may also be traced from the relative motion of local frames. Thus, instead of working backwards as we have done here for a charged particle with spin, we directly considered the relative motion of local frames, as a test particle (for simplicity without spin) carried out its world-line motion, as a means of describing gravitational effects. One of the essential ingredients in this development is just the point we have been trying to make here, that the tangent dynamics is not a complete probe of the relative motion of local particle frames and that the latter takes primacy, since it also enters into spin dynamical considerations.

As a general comment on gravitational theories, we may say the following. The fact that gravitation is omnipresent does not necessarily indicate that its effects should be placed in the metric of the space, as it may be that gravitational effects are more accurately described in terms of the relative motions of local particle frames which are only incompletely revealed in the tangent dynamics. Indeed, from our viewpoint, the possibility for the unified description of the effects of electromagnetic and gravitational effects via relative local frame motion is too attractive to be set aside without the most serious consideration.

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Symmetry and Degeneracy in Classical Mechanics*

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Recently it has been shown by several authors that both an $O(4)$ and an $SU(3)$ symmetry, heretofore associated with the nonrelativistic Kepler and three-dimensional isotropic harmonic-oscillator problems, respectively, are automatically possessed by all classical central potentials to the extent that the Poisson bracket forms of the Lie alegbras of these groups can be explicitly constructed. This result has been extended by Mukunda to be a property of all classical dynamical problems involving three degrees of freedom independent of the functional form of the Hamiltonian. We investigate the interrelations among the classical mechanical degeneracy, the simply periodic nature of motions, and the separability of Hamilton-Jacobi equations, and the question to what extent the invariance of the Hamiltonian of a classical system under the Poisson bracket forms of the Lie algebras constitutes a higher symmetry in the global and dynamical sense. We show that for a large class of classical systems the occurrence of degeneracies is a direct consequence of the separability of the Hamilton-Jacobi equations in *a continuous family* of coordinate systems and that, as such, Lie algebras do not by themselves automatically constitute higher symmetries unless a finitely multivalued realization of the corresponding group in the phase space of the system exists.

I. INTRODUCTION

THE recent success of higher-symmetry groups such as $SU(3)$ and $SU(6)$ in classifying the spectrum of elementary particles and the notion of noninvariance dynamical groups that characterize the entire spectrum have led to renewed investigations, from similar group-theoretic viewpoints, of some of exactly solvable dynamical systems which afford higher-symmetry groups and corresponding noninvariance dynamical groups in classifying energy levels.[1] Two outstanding examples of these systems are the nonrelativistic Kepler problem, whose higher symmetry group is $O(4)$ [2] and whose corresponding noninvariance dynamical group is the deSitter group $O(4,1)$ [3], and the three-dimensional isotropic oscillator which has $SU(3)$ [4] and $SU(3,1)$ [5] as its respective groups.

These two systems, which exhibit higher symmetries both in classical and quantum mechanics, occupy special places among central-potential problems in the sense that they possess symmetries higher than the apparent spherical symmetry under $O(3)$. Quantum-mechanically higher symmetry of these systems manifests itself through the occurrences of an "accidental" degeneracy such as n^2-fold degeneracy of the Kepler problem, each set of n^2 levels forming a basis for an irreducible representation of the group $O(4)$. In

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[1] See, for example, N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan, Phys. Rev. Letters **15**, 1041 (1965); contributions by E. C. G. Sudarshan and by Y. Ne'eman, in *Proceedings of the Third Coral Gables Conferences on Symmetry Principles at High Energy*, edited by A. Perlmutter *et al.* (W. H. Freeman & Company, San Francisco, 1966).

[2] W. Pauli, Z. Physik. **36**, 336 (1926); V. Fock, *ibid.* **98**, 145 (1935); V. Bargmann, *ibid.* **99**, 576 (1936).

[3] For example, see M. Y. Han, Nuovo Cimento **42B**, 367 (1966) and references therein; H. Bacry, *ibid.* **41A**, 222 (1966).

[4] J. M. Jauch and E. L. Hill, Phys. Rev. **57**, 641 (1940); D. M. Fradkin, Am. J. Phys. **33**, 207 (1965).

[5] See Ref. 1; R. C. Hwa and J. Nuyts, Phys. Rev. **145**, 1188 (1966).

classical mechanics corresponding degeneracies manifest themselves through the functional dependence of the Hamiltonian on the action variables. The occurrences of these classical "accidental" degeneracies are connected on the one hand with the properties of motions being simply periodic (i.e., all orbits are closed), and on the other hand with the separability of the Hamilton-Jacobi equations in more than one set of coordinate systems which are the two characteristics possessed only by the Kepler and oscillator problems amongst all central potentials.

