=\langle 0|(\chi^-\chi^+-\chi^+\chi^-)|0\rangle\,,$$
$$=-\langle 0|\chi^+\chi^-|0\rangle\,. \quad (A9)$$

But,

$$\langle 0|\chi^2(x)|0\rangle=\langle 0|\chi^+\chi^-|0\rangle\,. \quad (A10)$$

Therefore, it follows that

$$:\chi^2(x):=\chi^2(x)-\langle 0|\chi^2(x)|0\rangle\,, \quad (A11)$$

which is consistent with the result for the interacting field operator $f(x)$ given by

$$:f^2(x):=f^2(x)-2\langle 0|f(x)|0\rangle f(x)$$
$$-\langle 0|f^2(x)|0\rangle+2[\langle 0|f(x)|0\rangle]^2\,, \quad (A12)$$

if we note that $\langle 0|\chi(x)|0\rangle=0$.