Diseases of Infinite-Component Field Theories*

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Examples are presented of local, covariant field theories for which the familiar spin and statistics and *PCT* theorems do not apply. The field theories employ infinite-dimensional representations of the homogeneous Lorentz group and are formulated in terms of fields satisfying local commutation rules. Since the existing proofs of PCT and spin and statistics assume fields with a finite number of components, our conclusions do not violate these theorems. The existence of solutions with spacelike momenta is discussed.

I. INTRODUCTION

THE point of this note is to discuss some examples of local, Lorentz-covariant field theories which can independently violate the *PCT* theorem and the usual connection between spin and statistics. The theories are formulated in terms of fields $\psi_\alpha(x)$, where the index α labels a basis for an infinite dimensional representation of the homogeneous Lorentz group. Since the existing proofs of *PCT*, spin, and statistics, etc. assume fields with a finite number of components, our conclusions do not contradict these theorems.

We were led to the study of these field theories by a search for a linear, relativistic wave equation similar to the Dirac or Kemmer equations, but sufficiently generalized so that the solutions describe a set of particles with a nontrivial mass spectrum. It will be evident from our discussion, and also by referring to examples studied in recent literature, that there are several such theories.[1–4]

The most general first-order wave equation can be written

$$(i\beta^\mu\partial_\mu-\mathfrak{M})\psi=0\,, \tag{1.1}$$

where β^μ and $\mathfrak{M}$ are some (vector and scalar) matrices operating on the indexed field ψ. Consider an example with a degenerate mass spectrum: Let ψ be a doubly indexed field, where the indices label a basis for a tensor-product representation of the homogeneous Lorentz group. For instance, let the first index label the four-dimensional Dirac representation,[5] and for the time being leave the second representation unspecified. To get a degenerate mass spectrum, the wave function must satisfy the Dirac equation

$$(i\gamma^\mu\partial_\mu-m_0)\psi=0\,. \tag{1.2}$$

Let $\Gamma^{\mu\nu}$ be the generators of the second, unspecified, representation and $\frac{1}{2}\sigma^{\mu\nu}$ the generators of the Dirac representation. Notice that Eq. (1.2) is covariant under physical Lorentz transformations $J^{\mu\nu}$ if these are defined either by

$$J^{\mu\nu}=\tfrac{1}{2}\sigma^{\mu\nu}+i(x^\mu\partial^\nu-\partial^\nu x^\mu)\,, \tag{1.3}$$

or by

$$J^{\mu\nu}=\tfrac{1}{2}\sigma^{\mu\nu}+\Gamma^{\mu\nu}+i(x^\mu\partial^\nu-\partial^\nu x^\mu)\,. \tag{1.4}$$

That is, we can consider the $\Gamma^{\mu\nu}$ either as an $SL(2,C)$ internal symmetry completely decoupled from the spin, or as a part of the physical Lorentz transformations. If we generalize this simple theory to obtain a nontrivial mass spectrum, we shall find that we are always forced to choose definition (1.4).

In this paper two theories are discussed in detail. Both are presented in Sec. II; the first is characterized by an equation of motion similar in form to an equation proposed by Corben[2] to describe a relativistic rotator, and the second is based on a theory discovered by Majorana[1] in 1932 and recently revived by Fradkin.[6] In Sec. III we show how the second quantization of either theory can lead to the violation of the spin-statistics theorem, and in Sec. IV we discuss the *PCT* transformation. Section V includes a discussion of the spacelike solutions.

II. TWO WAVE EQUATIONS

As a first example, we study a field which can be defined by a Lagrangian density

$$\mathfrak{L}_0=\bar{\psi}(x)[i\gamma^\mu\partial_\mu-m_0-\tfrac{1}{2}m_1\sigma_{\mu\nu}\Gamma^{\mu\nu}]\psi(x)\,, \tag{2.1}$$

where γ^μ and $\sigma^{\mu\nu}$ are the usual Dirac operators, and the $\Gamma^{\mu\nu}$ are the generators of some representation of $SL(2,C)$. Let $\sigma^{ij}=\epsilon^{ijk}\sigma_k$, $\Gamma^{ij}=\epsilon^{ijk}\Sigma_k$, $\sigma^{i0}=\tau^i=i\alpha^i$, and $\Gamma^{i0}=\Lambda^i$. In our notation, $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ are Hermitian.

The equation of motion following from (2.1) is

$$[i\gamma^\mu\partial_\mu-m_0-\tfrac{1}{2}m_1\sigma_{\mu\nu}\Gamma^{\mu\nu}]\psi(x)=0\,, \tag{2.2}$$

where $\sigma_{\mu\nu}\Gamma^{\mu\nu}=2[\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}-\boldsymbol{\tau}\cdot\boldsymbol{\Lambda}]$.

It is evident that when $m_1\neq 0$, the $\sigma_{\mu\nu}\Gamma^{\mu\nu}$ term will be Lorentz invariant only if we choose definition (1.4) for the physical transformations.