Recently, however, it has been shown by several authors[6] that *both* an $O(4)$ and an $SU(3)$ symmetry are automatically possessed by *all* classical spherically symmetric potentials *to the extent* that the Poisson bracket form of the Lie algebras of these two groups can be explicitly constructed in terms of the primitive canonical variables q_i and p_i for all such potentials, every generator having zero Poisson bracket with the Hamiltonian. This result has been recently extended by Mukunda,[7] who showed that all classical dynamical problems involving n degrees of freedom automatically possess invariance under $O(n+1)$ and $SU(n)$ Lie algebras, independent of the functional form of the Hamiltonian, indicating that the automatic existence of such higher-symmetry algebras is purely a consequence of detailed properties of local canonical transformations and is totally independent of the dynamical contents of the system.

In this paper we investigate the question to what extent the invariance of the Hamiltonian of a classical system under the Poisson-bracket form of the Lie algebras of these groups constitutes a higher symmetry in the global and dynamical sense in relation to the existence of degeneracy. We have previously investigated[8] a closely related question in somewhat narrower scope, namely, the two-dimensional central-potential problems. For the two-dimensional Kepler and isotropic oscillator problems whose Hamiltonians are invariant under the identical Lie algebras, $O(3)$ and $SU(2)$, respectively, we have shown that the distinction between the two *groups* can be made only by considering the explicit realizations of corresponding groups in terms of *finite* canonical transformations.

In Sec. II we give the classical-mechanical definition of degeneracy in terms of the action-angle-variables formalism,[9] which shows how the "accidental" degeneracies of the Kepler and isotropic oscillator problems are related to their simply periodic properties.

In Sec. III we show that for separable multiply periodic classical systems that include all central-potential bound systems the occurrence of degeneracies is a direct consequence of the separability of the Hamilton-Jacobi (HJ) equation of the system in a *continuous family* of coordinate systems. Having first elucidated the method of proof for the well-known case of spherical symmetry, we prove that the "accidental" degeneracies of the Kepler and isotropic-oscillator systems are the consequences of the separability of the HJ equation in a continuous family of prolate spheroidal coordinate systems related to one another by single-valued transformations with arbitrary orientation of the unique axis and arbitrary interfocal distance, a coordinate system often used in its two-dimensional form, the elliptic coordinates, in connection with the problems of two centers of gravitation.[10]

In Sec. IV we show that the existence of a Poisson bracket form of a Lie algebra under which the Hamiltonian is invariant does not by itself automatically constitute a higher symmetry of the system unless a single-valued, or at most a finitely many-valued realization of the corresponding group in the phase space exists.

II. CLASSICAL DEFINITION OF DEGENERACY

The quantum states of bound systems are discrete, labeled by the energy of the state together with the eigenvalues of other observables constituting a complete set of commuting observables. A system possesses a degeneracy if more than one state has a given energy eigenvalue. In the classical description of a bound system there are no discrete states, so that the quantum definition of degeneracy is inadequate in classical mechanics. A classical bound system is usually multiply periodic, which means that any variable of the system can be expanded in a multiple Fourier series with fundamental frequencies $\nu_1, \nu_2, \cdots \nu_f$ when there are f degrees of freedom. If these f frequencies are all incommensurable, the system is nondegenerate. If there exist m relations among the frequencies of the form

$$\sum_{i=1}^{f} n_i^{(k)}\nu_i=0\,, \qquad (1)$$

$$k=1, \cdots m \quad (1\leqslant m\leqslant f-1)\,,$$

with integer n_i, the system is said to be m-fold degenerate.[11] When m is equal to $f-1$ (complete degeneracy), all frequencies are rational fractions of each other so that the motion is simply periodic. For the multiply periodic systems which are separable, these frequencies are given by

$$\nu_i=\partial H/\partial J_i\,, \qquad (2)$$

[6] D. M. Fradkin, Progr. Theoret. Physics (Kyoto) **37**, 798 (1967); H. Bacry, H. Ruegg, and J. Souriau, Commun. Math. Phys. **3**, 323 (1966).

[7] N. Mukunda, Phys. Rev. **155**, 1383 (1967). This has also been remarked by H. Bacry *et al.* (Ref. 6).

[8] M. Y. Han and P. Stehle, Nuovo Cimento **48A**, 180 (1967).

[9] H. C. Corben and P. Stehle, *Classical Mechanics* (John Wiley & Sons, Inc., New York, 1960), 2nd ed., Chap. 11.

