It is difficult at this stage to say whether the failure of Eq. (15) is due to the truncation of the integral, a significant residue for the fixed pole, or both.

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Branch Points, Fixed Poles, and Falling Trajectories in the Complex J Plane*

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By means of the N/D equations, analytically continued in complex angular momentum, we consider the details of the mechanism by which cuts in the angular-momentum plane eliminate Gribov-Pomeranchuk singularities. The implications of the two-body unitarity requirement are investigated. A fixed l-plane pole in the signatured partial-wave amplitude is shown to exist where the first Gribov-Pomeranchuk singularity would be expected in the absence of cuts. It is also shown in this case that moving (Regge) poles are not asymptotic to the position of the fixed pole.

I. INTRODUCTION

It is the purpose of this paper to gain some additional insight into the properties of cuts in the angular-momentum plane from a point of view that emphasizes unitarity and analyticity rather than a discussion of explicit diagrams. These cuts were introduced by Mandelstam[1] in order to alleviate a conflict with crossed-channel unitarity, arising when the spins of external particles in scattering reactions are high. The mechanism of producing such cuts persists even for lower values of the spin where there is no direct conflict with unitarity. The function of the cut is to prevent a fixed singularity in the angular-momentum plane, originally discovered by Gribov and Pomeranchuk,[2] from manifesting itself in the asymptotic behavior of the full amplitude. Such a behavior would, in the presence of high-external spin, violate the Froissart bound on the scattering amplitude. The cut removes the unwanted asymptotic term and may, as we shall discuss, eliminate the Gribov-Pomeranchuk essential singularity entirely.

The method of attack is to discuss both the Gribov-Pomeranchuk singularity and the Regge-Mandelstam cuts from what might be termed a more dynamical standpoint. We use the N/D equations including inelastic unitarity and show how both the Gribov-Pomeranchuk phenomenon and the Regge cuts enter into such equations. The former arises from the discontinuity across the left-hand cut in the partial-wave amplitude having a series of fixed poles in the angular momentum l, a property which follows directly from the Mandelstam representation. The Regge cuts enter through the inelasticity factor

$$R_l = \frac{\sigma_{\text{total}}^{l}}{\sigma_{\text{elastic}}^{l}}$$

in the N/D equations.

It should be stressed that the N/D method of unitarizing the amplitude is a particularly appropriate one for this study because it does not modify the original input left-hand cut. As mentioned above, the presence of fixed poles in the left-hand cut is believed to be a property of the exact amplitude and should not be changed as a result of our unitarization.

For simplicity we discuss explicitly the case where all scattering particles are spinless. We think the discussion generalizes to the case of spin and some remarks on this are made in the last section. In Sec. II, we show how the Gribov-Pomeranchuk singularity is generated in the N/D equations by elastic unitarity when no Regge cuts are present. The central fact in this and in all our dynamical considerations is the presence of the poles in the left-hand cut.

In Sec. III we demonstrate how to introduce Mandelstam-Regge cuts into the N/D equations. The essential properties of the cuts which we use are deduced by the requirement that they remove an asymptotic term of the Gribov-Pomeranchuk type from the full amplitude. The nature of the solution to the N/D equations with the Regge cuts is described in Sec. IV. We find a simple fixed pole remaining where the Gribov-Pomeranchuk essential singularity would have been.

* This work is supported in part through funds provided by the U. S. Atomic Energy Commission under Contract AT(30-1)-2098.

[1] S. Mandelstam, Nuovo Cimento **30**, 1148 (1963).

[2] V. N. Gribov and I. A. Pomeranchuk, Phys. Letters **2**, 239 (1962); denoted hereafter as GP.

Such a fixed pole in the signatured partial-wave amplitude is shown not to contribute to the asymptotic behavior of the physical amplitude. It is also conjectured that the Gribov-Pomeranchuk singularity may be absent altogether (i.e., even from other l-plane sheets) when the Regge cut is present. Finally, a mechanism is described which may produce an indefinite falling of the Regge trajectories for large negative energies.

Section V translates Mandelstam's original arguments[1] into our point of view and shows that the two treatments are consistent. Roughly, the relationship of the two is as follows: Mandelstam considered the double pole in the l plane of *one* diagram with a two-body intermediate state. He showed that this double pole is cancelled by an infinite sum of diagrams with three-body intermediate states. The present work investigates the additional implications of the two-body unitarity requirement.