The discrete mass spectrum arising from Eq. (2.2) can be obtained without difficulty. The Hamiltonian is

$$H=-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+m_0\gamma^0+m_1\gamma^0(\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}-\boldsymbol{\tau}\cdot\boldsymbol{\Lambda})\,, \tag{2.3}$$

and hence the masses m_κ are determined by the solu-

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[1] E. Majorana, Nuovo Cimento **9**, 335 (1932). See also I. M. Gel'fand and A. M. Yaglom, Zh. Eksperim. i Teor. Fiz. **18**, 703 (1948).
[2] H. C. Corben, Proc. Natl. Acad. Sci. US **48**, 1559 (1962).
[3] Y. Nambu (to be published).
[4] G. Feldman and P. T. Matthews, Ann. Phys. (N. Y.) **40**, 19 (1966); Phys. Rev. **151**, 1176 (1966); **154**, 1241 (1967).
[5] Harish-Chandra, Proc. Roy. Soc. (London) **A189**, 372 (1947).
[6] D. M. Fradkin, Am. J. Phys. **34**, 4 (1966); **34**, 314 (1966).

tions to

$$\mathfrak{M}'u_\kappa = m_\kappa u_\kappa\,, \tag{2.4}$$

where the mass operator $\mathfrak{M}'$ is

$$\mathfrak{M}' = m_0\gamma^0 + m_1\gamma^0(\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma} - \boldsymbol{\tau}\cdot\boldsymbol{\Lambda})\,. \tag{2.5}$$

The u_κ can also be chosen as eigenvectors of the total spin $J^2 = (\frac{1}{2}\boldsymbol{\sigma}+\boldsymbol{\Sigma})^2$ and of J_3. Each solution is then characterized by a definite mass, spin, and helicity.

The matrices for γ^0, $\gamma^0\boldsymbol{\sigma}$, and $\gamma^0\boldsymbol{\tau}$ are Hermitian. If the mass operator $\mathfrak{M}'$ is to be self-conjugate—thus insuring the reality of the m_κ—the matrices $\boldsymbol{\Sigma}$ and $\boldsymbol{\Lambda}$ must also be Hermitian. For the ordinary spin representations of the Lorentz group, $\boldsymbol{\Sigma}$ is Hermitian, but $\boldsymbol{\Lambda}$ is anti-Hermitian. We are therefore forced to take the unitary, infinite dimensional representations generated by Hermitian $\Gamma^{\mu\nu}$.

For simplicity, let us restrict ourselves to irreducible representations. The values of the Casimir operators $C_0 = \frac{1}{2}\Gamma_{\mu\nu}\Gamma^{\mu\nu} = \boldsymbol{\Sigma}^2 - \boldsymbol{\Lambda}^2$ and $C_1 = \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\Gamma_{\mu\nu}\Gamma_{\alpha\beta} = \boldsymbol{\Sigma}\cdot\boldsymbol{\Lambda}$ are characteristic of any irreducible representation. However, C_1 must be zero if there is to exist a parity operation which is a symmetry of the wave equation. That is, the reflexion operator P must commute with rotations $\mathbf{J}$ and anticommute with rapidity transformations $\mathbf{K}$ ($K^i \equiv J^{i0}$). But these operators have the form of Eq. (1.4), so that $P = \gamma^0\mathcal{P}R$, where R accomplishes the spatial reflexion, and $\mathcal{P}$ is a matrix in the space of the $\Gamma^{\mu\nu}$. Thus $\mathcal{P}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\mathcal{P}$ and $\mathcal{P}\boldsymbol{\Lambda} = -\boldsymbol{\Lambda}\mathcal{P}$ from which follows $\mathcal{P}(\boldsymbol{\Sigma}\cdot\boldsymbol{\Lambda})\mathcal{P} = -\boldsymbol{\Sigma}\cdot\boldsymbol{\Lambda}$. If the representation is irreducible, $\boldsymbol{\Sigma}\cdot\boldsymbol{\Lambda} = C_1 = 0$.

In order to solve for the m_κ, one need only remember that for the unitary representations the basis vectors can be labeled by j and m, which indicate the eigenvalues of $\boldsymbol{\Sigma}^2$ and Σ_3, respectively.[7] For each j there are $2j+1$ states which transform under $\boldsymbol{\Sigma}$ like the j representation of the rotation group. When $\boldsymbol{\Sigma}\cdot\boldsymbol{\Lambda}=0$, the $\boldsymbol{\Lambda}$ matrices connect states of different j, and all j occur which are integrally larger than some minimum j_0, which may be integral or half-integral. Since $\boldsymbol{\Lambda}$ is a vector under $\boldsymbol{\Sigma}$, it connects $|j,m\rangle$ only to $|j\pm1, m\rangle$ and $|j\pm1, m\pm1\rangle$. $\boldsymbol{\Lambda}$ can have no diagonal matrix elements since it has odd parity.