[10] Reference 9, p. 207.

[11] M. Born, *The Mechanics of the Atom* (Frederick Ungar Publishing Company, New York, 1960).

where H is the Hamiltonian and the J_i are the action variables given by

$$J_i = \oint p_i dq_i, \tag{3}$$

the integration being over one complete cycle of q_i.

A separable multiply periodic system is therefore degenerate if the Hamiltonian, expressed as a function of action variables, depends on certain of these variables only through a linear combination with integer coefficients. For example, the effect of the spherical symmetry is to ensure that the action variables J_θ and J_ϕ enter the Hamiltonian only in the combination $J_\theta + J_\phi$ so that $\nu_\theta = \nu_\phi$. The effect of the higher symmetries is to make the Hamiltonian a function of $nJ_r + J_\theta + J_\phi$ with rational n ($n=1$ for the Kepler problem and $n=2$ for the oscillator problem), and not of J_r and $J_\theta + J_\phi$ separately.

The connection between this definition of degeneracy and the quantum definition is made clear by the application of the semiclassical quantization rule, that stationary states of separable multiply periodic systems are obtained by equating the action variables to integer multiples of Planck's constant. The relation between the accidental degeneracies of the Kepler and the isotropic-oscillator problems and the simply periodic nature of their motions also becomes clear by this definition of degeneracy. Confining consideration to three degrees of freedom, all central-potential problems are at least singly degenerate ($\nu_\theta = \nu_\phi$), because of the spherical symmetry, and at most can be doubly degenerate, since the dimension is three ($m = f-1 = 2$). If for some central potentials the system is doubly degenerate, i.e., there exists accidental degeneracy, then the system is completely degenerate and is necessarily simply periodic. But it has been shown[12] that the only central potentials for which all motions are simply periodic are the Kepler and oscillator potentials.

III. RELATION OF SEPARABILITY TO DEGENERACY

For separable multiply periodic system we now show that if the Hamilton-Jacobi equation of a system is separable in a continuous family of coordinate systems, then the system has some degeneracy in the sense defined above by (1). We do this first in the familiar case of rotational invariance which is known to lead to the equality of frequency of the ϕ and θ motions in classical-central systems, and to the $(2l+1)$ degeneracy of quantum systems.

[12] J. Bertrand, Compt. Rend. **77**, 849 (1875). A theorem of Bertrand and Koenig shows this for potentials which vanish at infinity. In general one requires a motion near a circular motion to have a "betatron frequency" about that circular orbit which is an integer ν. Then $V \propto r^{\nu^2-2}$. For $\nu \geq 3$ there exist straight-line orbits through the force center which have a betatron frequency of 2, so in general the orbits between these two extremes have non-integer betatron frequencies and are therefore not closed.

The Hamilton-Jacobi equation for a system composed of a particle in a central potential is separable in spherical coordinates. The Hamiltonian for a central system is, taking unit mass,

$$H = \frac{1}{2}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right] + V(r). \tag{4}$$

The variable ϕ is cyclic, so

$$p_\phi = m, \quad J_\phi = 2\pi m, \tag{5}$$

and we have

$$p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} = 2r^2[E - V(r) - \tfrac{1}{2}p_r^2] = l^2, \tag{6}$$

where $l^2 \geqslant 0$ is the separation constant which, in the standard manner, is given by[9]

$$2\pi l = J_\theta + J_\phi, \tag{7}$$

demonstrating the existence of the degeneracy mentioned above. The connection between spherical symmetry and this degeneracy is revealed by the argument given below.

Let the system be described in a Cartesian coordinate system C. From this system we transform to a spherical coordinate system P, with its origin at the force center. In this new system the Hamilton-Jacobi equation separates, and we may consider an orbit for which $J^P{}_\theta = 0$. The time dependence of any variable of the system, such as one of the Cartesian coordinates of C, can be expressed as a multiple Fourier series which contains only the frequencies of the radial and azimuthal motions, ν_r and ν_ϕ:

$$x = \sum_{\rho,\sigma=-\infty}^{\infty} A_{\rho\sigma} \exp[-i(\rho\nu_r + \sigma\nu_\phi)t], \tag{8}$$

since θ is a constant for this orbit.