II. GRIBOV-POMERANCHUK SINGULARITY

It was first noted by Gribov and Pomeranchuk (GP)[2] that the discontinuity across the left-hand cut of an elastic, partial-wave amplitude (for spinless particles) possesses a series of fixed poles in the angular-momentum plane at $l=-1, -2, \cdots$. As a consequence of the elastic unitarity on the amplitude, this left-hand cut singularity was shown by them to generate an infinite number of Regge poles in the neighborhood of these negative integers for all values of the energy. We shall begin by discussing the Gribov-Pomeranchuk phenomenon from the point of view of the N/D equations, so that we can later show how the introduction of cuts into the N/D equations removes asymptotic difficulty associated with the GP singularity.

We consider elastic scattering of equal-mass, spinless particles, with signatured partial-wave amplitudes $B^{\pm}(l,\nu)$ having the following definition in terms of the Froissart-Gribov transform[3]:

$$\nu^l B^{\pm}(l,\nu)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt}{2\nu}Q_l\left(1+\frac{t}{2\nu}\right)D_t^{\pm}(t,\nu), \quad (2.1)$$

where ν is the three-momentum squared in the center of mass. The functions $B^{\pm}(l,\nu)$ are real analytic functions of ν because of the factor ν^l. The functions $D^{\pm}(t,\nu)$ are the t-channel absorptive parts of the amplitudes $A^{\pm}(s,z_s)$ defined in terms of the ordinary absorptive parts D_t and D_u in the t and u channels by the relations[4]:

$$A^{\pm}(s,z_s)=A_R(s,z_s)\pm A_L(s,-z_s),$$

$$A_R(s,z_s)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t(s,z_s)}D_t(t',s), \quad (2.2)$$

$$A_L(s,z_s)=\frac{1}{\pi}\int_{u_0}^{\infty}\frac{du'}{u'-u(s,z_s)}D_u(u',s),$$

where, as usual, $z_s=1+t/2\nu$ and $s=4(\nu+m^2)$.

If we denote discontinuity by the symbol $2i\Delta$, and the start of the left hand by $s=s_L$, then[5] $\Delta B^+(l,s)$ has poles at $l=-1, -3, -5, \cdots$, and $\Delta B^-(l,s)$ has poles at $l=-2, -4, -6, \cdots$. The presence of these fixed poles in the left-hand discontinuities follows from the existence of a third double spectral function as shown by Gribov and Pomeranchuk.[2] In much of the development that follows we shall suppress the signature labels in $B(l,s)$, except at times when signature considerations are important for the point under discussion.

The amplitude $B(l,s)$ can be written as $N(l,s)/D(l,s)$, where D carries the right-hand cut of B and N the left-hand cut. Assuming elastic unitarity for $s>4m^2$ we can determine the following integral equation for $N(l,s)$:

$$N(l,s)=V(l,s)+\frac{1}{\pi}\int_{4m^2}^{\infty}ds'K(l,s,s')N(l,s'), \quad (2.3)$$

where

$$K(l,s,s')=\frac{V(l,s)-V(l,s')}{s-s'}\rho(l,s'), \quad (2.4)$$

and

$$V(l,s)=\frac{1}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{s'-s}\Delta B(l,s'). \quad (2.5)$$

The function $\rho(l,s)$ is ordinary two-body phase space multiplied by ν^l. The D function is given by the equation

$$D(l,s)=1-\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\rho(l,s')N(l,s'). \quad (2.6)$$

We shall ignore detailed questions about the convergence of the integral in Eq. (2.3) since the only crucial fact for our purposes is that elastic unitarity hold over a finite interval of the range of integration in Eq. (2.3). Thus the kernel of Eq. (2.3) is assumed to be Fredholm (that is, square integrable) for most values of l. [Note that $V(l,s)$ has no singularities in the variable s, for $s>s_L$.]

As l approaches -1, the kernel (for the $+$ amplitude) becomes unbounded and we expect Fredholm poles to appear in the solution $N(l,s)$ as a function of l. The fact that the residue of the pole at $l=-1$ in the kernel has

[3] See, for example, G. F. Chew, *The Analytic S Matrix* (W. A. Benjamin, Inc., New York, 1966), p. 52.

[4] G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).