Thus, a complete set of states can be labeled by the eigenvalues of γ^0, σ_3, $\boldsymbol{\Sigma}^2$ and Σ_3, or alternatively, using an orthogonal transformation whose matrix elements are rotational Clebsch-Gordan coefficients, as $|\beta JMj\rangle$, where β and M are the eigenvalues of γ^0 and J_3. Since γ^0 does not commute with $\mathfrak{M}'$, these are not the eigenstates of the mass. However, $\mathfrak{M}'$ connects $|\beta, J, M, J\mp\frac{1}{2}\rangle$ only to itself and $|-\beta, J, M, J\pm\frac{1}{2}\rangle$, and can be shown to have the following form between two such states of spin J.

$$\mathfrak{M}' = \begin{pmatrix} m_0 - m_1[J+\frac{3}{2}] & -im_1[J(J+1)-C_0-\frac{3}{4}]^{1/2} \\ im_1[J(J+1)-C_0-\frac{3}{4}]^{1/2} & -m_0-m_1[J-\frac{1}{2}] \end{pmatrix}. \tag{2.6}$$

For each J there are two solutions for the mass m:

$$m = m_1(J+\tfrac{1}{2}) \pm[(m_0-m_1)^2 + m_1^2(J(J+1)-C_0-\tfrac{3}{4})]^{1/2}. \tag{2.7}$$

There are also two more solutions, with m replaced by $-m$, coming from the eigenvalues of another 2×2 block. The Casimir operator C_0 is related the minimum value $j_0(j_0+1)$ of $\boldsymbol{\Sigma}^2$ so that the square root is never complex. In fact for the case considered here ($\boldsymbol{\Sigma}\cdot\boldsymbol{\Lambda} = C_1 = 0$) we have $C_0 = j_0^2 - 1$ (unless $j_0 = 0$), so that

$$m = m_1(J+\tfrac{1}{2}) \pm\{(m_0-m_1)^2 + m_1^2[(J+\tfrac{1}{2})^2 - j_0^2]\}^{1/2}, \tag{2.8}$$

which is obviously real.

Notice that in one sequence of states the mass increases indefinitely with J, but that the second sequence, obtained by using the minus sign in Eq. (2.7), decreases asymptotically to $m=0$. Thus there is no discrete lowest mass; rather, the mass spectrum has an accumulation point.

Our second example is based upon Majorana's equation.[1] If there exists in the infinite-dimensional representation a set of 4 matrices Γ^μ which transform like a vector under the $\Gamma^{\mu\nu}$, we replace the Dirac γ^μ by these Γ^μ, and take the other representation to be the identity representation. The wave equation is then just

$$[i\Gamma^\mu\partial_\mu - m_0]\psi(x) = 0\,, \tag{2.9}$$

and this arises from the Lagrangian density

$$\mathcal{L}_0 = \psi^\dagger(i\Gamma^\mu\partial_\mu - m_0)\psi\,. \tag{2.10}$$

The spin index of $\psi(x)$ labels the basis of the unitary representation.

Since polynomials in $\Gamma^{\mu\nu}$ span the space (because the representation is irreducible), one might think that odd-rank tensors cannot exist. But this is not necessarily true, as the space is of infinite dimension and there exist matrices which are not polynomials (of course, they must be limits of polynomials).

Majorana wrote down explicitly the matrices $\Gamma^{\mu\nu}$ for the two representations with $C_1 = 0$ and $C_0 = -\frac{3}{4}$. Each of these has an infinite tower of spins, one starting from $j_0 = 0$ (integral spins) and the other from $j_0 = \frac{1}{2}$ (half-integral spins). For these representations Majorana exhibited a set of four Hermitian matrices Γ^μ which transform as the components of a four-vector.

It is not hard to show that such Γ^μ exist *only* for those

[7] M. A. Naimark, *Linear Representations of the Lorentz Group*, (Pergamon Press, Ltd., London, 1964).

two representations.[8] The remaining $C_1=0$ representations possess no Γ^μ. The Γ^0 matrix is diagonal (in either case) in spin, and has eigenvalues $(j+\frac{1}{2})$. The discrete mass spectrum of Eq. (2.9) is therefore $m_j=m_0/(j+\frac{1}{2})$.

We shall return to a theory based upn the Majorana field in the following sections. We note here only that these masses decrease asymptotically to zero, a feature possessed also by one branch of the spectrum of our first example.

III. SECOND QUANTIZATION

Next we discuss Eq. (2.2) as a quantized field theory. We can proceed in complete analogy to the usual discussion of the Dirac field. By applying the appropriate rapidity transformation to the $\mathbf{p}=0$ solutions of the wave equation, we can obtain functions describing particles of mass m, spin s, helicity λ, and momentum $\mathbf{p}$. Call these $u(m,\lambda,\mathbf{p})e^{-ip_m\cdot x}$. There are corresponding negative frequency solutions $v(m,\lambda,\mathbf{p})e^{ip_m\cdot x}$.