This same orbit may be described in another spherical coordinate system P' related to P by a rotation R. In this new system, the Hamilton-Jacobi equation is separable just as in P because the form of the Hamiltonian is invariant under R, but the orbit considered above with $J^P{}_\theta = 0$ corresponds to $J^{P'}{}_\theta \neq 0$. The Cartesian coordinate x is now to be expressed as a multiple Fourier series involving three frequencies:

$$x = \sum_{\rho,\sigma,\tau=-\infty}^{\infty} A_{\rho\sigma\tau} \exp[-i(\rho\nu_r + \sigma\nu_\phi + \tau\nu_\theta)t]. \tag{9}$$

The dependence of x upon r is not changed, so the terms of the series (9) describing the dependence of x on t through the dependence of r on t are unaltered. The over-all time dependence of x is also unaltered, so that the new angular dependence must involve the same frequencies as the old, which implies that $\nu_\theta = n\nu_\phi$ with integer n. This is a consequence of the fact that one complete cycle of ϕ in P corresponds to one complete cycle of ϕ' and θ' in P', the transformation from P to P' being *single valued*. A similar argument starting from an orbit with $J_\phi=0$ leads to $\nu_\phi=n'\nu_\theta$, so that both n and n' are unity and $\nu_\theta=\nu_\phi$.

Because the rotation R can be a *finite* rotation, the relation $\nu_\theta=\nu_\phi$ is established for all orbits and not just those in the neighborhood of $J_\theta=0$. The Hamiltonian can thus depend on J_θ and J_ϕ only through the linear combination

$$J_\Omega=J_\theta+J_\phi, \tag{10}$$

which demonstrates the presence of the degeneracy as a consequence of the invariance of the Hamiltonian under rotations. The ϕ-θ degeneracy is common to all central systems. It is known that there are only two central systems with higher degeneracy, as degeneracy of the r motion with the angular motion implies the closing of all orbits. Only the Kepler system with negative energies and the harmonic oscillator have closed orbits exclusively.[12]

It is well known that each of these systems has a Hamilton-Jacobi equation which is separable in two coordinate systems; the Kepler system is separable in spherical and parabolic coordinates, the oscillator in spherical and Cartesian coordinates. These two dynamical systems are also both separable in prolate-spheroidal coordinates, with arbitrary orientation of the unique axis and arbitrary interfocal distance. This separability in a continuous family of coordinate systems larger than the family of spherical coordinate systems gives rise to a further degeneracy, that of the radial motion with the angular motion.

Consider, for example, the three-dimensional isotropic harmonic oscillator

$$H=\tfrac{1}{2}(p^2{}_x+p^2{}_y+p^2{}_z)+\tfrac{1}{2}(x^2+y^2+z^2). \tag{11}$$

Prolate-spheroidal coordinates are introduced by

$$\begin{aligned} x&=c\sinh\xi\cos\eta\cos\phi,\\ y&=c\sinh\xi\cos\eta\sin\phi,\\ z&=c\cosh\xi\sin\eta. \end{aligned} \tag{12}$$

The surface $\phi=$const are planes through the z axis. The surfaces $\xi=$const are prolate spheroids with foci on the z axis at $\pm c$. The surfaces $\eta=$const are hyperboloids of two sheets with foci on the z axis at $\pm c$. This coordinate system reduces to the spherical and cylindrical coordinates in the limit $c\to 0$ and $c\to\infty$, respectively.[13] The conjugate momenta are

$$\begin{aligned} p_\xi&=c\cosh\xi\cos\eta(\cos\phi p_x+\sin\phi p_y)\\ &\qquad+c\sinh\xi\sin\eta p_z,\\ p_\eta&=-c\sinh\xi\sin\eta(\cos\phi p_x+\sin\phi p_y)\\ &\qquad+c\cosh\xi\cos\eta p_z,\\ p_\phi&=c\sinh\xi\cos\eta(-\sin\phi p_x+\cos\phi p_y). \end{aligned} \tag{13}$$

[13] For finite values of x, y, z we consider the limit $c\to 0$. The circular functions being bounded, the value of $|\xi|\to\infty$. We take $\xi>0$ and write $\xi=\xi'\ln(2/c)$ so that ξ' remain finite in the limit. Then $r=\exp[\xi']$, $\theta=\frac{1}{2}\pi-\eta$, ϕ are spherical coordinates. Similarly the limit $c\to\infty$ may be taken for finite x, y, z. In this case only infinitesimal values of ξ and η occur, so we introduce $\xi'=\xi/c$, $\eta'=\eta/c$ and keep only leading terms in (12). Then ξ', ϕ, η' are the coordinates ρ, ϕ, z of a cylindrical system. The polar coordinates ξ', ϕ may then be considered as the polar limit $c'=0$ of a set of elliptical coordinates in the xy plane. On letting the interfocal distance c' of this system approach ∞, Cartesian coordinates are reached.