[5] See, for example, C. E. Jones and V. L. Teplitz, Nuovo Cimento **31**, 1079 (1963).

a left-hand cut means that it is not a separable function of the variables s and s' and, hence, there is an infinite accumulation of Fredholm poles as N nears $l=-1$.[6]

We see from Eq. (2.6) that all the Fredholm poles of N also occur in D so that the amplitude has no fixed poles in l. However, there are in general Regge trajectories $\alpha_i(s)$ which approach the location of these Fredholm poles as $s\rightarrow\infty$. To see this we write $N(l,s)$ in the neighborhood of one of its Fredholm poles,

$$N(l,s)\approx\frac{f(s)}{l-l_i}\quad\text{as}\quad l\rightarrow l_i. \tag{2.7}$$

For $l\rightarrow l_i$, the D function becomes

$$D(l,s)\approx1-\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\rho(l_i,s')\frac{f(s')}{l-l_i}. \tag{2.8}$$

The Regge trajectory of interest near $l=l_i$ is a solution to $D(l,s)=0$. But from Eq. (2.8) we see that as $l\rightarrow l_i$ the zero of $D(l,s)$ must recede to infinity. If the integral in Eq. (2.8) goes like $1/s$ as $s\rightarrow\infty$ we have

$$1-\frac{1}{\pi}\int ds'\frac{\rho(l_i,s')f(s')}{s(l-l_i)}=0. \tag{2.9}$$

Since there exist an infinite number of such Fredholm poles in the neighborhood of -1, we, in turn, have an infinite number of Regge trajectories near -1 giving the Gribov-Pomeranchuk phenomenon for the scattering of two spinless particles.

In potential scattering, where there is no third double spectral function, the function $V(l,s)$ and hence the kernel in Eq. (2.3) has fixed poles at $l=-1$, -2, -3, $\cdots$ coming from the Born term. However, such poles are not present in the discontinuity across the left-hand cut. The residue of the pole in the kernel at $l=-1$ is separable in this case and the fixed pole is simply converted into a moving Regge pole by unitarity and, as is well known, each fixed pole in the Born term becomes the high-energy limit for a single Regge trajectory.

III. REGGE CUTS

In the previous section, we have seen that the Gribov-Pomeranchuk singularity[2] at $l=-1$ (infinite accumulation of Regge poles near $l=-1$) seems to be required in order to reconcile the presence of a third double-spectral function with elastic unitarity in the s channel. The difficulty with the GP singularity, as Mandelstam has discussed, occurs when there is spin.[7] If, for example, the external particles have spins σ_1 and σ_2, the GP singularity is then promoted in total angular momentum J to the point $J=\sigma_1+\sigma_2-1$. If $\sigma_1+\sigma_2>2$, this will conflict with the Froissart bound on the asymptotic behavior of the amplitude in the t channel.[8] The GP singularity at $J=\sigma_1+\sigma_2-1$ in the s channel produces a lower bound on the asymptotic behavior $t^{\sigma_1+\sigma_2-1}$ at high t. But according to Froissart,[7] unitarity in the t channel demands that the amplitude increase no faster than linearly with t (to within logarithms). Thus there is a contradiction if $\sigma_1+\sigma_2>2$. Such difficulties also arise in the case of the scattering of spinless particles where intermediate states possess high spin.

In order to avoid these contradictions, Mandelstam has proposed the existence of cuts in the angular-momentum plane that shield the GP singularities in such a way that they will not directly affect the asymptotic behavior in t.[1] He has shown explicitly that certain Feynman diagrams possess such cuts which place fixed poles in other diagrams on sheets far from the physical region in the J plane.

Mandelstam's mechanism for shielding singularities applies equally well to the GP phenomenon which occurs at $l=-1$ for the scattering of spinless particles. This means that no $1/t$ behavior is expected at high t for the physical scattering amplitude.

For purposes of simplicity and illustration, we consider this spinless case and discuss the GP and cut phenomenon associated with the point $l=-1$ in the spinless problem. As a concrete example, we may think of π-π scattering.

A branch point in the angular-momentum plane of the type discussed by Mandelstam has a position in the l plane $\alpha_c(s)$ that depends on s, the total energy squared in the center-of-mass system. The function $\alpha_c(s)$ is real below the first inelastic threshold $s=s_I$ and is complex above it. We may also study the same branch point in the energy plane where it is a branch point in the s variable whose position depends on the angular momentum l, $s=s_c(l)$. Since we wish to write N/D equations which involve dispersion integrals in s, we must discuss the location of the branch point $s_c(l)$ as l varies.

For $\text{Re}\,l>n$ where n is some positive number, the Froissart-Gribov formula (2.1) shows that there are no singularities of any type in the l plane. When this statement is interpreted in the s plane, it means that the branchpoint $s_c(l)$ must be located in an unphysical region reached by continuing through a physical sheet cut.

General arguments can be given to show that for large l the branch point $s_c(l)$ must be on a sheet reached by continuing through the inelastic part of the right-hand cut.[9] This fact can also be verified explicitly for the cut in the Mandelstam diagram. As l is decreased the moving branch points will generally emerge from the inelastic threshold. In Fig. 1, we trace the branch point of interest. At $l=l_0$, $s_c(l_0)=s_I$ and when $l=-1$,

[6] See the Appendix for a fuller discussion of this point.
[7] S. Mandelstam, Nuovo Cimento **30**, 1113 (1963).
[8] M. Froissart, Phys. Rev. **123**, 1053 (1961).
[9] R. Oehme, *Complex Angular Momentum in Elementary Particle Scattering* (Oliver and Boyd, Edinburgh, 1964), p. 129.