Let $a(m,\lambda,\mathbf{p})$ and $b(m,\lambda,\mathbf{p})$ be destruction operators for particles and antiparticles, respectively, with the indicated mass, helicity and momentum (the spin s is determined by the mass). Similarly, let $a^\dagger(m,\lambda,\mathbf{p})$ and $b^\dagger(m,\lambda,\mathbf{p})$ refer to the corresponding creation operators. The most general solution to Eq. (2.2) can then be written in a momentum expansion as

$$\psi(x)=\sum_{m,\lambda}\int\frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m}{p_m{}^0}\right)^{1/2}\times[a(m,\lambda,\mathbf{p})u(m,\lambda,\mathbf{p})e^{-ip_m\cdot x}+b^\dagger(m,\lambda,\mathbf{p})v(m,\lambda,\mathbf{p})e^{ip_m\cdot x}],\quad(3.1)$$

where

$$\mathbf{p}_m=\mathbf{p}\quad\text{and}\quad p_m{}^0=(\mathbf{p}^2+m^2)^{1/2},\qquad(3.2)$$

unless there are also space-like solutions which would have to be included in the sum. We shall return to the question of spacelike solutions in Sec. V.

If we require the creation and destruction operators to satisfy usual commutation or anticommutation rules

$$[a(m,\lambda,\mathbf{p}),a^\dagger(m',\lambda',\mathbf{p}')]_\mp=[b(m,\lambda,\mathbf{p}),b^\dagger(m',\lambda',\mathbf{p}')]_\mp=\delta_{mm'}\delta_{\lambda\lambda'}\delta^3(\mathbf{p}-\mathbf{p}'),\quad(3.3)$$

then the field ψ and its canonical conjugate $\pi=\partial\mathcal{L}_0/\partial(\partial_t\psi)=i\psi^\dagger$ satisfy at equal times

$$[\psi(\mathbf{x},t),\pi(\mathbf{y},t)]_\mp=i\int\frac{d^3p}{(2\pi)^3}\left\{\sum_{m\lambda}\frac{m}{p_m{}^0}[u(m,\lambda,\mathbf{p})u^\dagger(m,\lambda,\mathbf{p})\mp v(m,\lambda,-\mathbf{p})v^\dagger(m,\lambda,-\mathbf{p})]\right\}e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}.\quad(3.4)$$

Just as for the solutions to the Dirac equation we can choose the orthonormality conditions

$$u^\dagger(m,\lambda,\mathbf{p})u(m',\lambda',\mathbf{p})=v^\dagger(m,\lambda,\mathbf{p})v(m',\lambda',\mathbf{p})=m^{-1}p_m{}^0\delta_{mm'}\delta_{\lambda\lambda'},\quad(3.5)$$
$$u^\dagger(m,\lambda,\mathbf{p})v(m,\lambda,-\mathbf{p})=v^\dagger(m,\lambda,-\mathbf{p})u(m,\lambda,\mathbf{p})=0.$$

With the choice of anticommutation rules in Eqs. (3.3) and (3.4), and if the timelike solutions are complete, the expression inside the curly bracket of Eq. (3.4) becomes the unit matrix. The canonical anticommutation rules,

$$[\psi(\mathbf{x},t),\pi(\mathbf{y},t)]_+=i\delta^3(\mathbf{x}-\mathbf{y}),\qquad(3.6)$$

are then obtained and the theory is local and formally similar to the Dirac field theory. In contrast, the choice of commutation rules would not have led to a local theory.

The theory which results from this discussion is a local, Lorentz-covariant field theory of fermions. The spins of these particles are obtained by combining the Dirac spin of $\frac{1}{2}$ with the "unitary spin" which increases integrally from some lowest value of j_0. In the case that j_0 is half-integral, the theory describes integral spin particles satisfying Fermi statistics. If j_0 is integral, the usual connection between spin and statistics is obtained.

Let us now discuss the second quantization of the free Majorana theory which has the field Eq. (2.14). As we have mentioned, the one particle mass spectrum is $m_s=m_0(s+\frac{1}{2})^{-1}$, where s is the particle spin. A particular feature of this theory is that there are no negative frequency timelike solutions, and hence no antiparticles.

The general solution to Eq. (2.9) involving timelike momenta can be decomposed as

$$\psi(x)=\frac{1}{(2\pi)^{3/2}}\sum_{s\lambda}\int\frac{d^3p}{(2p_s{}^0)^{1/2}}a(\mathbf{p},s,\lambda)u(\mathbf{p},s,\lambda)e^{-ip_s\cdot x},\quad(3.7)$$

where

$$p_s{}^0=(\mathbf{p}^2+m_s{}^2)^{1/2},$$

and the $u(\mathbf{p},s,\lambda)$ satisfy

$$(\Gamma\cdot p_s-m_0)u(\mathbf{p},s,\lambda)=0.\qquad(3.8)$$

From Eq. (2.10), the momentum canonically conjugate to $\psi(x)$ is $\pi=\partial\mathcal{L}_0/\partial(\partial_0\psi)=i\psi^\dagger\Gamma_0$. Applying the usual rules (3.3) for the particle creation and destruction operators, we obtain from (3.7) the following expression for the equal time commutation (anticommutation) relations of the field ψ and its canonical momentum π:

$$[\psi(\mathbf{x},t),\pi(\mathbf{y},t)]_\mp=\frac{i}{(2\pi)^3}\int d^3p\times\left[\sum_{s\lambda}\frac{1}{2p_s{}^0}u(\mathbf{p},s,\lambda)u^\dagger(\mathbf{p},s,\lambda)\Gamma^0\right]e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}.\quad(3.9)$$