The Hamiltonian takes the form

$$H(\xi,\eta,\phi,p_\xi,p_\eta,p_\phi)=\frac{p^2{}_\xi+p^2{}_\eta}{2c^2(\sinh^2\xi+\cos^2\eta)}+\frac{p^2{}_\phi}{2c^2\sinh^2\xi\cos^2\eta}+\tfrac{1}{2}c^2(\sinh^2\xi-\cos^2\eta+1). \tag{14}$$

The Hamilton-Jacobi equation is separable for arbitrary c.

(i). ϕ is cyclic, p_ϕ a constant, $p_\phi=m$.

(ii).
$$\begin{aligned} p^2{}_\xi+p^2{}_\eta&=2c^2E(\sinh^2\xi+\cos^2\eta)\\ &\quad-m^2\left(\frac{1}{\cos^2\eta}+\frac{1}{\sinh^2\xi}\right)\\ &\quad-c^4(\sinh^4\xi+\sinh^2\xi-\cos^4\eta+\cos^2\eta), \end{aligned} \tag{15}$$

in which there are no terms containing both ξ and η. For $m\neq 0$ there are turning points in both the ξ and η motions determined by

$$\begin{aligned} 0=p_\xi{}^2&=-c^4\sinh^4\xi+(2c^2E-c^4)\sinh^2\xi-\frac{m^2}{\sinh^2\xi}-\alpha,\\ 0=p_\eta{}^2&=c^4\cos^4\eta+(2c^2E-c^4)\cos^2\eta-\frac{m^2}{\cos^2\eta}+\alpha, \end{aligned} \tag{16}$$

α being the separation constant.

For $m=0$ the problem reduces to that of the two-dimensional oscillator in a plane containing the z axis. The character of the turning points depends on whether the orbit cuts the interfocal line or not. We shall return to this point later.

Now consider an oscillator described in a Cartesian coordinate system C. A set of prolate spheroidal coordinates (PS) is introduced, and an orbit with $J_\xi=J_\eta=0$ selected, with $J_\phi=2\pi m\neq 0$. Any single-valued variable of the system, the Cartesian coordinate x, for example, may be expressed as a simple Fourier series in the time with the frequency ν_ϕ:

$$x=\sum_{\rho=-\infty}^{\infty}a_\rho\exp[-i\rho\nu_\phi t]. \tag{17}$$

Instead of using the system PS we can use another system PS′ with a different orientation of the axis. (Changing the interfocal distance without a rotation makes no difference here because the orbit is in the equatorial plane of PS.) This introduces ξ and η motions, but cannot change the time dependence of x, so can introduce no new fundamental frequency. This does not establish complete degeneracy because only circular orbits were allowed when $J_\xi=J_\eta=0$, establishing only the spherical degeneracy which here connects the ϕ motion with a certain combination of ξ-η motions. The degeneracy of the ξ and η motions is established by considering an orbit given by PS by $J_\xi=J_\phi=0$, $J_\eta\neq 0$. This is an ellipse with the z axis as major axis described by $\xi=\text{const}$, $\phi=\text{const}$. On changing the interfocal distance the Hamilton-Jacobi equation remains separable, but this orbit now has $J_\xi\neq 0$. This introduces the frequency ν_ξ into the Fourier series for x which previously contained only ν_η. Thus ν_ξ and ν_η are degenerate. Together with the spherical degeneracy shown above, this establishes the complete degeneracy of the isotropic harmonic oscillator.

The Kepler system can be discussed in the same way. It is more convenient to define the prolate spheroidal coordinates by[9]

$$\begin{aligned} x&=c\sinh\xi\sin\eta\cos\phi\,,\\ y&=c\sinh\xi\sin\eta\sin\phi\,,\\ z&=c\cosh\xi\cos\eta\,,\end{aligned}\tag{18}$$

as this puts the origin of the spheroidal system at point $x=y=0$, $z=c$, which is a focus of the conics of the coordinate system. This reduces, in the limit $c\to 0$, to spherical coordinates. In the limit $c\to\infty$ we get paraboloidal coordinates.