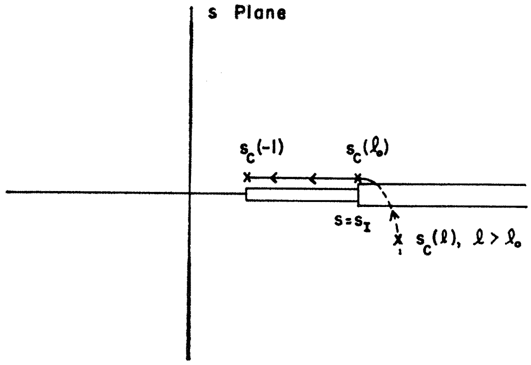

FIG. 1. Moving branch point in the energy plane.

$s_c(-1)=4m^2$, the elastic threshold. So as l approaches -1, the elastic unitarity cut becomes completely blanketed. This is the essential reason why the pole in the potential $V(l,s)$ is no longer a problem; it presents no conflict with elastic unitarity because there is no region over which elastic unitarity is valid.

We now formulate the N/D equations in the presence of the Regge cut $s_c(l)$. Again we endow N with only the left-hand cut and D with the right-hand cut. For l sufficiently large that no Regge cuts yet appear on the physical sheet in the s plane, the N/D equations are the same as those given in Eqs. (2.3)–(2.6) except that wherever the phase space factor ρ appears it should be replaced by ρR where R describes the inelasticity and is equal to one for $s<s_I$.

For $-1<l<l_0$, the branch point $s_c(l)$ moves onto the physical sheet between the thresholds $s=4m^2$ and $s=s_I$. Now explicit account must be taken of the fact that elastic unitarity no longer gives the discontinuity of the amplitude in the interval $s_c(l)<s<s_I$. This may be achieved by writing the N/D equations in the form

$$N(l,s)=V(l,s)+\frac{1}{\pi}\int_{4m^2}^{s_c(l)} ds' k(l,s',s)\rho(l,s')N(l,s') +\frac{1}{\pi}\int_{s_c(l)}^{\infty} ds' k(l,s',s)\left[-\Delta\frac{1}{B(l,s')}\right] \times N(l,s')\,, \quad (3.1)$$

$$D(l,s)=1+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\left[\Delta\frac{1}{B(l,s')}\right]N(l,s')\,, \quad (3.2)$$

$$2i\Delta\frac{1}{B(l,s)}=\frac{1}{B(l,\,s+i\epsilon)}-\frac{1}{B(l,\,s-i\epsilon)}\,, \quad (3.3)$$

$$k(l,s',s)=\frac{V(l,s')-V(l,s)}{s'-s}\,. \quad (3.4)$$

The discontinuity of $1/B$ in Eq. (3.4) is evaluated by going above all cuts in the s plane (including the Regge cut) and then around the threshold $s=4m^2$ and below all cuts. It may be verified that (1) $N(l,s)$ has no branch point at $s_c(l)$, and (2) $N(l,s)$ and $D(l,s)$ have no singularity in l at $\alpha(s_I)=l_0$.

It will be useful to derive here the relation between $\Delta(1/B)$ and the corresponding discontinuity $\Delta_c(1/B)$ across the Regge cut only. In order to clarify the situation, the relevant cut structure is drawn in detail in Fig. 2. We see then that

$$2i\Delta(1/B)=[1/B(s_+)]-[1/B(s_-)]\,,$$
$$2i\Delta_c(1/B)=[1/B(s_+)]-[1/B(s_{+'})]\,. \quad (3.5)$$

Furthermore, the discontinuity in the region $4m^2<s<s_c(l)$ can be calculated by elastic unitarity to give

$$\Delta(1/B)=-\rho\,, \quad 4m^2<s<s_c(l)\,. \quad (3.6)$$

The analytic continuation of Eq. (3.6) to the region $s>s_c(l)$ relates the points $s_{+'}$ and s_- as follows:

$$[1/B(s_{+'})]-[1/B(s_-)]=-2i\rho\,. \quad (3.7)$$

Combining Eqs. (3.5) and (3.7) gives

$$\Delta_c(1/B)=\Delta(1/B)+\rho\,. \quad (3.8)$$

Equation (3.8) enables us to relate the partial-wave amplitude $B(l,s)$ on the two sides of the Regge cut. To see this we recall that $N(l,s)$ does not have the Regge cut at $s_c(l)$ (it has only the left-hand cut). Thus the difference of $B(l,s)$ on the two sides of the Regge cut comes entirely from $D(l,s)$. Referring again to Fig. 2, we see that

$$D(l,s_+)-D(l,s_{+'})=N(l,s)2i\Delta_c[1/B(l,s)]\,. \quad (3.9)$$

This relation may be analytically continued to give $D(l,s)$ and hence $B(l,s)$ at any point on the second sheet of the Regge cut when Δ_c is known.