If the timelike solutions were complete, the expression inside the square bracket of Eq. (3.9) would be the unit matrix (see Appendix A) and there would result

[8] I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (Pergamon Press, Inc., New York, 1963), pp. 274 ff.

the canonical commutation rules

$$[\psi(\mathbf{x},t),\pi(\mathbf{y},t)]_{\mp}=i\delta^3(\mathbf{x}-\mathbf{y}). \quad (3.10)$$

The Hamiltonian density from Eq. (2.15) is

$$\mathcal{H}=:\psi^\dagger(i\mathbf{\Gamma}\cdot\nabla+m_0)\psi:, \quad (3.11)$$

and therefore, substituting Eq. (3.7) into Eq. (3.11) and using the orthonormality properties of the $u(\mathbf{p},s,\lambda)$ discussed in the Appendix, we would obtain for the total energy operator

$$H=\sum_{\lambda s}\int d^3p(\mathbf{p}^2+m_s^2)^{1/2}a^\dagger(\mathbf{p},s,\lambda)a(\mathbf{p},s,\lambda). \quad (3.12)$$

Thus we would obtain local commutation or anticommutation rules and a positive-definite energy, without a restriction on the connection between spin and statistics.

IV. THE *PCT* THEOREM

The failure of the usual connection between spin and statistics for the theories discussed in Secs. II and III suggests that the *PCT* theorem might also fail. We show in this section that this can be the case for our theories (and for others like them) even for those cases where the spin-statistics theorem holds.

For the field of Eq. (2.2), let us imitate the usual proof for the Dirac field. Since in the free-field theory we know that for every particle state there exists the corresponding antiparticle state, it follows that a *PCT* operator can be defined on the fields, which leaves the free Lagrangian invariant. We now investigate the properties any such operator must have, and then will show that in contrast to usual theories, it is possible to construct local interactions which break *PCT* invariance, although the operator can still be defined on the interacting fields.

Any *PCT* operator must be an antiunitary operator satisfying

$$(PCT)\psi(x)(PCT)^{-1}=\theta^*\psi^\dagger(-x), \quad (4.1)$$

where θ is a matrix operating on the indices. First apply *PCT* to the wave Eq. (2.2):

$$(PCT)[i\gamma^\mu\partial_\mu-m_0-\tfrac{1}{2}m_1\sigma_{\mu\nu}\Gamma^{\mu\nu}]\psi(x)(PCT)^{-1}=0. \quad (4.2)$$

Since *PCT* is anti-unitary, it complex-conjugates all c numbers; therefore

$$[-i\gamma^{\mu*}\partial_\mu-m_0-\tfrac{1}{2}m_1\sigma^{\mu\nu*}\Gamma_{\mu\nu}^*]\theta^*\psi^\dagger(-x)=0. \quad (4.3)$$

Next take the Hermitian conjugate of this equation, change x to $-x$ (and therefore ∂_μ to $-\partial_\mu$), and multiply by θ^{-1}:

$$\theta^{-1}[-i\gamma^\mu\partial_\mu-m_0-\tfrac{1}{2}m_1\sigma_{\mu\nu}\Gamma^{\mu\nu}]\theta\psi(x)=0. \quad (4.4)$$

This is consistent with the wave equation only if $\theta^{-1}\gamma^\mu\theta=-\gamma^\mu$ and $\theta^{-1}\sigma_{\mu\nu}\Gamma^{\mu\nu}\theta=\sigma_{\mu\nu}\Gamma^{\mu\nu}$. The first condition implies $\theta=i\gamma_5\Theta$, where Θ is a matrix in the unitary space only. Since γ_5 commutes with $\sigma^{\mu\nu}$, it follows that $\Theta^{-1}\Gamma^{\mu\nu}\Theta=\Gamma^{\mu\nu}$. Because the $\Gamma^{\mu\nu}$ generate an irreducible representation, by Schur's lemma, Θ must be a multiple of the identity.

The standard proof continues by demonstrating that all tensor densities [$:\bar\psi(x)\gamma^\mu\psi(x):$, $:\bar\psi(x)\partial^\mu\psi(x):$, $:\bar\psi(x)\sigma^{\mu\nu}\psi(x):$, etc.] are even or odd under *PCT* with the rank of the tensor; and therefore that all invariants, like the Lagrangian, are even.

However, we may choose the $\Gamma^{\mu\nu}$ to generate one of the two Majorana representations. Then there exist four-vector matrices Γ^μ in the space spanned by the $\Gamma^{\mu\nu}$. But Θ is a multiple of the identity, and so $\Theta^{-1}\Gamma^\mu\Theta=\Gamma^\mu$. Thus $\theta^{-1}\Gamma^\mu\theta=\Gamma^\mu$, whereas $\theta^{-1}\gamma^\mu\theta=-\gamma^\mu$. Now we can construct a vector density $\bar\psi(x)\Gamma^\mu\psi(x)$, which is even under *PCT*.