One point remains to be commented on. In Cartesian coordinates the oscillator Hamiltonian is

$$H^C=\nu(J_x+J_y+J_z)\,,\tag{19}$$

while in spherical coordinates it is

$$H^P=\nu(2J_r+J_\theta+J_\phi)\,,\tag{20}$$

even though we have shown that there is a continuous family of coordinate systems interpolating between these two limits. The factor 2 multiplying J_r arises in the transition from $c'=0$ to $c'=\infty$ in going from the cylindrical coordinates ($c=\infty$, $c'=0$) to Cartesian coordinates ($c=\infty$, $c'=\infty$).[13] In the cylindrical coordinates any orbit with $m\neq 0$ has a trace on the xy plane enclosing the origin, or enclosing the foci of the elliptic-coordinate system with sufficiently small c'. As c' tends to ∞, the foci cross the orbit and are no longer enclosed by it. At this point the ranges of the coordinates change, causing a change in the definition of the action variables. In the xy plane elliptic coordinates can be defined by

$$\begin{aligned} x&=c'\cosh\bar{\xi}\sin\bar{\eta}\,,\\ y&=c'\sinh\bar{\xi}\cos\bar{\eta}\,.\end{aligned}\tag{21}$$

The coordinate curve $\bar{\xi}=0$ is the line segment between the two foci, points on this segment being distinguished by values of $\bar{\eta}$, which has the range $-\pi/2\leq\bar{\eta}\leq\pi/2$. Points just below this segment then have values of $\bar{\xi}<0$. Thus in describing an orbit which cuts this line segment between the foci, the coordinates vary continuously with the time when $-\infty\leq\bar{\xi}\leq\infty$, $-\pi/2\leq\bar{\eta}\leq\pi/2$. When the orbit encloses the foci, $\bar{\xi}$ is never zero along the orbit, so that to have a continuous variation of coordinates along the orbit $\bar{\xi}$ must remain of one sign, say positive, and the range of $\bar{\eta}$ must be increased to $-\pi\leq\bar{\eta}\leq\pi$ to provide positive and negative values of y as well as of x. It is this change of range which brings in the factor 2.

The argument used in the special case of the harmonic oscillator can be applied to any system whose Hamilton-Jacobi equation is separable in a continuous family of coordinate systems connected to each other by single-valued transformations. In one member of the family, an orbit is selected for which only one J_i is finite. In another member of the family this same orbit corresponds to nonzero values of more than one J, and the frequencies admixed in this way must be integer multiples of frequencies already present, namely ν_i, so that a degeneracy is present. The transformations involved need not be only extended-point transformations but may be more general canonical transformations.

IV. RELATION OF SYMMETRY TO DEGENERACY

If the Hamiltonian of a system is such that it is invariant under a group G with generators X_i, and if there exists one coordinate system in which the Hamilton-Jacobi equation is separable, then there exists a continuous family of coordinate systems in which the Hamilton-Jacobi equation is separable. The invariance condition

$$(H,X_i)_{\text{PB}}=0\tag{22}$$

means that the form of the Hamiltonian is the same in coordinates q, p and q', p' related to each other through X_i, namely by

$$\begin{aligned} q'&=q+\epsilon(\partial X_i/\partial p)\,,\\ p'&=p-\epsilon(\partial X_i/\partial q)\,,\end{aligned}\tag{23}$$

so that if it leads to a separable Hamilton-Jacobi equation in the variables q, p, it leads to an equation separable in exactly the same way in the variables q', p'. These are not necessarily related by an extended-point transformation. It remains to investigate the implications of this for the existence of degeneracies.

Equation (23) characterizes the coordinate transformation locally. The argument establishing degeneracy as a consequence of separability in a continuous family of coordinate systems was based on global properties of the transformation, namely, the fact that a complete cycle of motion described in one coordinate system corresponded to a complete cycle in any other coordinate system of the family. Therefore, if the global transformations corresponding to (23) are single valued, the existence of the X_i establishes degeneracy. If these transformations are infinitely many valued, then the argument establishing degeneracy from separability is inapplicable, and no degeneracy results from the existence of this particular group. If these transformations are finitely many valued, a slightly modified argument shows that degeneracy again results with the frequencies involved being rationally related rather than equal.