IV. SOLUTION TO N/D EQUATIONS WITH REGGE CUTS

We now undertake to discuss the solutions to the N/D equations with Regge cuts given by Eqs. (3.1) and (3.2). First we show that the solution of Eqs. (3.1) and (3.2) gives us the amplitudes on the "physical

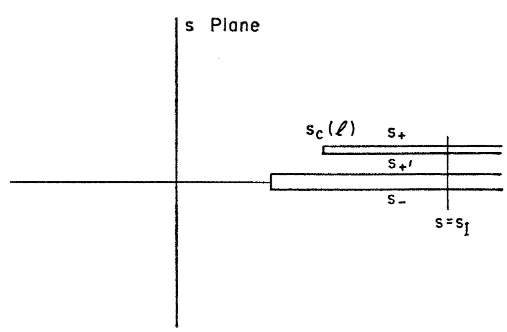

FIG. 2. Points in energy plane related by discontinuity formulas.

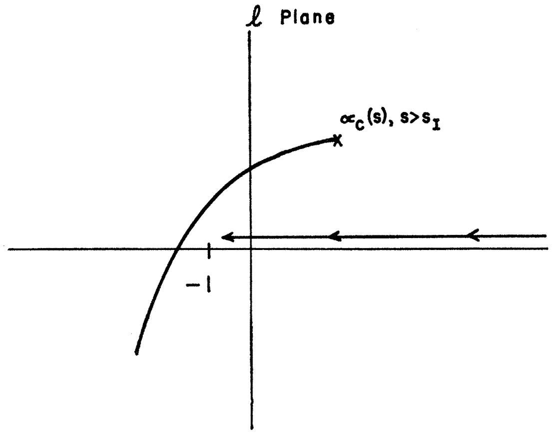

FIG. 3. Path of continuation to the point $l=-1$.

sheet" in the l plane. What is meant by physical sheet in the l plane is indicated in Fig. 3 where the Mandelstam branch point $\alpha_c(s)$ is shown for $s>s_I$. To reach the point $l=-1$, we continue l along the real axis as shown in Fig. 3. In other words, the physical sheet of the l plane consists of those points in the l plane reached in pushing the Sommerfeld-Watson contour to the left (see Fig. 4).

For $s<s_I$, $\alpha_c(s)$ becomes real; however, by making s slightly complex with a positive imaginary part the branch point $\alpha_c(s)$ will remain slightly above the real axis and the path to $l=-1$ given in Fig. 3 remains well defined. Such a continuation to $l=-1$ gives rise to Eqs. (3.1) and (3.2) as the l plane physical sheet values of N and D. The values of N and D near $l=-1$ on the opposite side of the cut in Fig. 3 are obtained by using the discontinuity for D given in Eq. (3.9). The N function is the same on both sides of the cut.

We notice that in the integral equation for N, (3.1), that $k(l,s,s')$ has a pole at $l=-1$ as in the elastic case discussed in Sec. II. But the preceding remarks about being on the physical sheet in the l plane require that N not have an infinite accumulation of Fredholm poles at $l=-1$ as it did in the case where the Regge cuts are ignored. The first term of Eq. (3.1) in which $k(l,s,s')$ appears gives no problem because $s_c(l)$ approaches $4m^2$ as l approaches -1. However, the second integral term involving $k(l,s,s')$ will give rise to the unwanted accumulation of poles unless[10]

$$\Delta[1/B(l,s)] \xrightarrow[l\to -1]{} (l+1)g(s). \qquad (4.1)$$

[It should be emphasized that we are now discussing only $B^+(l,s)$, the positive signature amplitude, since $B^-(l,s)$ does not have poles in $V^-(l,s)$ until $l=-2$.]

With the property (4.1) the resolvent kernel for the integral Eq. (3.1) will be regular at $l=-1$. However, N^+ will still have a simple pole at $l=-1$ because of the inhomogeneous term $V^+(l,s)$. From Eq. (3.2) $D^+(l,s)$ obviously has no pole at $l=-1$, so $B^+(l,s)$ is left with a simple pole at $l=-1$. This, of course, is consistent with Eq. (4.1).