Thus we can construct interaction Lagrangians, local in these fields, which must violate *PCT* invariance. For example,

$$\mathcal{L}_I=e:\bar\psi\Gamma^\mu\psi A_\mu:, \quad (4.5)$$

where A_μ is any ordinary vector field. Or,

$$\mathcal{L}_I=g:\bar\psi\Gamma^\mu\psi\bar\psi\gamma_\mu\psi:. \quad (4.6)$$

Both of these are odd under *PCT*, whereas the free Lagrangian is even.

The free quantized Majorana theory violates *PCT* in a more direct but physically less interesting way: there simply are no antiparticles, so clearly no *PCT* operator exists in the theory.

V. DISCUSSION

The conventional proof of the connection between spin and statistics is based upon comparing the Wightman function $M_{\alpha\beta}(x)\equiv\langle|\phi_\alpha(x)\phi_\beta(0)|\rangle$ to $M_{\beta\alpha}(-x)=\langle|\phi(0)\phi(x)|\rangle$.[9] The functions $M_{\alpha\beta}(x)$ and $M_{\alpha\beta}(-x)$ are connected by a complex Lorentz transformation, which can be taken to be $e^{i\pi(J+iK)_3}$. For finite-dimensional representations the matrices are the usual $D^{(s_1,s_2)}$, or an appropriate sum of these, and the index can label the eigenvalues (m_+,m_-) of $(J\pm iK)_3$, which are always integer or half integer for the nonunitary representations. Therefore, the matrix part of $e^{i\pi(J+iK)_3}$ is diagonal in this basis, and is $\pm i$ or ± 1 according as the spin is integral or half integral. Thus

$$M_{\alpha\beta}(x)=(-1)^{2s}M_{\alpha\beta}(-x). \quad (5.1)$$

The axiomatic proof therefore depends crucially on the use of the $D^{(s_1,s_2)}$ representations, and does not apply to our examples.[10] Similar remarks apply to the axiomatic proofs of the *PCT* theorem.

Let us now consider the question of space-like solutions of our field equations. The existence of such solutions of the Majorana equation has been known for

[9] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (W. A. Benjamin and Company, Inc., New York, 1964).

[10] In our case, $\mathbf{J}+i\mathbf{K}$ generate an infinite-dimensional representation of $SU(2)$, and therefore the eigenvalues are not necessarily integers or half integers.

many years,[1] and in Appendix B we show that these space-like momenta must be included in the spectrum for completeness. Although we have not shown it explicitly, we suspect that space-like solutions also are required in the first theory of Sec. II. Equation (3.1) and Eqs. (3.3)–(3.5), as well as Eqs. (3.8), (3.9), would then have to be extended to include the contributions from the elementary creation and destruction operators which refer to these space-like modes. The canonical commutation, or anticommutation, rules (3.6) and (3.10) would then again be obtained, and our conclusions concerning spin and statistics and *PCT* violation would remain unaltered.

The presence of space-like momenta in the asymptotic fields of our two example theories suffices to eliminate them from any direct physical applications. In fact, it may be that all Lorentz-covariant field theories with first-order wave equations and nondegenerate mass spectra which employ a unitary representation suffer from this same disease; although this is certainly not true for second-order equations. It is amusing to note that to guarantee a real spectrum of the mass operator in Eq. (2.5), we were led to the use of infinite dimensional, unitary representations of the $\mathbf{\Sigma}$ and $\mathbf{\Lambda}$ only to be faced with the space-like momenta, which are no more physically acceptable than complex masses.

Despite the existence of space-like momenta in the two theories which we have studied, it nevertheless may be possible to construct a *PCT*-violating theory without these pathological solutions. If such theories could be constructed, *PCT* violation would be compatible with local, covariant field theory; on the other hand, a proof that this is not possible would extend the existing *PCT* theorem to theories with infinite component fields.

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APPENDIX A

In this Appendix we outline the few steps leading from Eq. (3.9) to Eq. (3.10). Equation (3.8) can be written as

$$(\Gamma^0)^{-1}(\mathbf{\Gamma}\cdot\mathbf{p}+m_0)u(\mathbf{p},s,\lambda)=p_s{}^0u(\mathbf{p},s,\lambda)\,. \qquad \text{(A1)}$$

Defining $\mathcal{E}(\mathbf{p})\equiv(\Gamma^0)^{-1}(\mathbf{\Gamma}\cdot\mathbf{p}+m_0)$, it is easy to see that

$$\Gamma^0\mathcal{E}(\mathbf{p})=\mathcal{E}^\dagger(\mathbf{p})\Gamma^0\,, \qquad \text{(A2)}$$

and this relation guarantees that the eigenvectors of $\mathcal{E}(\mathbf{p})$ belonging to different eigenvalues are orthogonal with respect to the metric Γ^0. Since there is only one mass associated with each spin s, we have therefore

$$u^\dagger(\mathbf{p},s,\lambda)\Gamma^0u(\mathbf{p},s',\lambda')\sim\delta_{ss'}\,. \qquad \text{(A3)}$$