The theory is illustrated by the two dimensional oscillator. Consider the anisotropic oscillator whose Hamiltonian is

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+\tfrac{1}{2}(\alpha^2x^2+\beta^2y^2). \tag{24}$$

The transformation to action-angle variables J and w is carried out for motion in the y direction, the generating function being[9]

$$\begin{gathered} S(y,w)=\tfrac{1}{2}\beta y^2\cot 2\pi w\,, \\ p_y=(\partial S/\partial y)\,,\quad J=-(\partial S/\partial w)\,, \end{gathered} \tag{25}$$

from which follow

$$\begin{aligned} J&=\frac{2\pi}{\beta}\left(\frac{p_y^2+\beta^2y^2}{2}\right) \\ &=\frac{2\pi}{\beta}H_y\,, \\ w&=\frac{1}{2\pi}\tan^{-1}\left(\frac{\beta y}{p_y}\right) \\ &=\frac{1}{2\pi}\sin^{-1}\left(\frac{\beta y}{\sqrt{(2H_y)}}\right), \end{aligned} \tag{26}$$

so that

$$\begin{aligned} y&=\left(\frac{J}{\pi\beta}\right)^{1/2}\sin 2\pi w\,, \\ p_y&=\left(\frac{J\beta}{\pi}\right)^{1/2}\cos 2\pi w\,. \end{aligned} \tag{27}$$

Now we carry out the canonical transformation

$$J\to J'=(\beta/\alpha)J\,,\quad w\to w'=(\alpha/\beta)w \tag{28}$$

and then replace J', w' by y', p_y' according to (26) with β replaced by α. Thus

$$\begin{aligned} y&=\left(\frac{\sqrt{(2H_y)}}{\beta}\right)\sin(\beta/\alpha)\sin^{-1}\frac{\alpha y'}{\sqrt{(2H_y')}}\,, \\ p_y&=\sqrt{(2H_y)}\cos(\beta/\alpha)\sin^{-1}\frac{\alpha y'}{\sqrt{(2H_y')}}\,, \end{aligned} \tag{29}$$

with $2H_y'=p_y'^2+\alpha^2y'^2$. These may be solved for y', p_y' to yield

$$\begin{aligned} y'&=\frac{\sqrt{(2H_y)}}{\alpha}\sin(\alpha/\beta)\sin^{-1}\left(\frac{\beta y}{\sqrt{(2H_y)}}\right), \\ p_y'&=\sqrt{(2H_y)}\cos(\alpha/\beta)\sin^{-1}\left(\frac{\beta y}{\sqrt{(2H_y)}}\right). \end{aligned} \tag{30}$$

Equation (30) specifies a time-independent canonical transformation which, therefore, leaves the value of the Hamiltonian invariant. In terms of x' $(=x)$, p_x' $(=p_x)$, y', and p_y' the Hamiltonian has the form

$$H=\tfrac{1}{2}(p_x'^2+p_y'^2)+\frac{\alpha^2}{2}(x'^2+y'^2)\,, \tag{31}$$

which is that of an isotropic oscillator in the space x', y'.

The Hamiltonian (31) is invariant under $SU(2)$ [8,14] applied in the space x', p_x', y', p_y', and in this space the realization of $SU(2)$ is single valued so that the isotropic oscillator is degenerate. It follows that the Hamiltonian (24) is also invariant under $SU(2)$, the generators of $SU(2)$ in the space x, p_x, y, p_y being the transforms of those for the isotropic oscillator according to (30). For irrational α/β, however, the transformation (30) is infinitely many valued, so no identification of frequencies can be made before and after this transformation, and hence no degeneracy results. When α/β is rational, the transformation (30) is finitely many valued, and this does lead to a degeneracy. To see this in detail the transformation from one coordinate system in which the Hamilton-Jacobi equation separates to another such system is carried out in three steps.

(i). The anisotropic oscillator is made "isotropic" by the transformation $x, p_x, y, p_y\to x', p_x', y', p_y'$.

(ii). An element of the symmetry group, $SU(2)$, is used to effect the transformation $x', p_x', y', p_y'\to x''', p_x''', y''', p_y'''$, which leaves the Hamilton-Jacobi equation separated as before.

(iii). The inverse of the transformation (i), i.e., Eq. (29) is applied to the variables x''', etc., leading back to the anisotropic oscillator described in new variables x'', p_x'', y'', p_y''. The entire transformation $x, p_x, y, p_y\to x'', p_x'', y'', p_y''$, is a realization of an element of $SU(2)$ for the anisotropic oscillator.