We now also can see how the cut introduces a mechanism by which Regge trajectories may drop indefinitely as $s\to-\infty$. From Sec. II we saw that trajectories tend to asymptote to fixed poles in the D function. Now, however, the fixed poles in D previously introduced by the poles in N are cancelled by $\Delta(1/B)$ which possesses zeroes at these points. Thus a mechanism is provided for $\alpha(s)\to-\infty$ as $s\to-\infty$. Experimentally, Regge trajectories appear to be rising indefinitely for s in the positive (resonance or bound state) region and it would be presumed that they also drop indefinitely for negative s. A model that includes Regge cuts appears capable of producing such trajectories.

One may also ask if the potential $V(l,s)$ may not have fixed poles whose residues are factorizable, which do not arise from the presence of a third-double spectral function, and are thus similar to the poles in the Born approximation to potential scattering. Such poles might provide asymptotic limits for Regge trajectories.[11] There are, in the relativistic problem, no fixed t poles like the Born term in potential theory and the Regge poles in the t channel, which give rise to the bound state and resonance pole in t may have residues which go exponentially to zero for asymptotic t, producing no fixed singularities in the l plane.

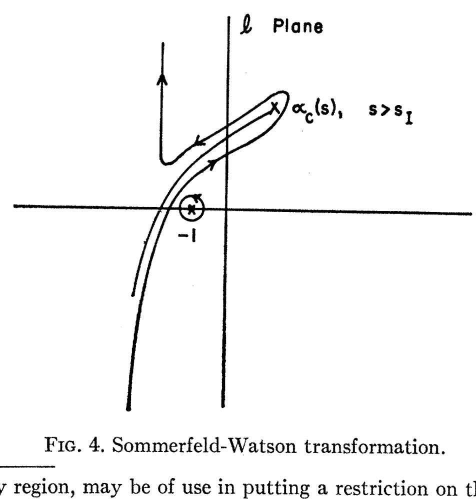

FIG. 4. Sommerfeld-Watson transformation.

[10] Higher-order vanishing of this discontinuity, i.e.,

$$\Delta(1/B)\to(l+1)^m f(s),$$

cannot here be ruled out on the basis of the present discussion. If $m>1$, there is no change in our conclusion that the GP singularity is replaced by a fixed pole. The case $m>1$ does, however, necessitate a careful study of analytically continued partial-wave threshold behavior in order to arrive at a mechanism by which moving Regge poles pass through $l=-1$. It is interesting to note that the condition $\Delta(1/B)\to 0$, when extrapolated to the negative energy region, may be of use in putting a restriction on the form of phenomenological cut contributions to high-energy crossed-channel scattering.

[11] It is also possible that even if Born-term–type fixed poles are present at the negative integers that they will also be rendered harmless by the cut mechanism.

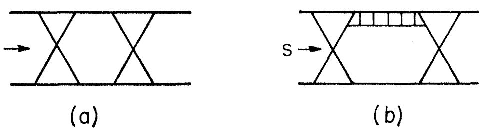

FIG. 5. The Mandelstam diagrams.

One final remark concerns the nature of the singularity at $l=-1$ on the opposite side of the Regge cut in Fig. 3. As we have mentioned, $N^+(l,s)$ is the same on both sides of the cut and, hence, retains its simple pole at $l=-1$. If we denote by $D_{\text{II}}^+(l,s)$ the value of D^+ on the other side of the Regge cut, we can write, using Eqs. (3.8) and (3.9):

$$D_{\text{II}}^+(l,s)=D^+(l,s)-N^+(l,s)2i[\Delta[1/B^+(l,s)]+\rho], \quad (4.2)$$

where the discontinuity $\Delta[B^+(l,s)]^{-1}$ is continued to complex s. The remarkable property of Eq. (4.2) is that it appears to indicate that it is at least consistent for *no* essential singularity (or infinite accumulation of poles) to exist on this sheet. If $\Delta[B^+(l,s)]^{-1}$ goes linearly to zero as $l\to -1$ for complex s (as we have assumed that it does for real s) and if this vanishing is independent of the approach to $l=-1$, D_{II}^+ from Eq. (4.2) will have merely a simple pole at $l=-1$ and $B_{\text{II}}^+(l,s)$ will be analytic there; in this case a single trajectory will asymptote to $l=-1$.

In order to show that this suggestion does not conflict with Mandelstam's work, we shall demonstrate in the next section that if the integral equations are iterated to second order in $V^+(l,s)$ that this approximation has a double pole in $B_{\text{II}}^+(l,s)$ at $l=-1$.