Let L be a Lorentz transformation and $D(L)$ its realization in the Majoranna representation. Then

$$u(\mathbf{p},s,\lambda)=\sum_{\lambda'}R_{\lambda\lambda'}D(L)u(\mathbf{p}',s,\lambda')\,, \qquad \text{(A4)}$$

where R is a unitary matrix, and p' is the momentum obtained from $(p,p^0=(\mathbf{p}^2+m_s{}^2)^{1/2})$ by the Lorentz transformation L^{-1}; that is,

$$p^\mu=L^\mu{}_\nu p'^\nu\,. \qquad \text{(A5)}$$

From Eq. (A4), and the fact that

$$D^\dagger(L)\Gamma^\mu D(L)=L^\mu{}_\nu\Gamma^\nu\,, \qquad \text{(A6)}$$

it follows that

$$u^\dagger(\mathbf{p},s,\lambda)\Gamma^\mu u(\mathbf{p},s,\lambda')$$
$$=L^\mu{}_\nu\sum_{\lambda'',\lambda'''}R^\dagger{}_{\lambda'''\lambda}R_{\lambda'\lambda''}u^\dagger(\mathbf{p}',s,\lambda''')\Gamma^\nu u(\mathbf{p}',s,\lambda''). \qquad \text{(A7)}$$

Choose the Lorentz transformation in (A5) to lie along the direction of $\mathbf{p}$ and of such a magnitude that $\mathbf{p}'=0$. Then $R_{\lambda\lambda'}=\delta_{\lambda'\lambda}$, and Eq. (A7) reads

$$u^\dagger(\mathbf{p},s,\lambda)\Gamma^\mu u(\mathbf{p},s,\lambda)=L^\mu{}_\nu u^\dagger(0,s,\lambda)\Gamma^\nu u(0,s,\lambda')\,. \qquad \text{(A8)}$$

Since the $\Gamma^i(i=1,2,3)$ have no diagonal elements in the Majoranna representation (that is, they have no matrix elements between states with the same eigenvalue of Γ^0), only $\nu=0$ contributes to the right-hand side of Eq. (A8). If the $u(0,s,\lambda)$ are normalized according to

$$u^\dagger(0,s,\lambda)\Gamma^0u(0,s,\lambda')=2m_s\delta_{\lambda\lambda'}\,, \qquad \text{(A9)}$$

Eq. (A8) takes the form

$$u^\dagger(\mathbf{p},s,\lambda)\Gamma^\mu u(\mathbf{p},s,\lambda')=\delta_{\lambda\lambda'}L^\mu{}_0 2m_s\,,$$

and by comparing this with Eq. (A5), we have

$$u^\dagger(\mathbf{p},s,\lambda)\Gamma^\mu u(\mathbf{p},s,\lambda')=2p^\mu\delta_{\lambda\lambda'}\,. \qquad \text{(A10)}$$

Combining (A10) with (A3) finally leads to

$$u^\dagger(\mathbf{p},s,\lambda)\Gamma^0u(\mathbf{p},s',\lambda')=2p_s{}^0\delta_{ss'}\delta_{\lambda\lambda'}\,. \qquad \text{(A11)}$$

This normalization and the assumed completeness of the $u(\mathbf{p},s,\lambda)$ imply that the expression inside the bracket of (3.9) is the unit matrix. Apparently, it is the s-dependent normalization of the $u(\mathbf{p},s,\lambda)$ given in Eq. (A11) which allows us to contradict one of the conclusions of Ref. 4.

APPENDIX B

The existence of spacelike solutions to Eq. (2.9) was first pointed out by Majorana himself.[1] In our language, the set of infinite-component vectors $u(\mathbf{p})$ which satisfy

$$(\Gamma_0p_0-\mathbf{\Gamma}\cdot\mathbf{p}-m_0)u(\mathbf{p})=0 \qquad \text{(B1)}$$

for fixed p and $p_0^2-\mathbf{p}^2>0$ is not complete with the Γ_0 metric we used in Sec. III; i.e., the quantity in the brackets in Eq. (3.9),

$$\sum_{s,\lambda} 2p_s{}^0 u(\mathbf{p},s,\lambda)u^\dagger(\mathbf{p},s,\lambda)\Gamma_0 \tag{B2}$$

is not the identity matrix. The $u(\mathbf{p},s,\lambda)$ are the eigenvectors of $\mathcal{E}\equiv\Gamma_0^{-1}(\mathbf{\Gamma}\cdot\mathbf{p}+m_0)$; we can show that in addition to the discrete timelike values $p_0=(\mathbf{p}^2+m_s{}^2)^{1/2}$, $\mathcal{E}$ has a continuous spectrum, as follows:

Since $\mathcal{E}$ is Hermitian in the Γ^0 metric, its spectrum must be real, and to show that p_0 is in the spectrum it sufficies to construct a sequence of vectors u_N which are in the Hilbert space (i.e., such that $u_N{}^\dagger\Gamma^0u_N<\infty$) and such that

$$\frac{u_N{}^\dagger(\mathcal{E}-p_0)\Gamma^0(\mathcal{E}-p_0)u_N}{u_N{}^\dagger\Gamma^0u_N}\longrightarrow 0 \tag{B3}$$

as $N\rightarrow\infty$.