An orbit is picked such that in the original coordinate system $J_x\neq 0$, $J_y=0$. An observable O, a single-valued function of x, p_x, y, p_y, has a time dependence given by a Fourier series such as (17) with the fundamental frequency $\nu_x=\alpha/2\pi$, since for this orbit the y motion has zero amplitude. The transformation (i) is the identity transformation for x, p_x, but is generally multivalued in y, p_y, so that after (i) has been carried out O is a multivalued function of y', p_y', but its time

[14] Reference 4; V. A. Dulock and H. V. McIntosh, Am. J. Phys. 33, 109 (1965).

dependence is unchanged. The transformation (ii) mixes the x' and y' motions but leaves the Hamilton-Jacobi equation separable, and thus by the argument given previously establishes the degeneracy of ν_x''' and ν_y''', in the form $\nu_y'''=n\nu_x'''$ with integer n. It can easily be shown as in the paragraph following (9) that n must be unity. The Fourier series of O $(x''', p_x''', y''', p_y''')$ has the single fundamental frequency $\nu_x'''=\nu_x$ for the orbit considered. The transformation (iii) leaves x''' unchanged, i.e., $x''=x'''$, $p_x''=p_x'''$, but makes a multivalued transformation of y''' to y''. This means that the Fourier series for O expressed in the variables x'', p_x'', y'', p_y'' involves two distinct fundamental frequencies, ν_x from the x'', p_x'' dependence, and a multiple of this from the y'', p_y'' dependence. Only when the transformation $y'''\rightarrow y''$ is finitely multivalued, which means that a finite number m of cycles of y''' corresponds to a number n of cycles of y'', can these two frequencies be combined to yield a single fundamental frequency for the Fourier series for O. Thus, when $\alpha/\beta=m/n$ with integer m, n, the existence of the symmetry $SU(2)$ establishes the existence of a degeneracy in the two-dimensional harmonic oscillator, because this renders the transformation (31) finitely multivalued.

The method of argument can be applied to any system separable in some one-coordinate system. Step (i) transforms the system into a system having the symmetry of interest as a single-valued transformation group, in step (ii) a symmetry operation of the group is carried out; in (iii) the transformation inverse to (i) is made. Only when this transformation is finitely multivalued does the degeneracy of the symmetric system survive the inverse transformation. Our conclusion is, then, that the existence of a semisimple group G whose generators X_i have zero Poisson bracket with the Hamiltonian of a separable dynamical system implies the existence of a degeneracy of that system only when the group G can be realized in the phase space of the system by finitely multivalued transformations.

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Electromagnetic Interaction of the Bargmann-Wigner Field with Spin ½*

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It is shown that the Bargmann-Wigner multispinor field for particles of spin ½ gives the usual result for the magnetic moment, provided it is ensured that the auxiliary field appearing in the Lagrangian formulation remains vanishing in the presence of electromagnetic interaction.

ALTHOUGH the Bargmann-Wigner multispinor field equations for elementary particles are found to be particularly suitable for the treatment of the symmetries of strong interactions,[1] the electromagnetic interaction of such fields has not yet been fully explored. Indeed, recently it has been claimed[2] that the multispinor formulation leads to particles of spin ½ without any intrinsic magnetic moment, which casts doubt on the validity of such a formalism for the description of charged particles. We shall, however, show that it is possible to establish complete equivalence between the multispinor field with spin ½ and the usual Dirac field by following a more judicious treatment of the electromagnetic interaction.

The Lagrangian density for a Bargmann-Wigner field with spin ½ can be expressed in terms of two third-rank multispinors $\psi_{\alpha\beta\gamma}$ and $\Omega_{\alpha\beta\gamma}$ as[3,4]

$$L=-\bar{\psi}[(\gamma\partial)_1+m]\psi-\bar{\Omega}[(\gamma\partial)_3-\tfrac{1}{2}m]\Omega +\tfrac{1}{2}[\bar{\psi}(\gamma\partial)_3\Omega+\bar{\Omega}(\gamma\partial)_3\psi], \quad (1)$$

with

$$\bar{\psi}^{\alpha\beta\gamma}=\psi^{*\alpha'\beta'\gamma'}(\gamma_4)_{\alpha'}{}^{\alpha}(\gamma_4)_{\beta'}{}^{\beta}(\gamma_4)_{\gamma'}{}^{\gamma}, \quad [(\gamma\partial)_1\psi]_{\alpha\beta\gamma}=(\gamma\partial)_{\alpha}{}^{\alpha'}\psi_{\alpha'\beta\gamma},\ \text{etc.}, \quad (2)$$

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[1] A. Salam, R. Delbourgo, and J. Strathdee, Proc. Roy. Soc. (London) **A284**, 146 (1965); B. Sakita and K. C. Wali, Phys. Rev. **139**, B1355 (1965).
[2] S. Chang, Phys. Rev. Letters **17**, 597 (1966); **17**, 894(E) (1966).
[3] G. S. Guralnik and T. W. B. Kibble, Phys. Rev. **139**, B712 (1965).
[4] We denote the space-time coordinates as $x_\mu=(x_i,ix_0)$, and take the γ_μ as Hermitian matrices with $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$.