V. COMPARISON WITH MANDELSTAM

Our purpose in this section is to show that our results are consistent with Mandelstam's conclusions about cuts in the l plane.[1] Mandelstam studied the two diagrams shown in Fig. 5. [Note that Fig. 5(b) is, in fact, an infinite sum of diagrams, and Mandelstam considered just the contributions of this infinite sum to the three-particle intermediate state.] He did not consider explicitly the effects of two-body unitarity.

For purposes of comparison we may regard Fig. 6 as representing the significant part of our $V^+(l,s)$. The crossed box given in Fig. 6 is the simplest diagram that has a third double spectral function and it has a simple pole at $l=-1$ in the amplitude with positive signature. Figure 5(a) is simply an iteration of Fig. 6; it thus has a double pole at $l=-1$. Mandelstam showed that the diagram in Fig. 5(b) possesses a cut in the l plane whose discontinuity has a double pole at $l=-1$, and that the sum of diagrams 5(a) and 5(b) has no double pole at $l=-1$ on the physical sheet of the angular-momentum plane.[12] There *is* a double pole on the opposite unphysical side of the l plane cut. We emphasize that the sum of Figs. 5(a) and 5(b) may possess a simple pole at $l=-1$ on the physical sheet; this does not imply an asymptotic behavior $1/t$ in the physical amplitude as we shall presently show. In any event, the diagram in Fig. 6 appears to have a fixed (uncancelled) simple pole at $l=-1$.

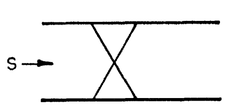

FIG. 6. Simplest diagram with third-double spectral function.

Before the N/D decomposition, Eqs. (3.1) and (3.2) are equivalent to the nonlinear equation:

$$B(l,s)=V(l,s)+\frac{1}{\pi}\int_{4m^2}^{s_c(l)}\frac{ds'}{s'-s}\rho(l,s')|B(l,s')|^2 + \frac{1}{\pi}\int_{s_c(l)}^{\infty}\frac{ds'}{s'-s}|B(l,s')|^2\left[-\Delta\frac{1}{B(l,s')}\right]. \quad (5.1)$$

We calculate the first iteration of Eq. (5.1) in the potential $V(l,s)$, which we denote by $B^1(l,s)$:

$$B^1(l,s)=V(l,s)+\frac{1}{\pi}\int_{4m^2}^{s_c(l)}\frac{ds'}{s'-s}\rho(l,s')|V(l,s')|^2 + \frac{1}{\pi}\int_{s_c(l)}^{\infty}\frac{ds'}{s'-s}|V(l,s')|^2\left[-\Delta\frac{1}{B(l,s')}\right]. \quad (5.2)$$

The amplitude $B^1(l,s)$ given by Eq. (5.2) is an approximation to the sum of diagrams 5(a), 5(b), and 6. The general structure of Eq. (5.2) as l approaches -1 is that all terms on the right give simple poles at $l=-1$. Although $|V|^2$ has a double pole in the last two integrals, the fact that $s_c(l)\to 4m^2$ and $\Delta(1/B)\to 0$ as $l\to -1$ combine to produce only a simple pole. To evaluate $B^1(l,s)$ on the opposite unphysical side of the Mandelstam cut, denoted $B_{\text{II}}^1(l,s)$, we use Eq. (3.8) together with Eq. (5.2) and obtain

$$B_{\text{II}}^1(l,s)=B^1(l,s)-2i|V(l,s)|^2\left[\Delta\frac{1}{B}+\rho\right]. \quad (5.3)$$

The second term on the right of Eq. (5.3) clearly possesses a double pole at $l=-1$ and, thus, our results are consistent with those of Mandelstam.[1]

To complete our comparison with Mandelstam, we demonstrate that a fixed pole at $l=-1$ does not introduce a $1/t$ term in the asymptotic expansion of the physical amplitude $A(s,z_s)$, even though such a term is present in the signatured amplitude $A^+(s,z_s)$. The point is that only the even part of $A^+(s,z_s)$ contributes to $A(s,z_s)$, but at high t, $A^+(s,z_s)\to 1/t$; therefore,

$$A^+(s,z_s)+A^+(s,z_s)\to (1/t)+(-1/t),$$

so the asymptotic term $1/t$ is absent from the physical

[12] In Mandelstam's discussion he assumed that the ladder in Fig. 5(b) produced a p-wave bound state; in our case the latter yields an s-wave bound state.

amplitude. This accords with Mandelstam's conclusions on the asymptotic behavior of the diagrams given in Fig. 5.