Let $v(s)$ be the basis vectors for the Majorana representation having eigenvalues $s(s+1)$ and 0 of Σ^2 and Σ_Z, respectively [which we have called $u(0,s,0)$]. The matrix Γ_3 is known explicitly[1]:

$$\begin{aligned}\Gamma_3v(s)&=\tfrac{1}{2}i[sv(s-1)-(s+1)v(s+1)], \quad s>0\\ \Gamma^3v(0)&=[-iv(1)/2].\end{aligned} \tag{B4}$$

A natural way to find a sequence u_N is to construct a vector $u=\sum_0^\infty a_sv(s)$ which is an eigenvector of $\mathcal{E}$. No generality is lost by choosing $p_1=p_2=0$. Thus we seek a solution to

$$(\Gamma_3p+m_0-\Gamma_0p_0)u(p)=0 \tag{B5}$$

for $|p_0|<|p|$. The vector $u(p)$ satisfying (B5) will not be normalizable, since all the discrete eigenvalues are timelike; but we may choose

$$u_N=\sum_{s=0}^N a_sv(s) \tag{B6}$$

and hope to satisfy (B3).

Equation (B5) reads

$$\sum_{s=0}^\infty [\tfrac{1}{2}isv(s-1)-\tfrac{1}{2}i(s+1)v(s+1) +\alpha\sqrt{s}-(s+\tfrac{1}{2})\beta\sqrt{s}]=0, \tag{B7}$$

where $\alpha=m_0/p$, $\beta=p_0/p$. The eigenvalue is spacelike, lightlike or timelike according as $\beta<1$, $\beta=1$, or $\beta>1$. Equation (B7) becomes a recursion relation for the a_s:

$$\tfrac{1}{2}i(s+1)a_{s+1}-\tfrac{1}{2}isa_{s-1}+\alpha a_s-\beta(s+\tfrac{1}{2})a_s=0. \tag{B8}$$

The only feature of the above rule, which defines the a_s, that we shall need is that for very large s, it becomes a second-order difference equation for a_s with constant coefficients:

$$a_{s+1}-a_{s-1}+2i\beta a_s=0, \quad s\gg1. \tag{B9}$$

A solution is $a_s=\gamma^s$, where

$$\gamma^2+2i\beta\gamma-1=0 \tag{B10}$$

or

$$\gamma=-i\beta\pm(1-\beta^2)^{1/2}. \tag{B11}$$

The two solutions may be labeled γ_1 and γ_2, and the most general solution to (B9) is

$$a_s=a_0\gamma_1{}^s+b_0\gamma_2{}^s. \tag{B12}$$

For spacelike eigenvalues, $\beta<1$, and $|\gamma_1|=|\gamma_2|=1$, so that

$$a_s=i^s(a_0e^{is\phi}+b_0e^{-is\phi}), \tag{B13}$$

where ϕ is real and determined by β.

One of the constants is determined by (B8) for $s=0$ and the "boundary condition" that $a_{s-1}=0$. The other is an arbitrary normalization, so that we are free to choose b_0/a_0 real. Thus, a_s is bounded:

$$|a_s|=[|a_0+b_0|^2\cos^2s\phi+|a_0-b_0|^2\sin^2s\phi]^{1/2}. \tag{B14}$$

[For timelike eigenvalues, $\beta>1$, ϕ is not real, the a_s are not bounded, and (B3) will not be satisfied except by some miraculous cancellation which presumably occurs for $p_0=+(p^2+m_s{}^2)^{1/2}$.]

Now we can evaluate (B3). With u_N defined by (B6),

$$(\Gamma^3+\alpha-\beta\Gamma_0)u_N =-\tfrac{1}{2}i(N+1)[a_{N+1}v(N)+a_Nv(N+1)] \tag{B15}$$

and the fraction in (B3) becomes

$$\frac{1}{p^2}\frac{u_N{}^\dagger(\Gamma_3+\alpha-\beta\Gamma_0)\Gamma_0{}^{-1}(\Gamma_3+\alpha-\beta\Gamma_0)u_N}{u_N{}^\dagger\Gamma_0u_N} =\frac{\tfrac{1}{4}(N+1)^2[|a_{N+1}|^2/(N+\tfrac{1}{2})+|a_N|^2/(N+\tfrac{3}{2})]}{\sum_{s=0}^N|a_s|^2(s+\tfrac{1}{2})}. \tag{B16}$$

Because $|a_N|^2$ is bounded from above, the numerator grows asymptotically like N. Because $|a_s|$ equals (or, if ϕ/π is irrational, comes arbitrarily close to) its maximum with periodic regularity, the denominator grows asymptotically like N^2, and therefore the fraction becomes small like $1/N$ as $N\rightarrow\infty$, and p_0 is in the continuous spectrum of $\mathcal{E}$.