VI. CONCLUSION

The results of the preceding sections may be summarized as follows: By means of the analytic continuation of the N/D equations (in the presence of one l-plane branch point) we show: (1) $N_l(s)$ has no singularity at $s_c(l)$; (2) $B_l(s)$ has no singularity at $\alpha_c(s_I)$. From the hypothesis that the contour in the Sommerfeld-Watson transformation encounters no essential singularity at the l for which the discontinuity across the left-hand cut has a pole ($l=-1$), we establish that $[B(s+i\epsilon)]^{-1}-[B(s-i\epsilon)]^{-1}\sim(l+1)$ as $l\to -1$. The N/D equations then yield as corollaries: (1) N has a pole at $l=-1$, (2) D is finite there, (3) zeros of D at $s=s_p(l)$ do not go to infinity in s as l approaches -1, (4) on a sheet reached by circling the l-plane branch point, at least one trajectory must go to infinity as l approaches -1, and (5) the above hypotheses are consistent with no essential singularity existing on any sheet in the l plane.

We believe that our technique may be extended to situations involving spin and unequal masses with analogous results.

We believe that our results (returning to the spinless even-signature case) apply equally well to the points $l=-3, -5, \cdots$, where the left-hand cut discontinuity also has poles. Here Regge cuts arising in higher order presumably are available for precisely masking the elastic unitarity cut for these values of l. We thus see a possible mechanism for indefinitely falling trajectories.

Finally we note that the presence of fixed poles in $B(l,s)$ does not require that a trajectory $\alpha(s)$ passing through $l=-1$ at $s=s_1$ have a residue function $\beta(s)$ with a pole at $s=s_1$. The presence or absence of such a pole is not decided by our considerations. If present, however, such a pole would invalidate the mechanism which produces dips in high-energy cross sections at values of momentum transfer for which trajectories pass through wrong-signature nonsense values.[13]

APPENDIX

We discuss here briefly the solution of Eq. (2.3) in the neighborhood of the pole at $l=-1$ in the kernel, $K(l,s,s')$. We assume as in Sec. II that K is a Fredholm kernel for $l\neq -1$.

[13] See, for example, C. B. Chiu and J. D. Stack, Phys. Rev. **153**, 1575 (1967). We are grateful to Dr. John H. Schwarz for stimulating discussions on this subject.

Given that K is Fredholm, the solution $N(l,s)$ of Eq. (2.3) will in general have poles in l at values of l such that $\lambda=(l+1)^{-1}$ is an eigenvalue of the equation

$$\lambda\int_{4m^2}^{\infty} ds'\bar{K}(s,s')\varphi(s')=\varphi(s)\,,$$

where $\bar{K}(s,s')$ is the residue of $K(l,s,s')$ at the pole at $l=-1$. Since $\bar{K}$ is nonseparable, an infinite number of these eigenvalues cluster at ∞ in λ or near -1 in l when $\bar{K}\neq 0$.

One further point must be made. The solution, $N(l,s)$ is not required to have a pole in l at an eigenvalue of $\bar{K}$ if V is orthogonal to the corresponding eigenfunction. This follows simply from writing

$$N=(1-\lambda\bar{K})^{-1}V$$
$$=\sum_i (1-\lambda/\lambda_i)^{-1}(\varphi_i,V)\,,$$

where

$$(\varphi_i,V)=\int_{4m^2}^{\infty}\varphi_i(s')V(s')ds'\,.$$

We can, however, show that V is nonorthogonal $[(\varphi_i,V)\neq 0]$ to an infinite number of eigenfunctions φ_i of $\bar{K}$, and, hence, N must have an infinite number of poles near $l=-1$. Suppose we have along the right-hand cut

$$V(s)=\sum_{i=1}^{N} a_i\varphi_i(s)\,. \tag{A1}$$

Then, using the homogeneous equation associated with (2.3) and the fact that the sum on the right is over a finite number of terms, we may perform the necessary continuation to obtain

$$\mathrm{disc}V(s)=-\left[\sum_{i=1}^{N} a_i\frac{\lambda_i}{\lambda}\int_{4m^2}^{\infty}\frac{\rho(s')\varphi_i(s')}{s'-s}ds'\right]\mathrm{disc}V(s)\,. \tag{A2}$$

The term in square brackets in Eq. (A2) must be equal to minus one. However, this cannot be true because this term has a right-hand cut whose discontinuity cannot vanish because of the linear independence of the φ_i's. Hence, V is not orthogonal to infinite number of the φ_i's, and $N(l,s)$ must have an infinite number of poles near $l=-1$.[14]

[14] When $V(l,s)$ is represented as an infinite sum of the φ_i's along the right-hand cut, this argument must break down, presumably because the continuation of the infinite sum over to the left-hand cut must fail to converge.