Material Sources for the Kerr Metric

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The Kerr metric is a vacuum solution of the Einstein field equations which appears to be the field exterior to some axially symmetric, rotating body. A method based on Synge's g method (guess $g_{\mu\nu}$ and calculate $T_{\mu\nu}$) is given for constructing interior solutions of the field equations which describe rotating, *nonperfect-fluid* bodies which might serve as sources of the Kerr metric. Finally, an argument is given which indicates that *reasonable perfect-fluid*–type solutions which might serve as sources of the Kerr metric may *not* exist.

I. INTRODUCTION

In 1963 Kerr[1] exhibited a vacuum solution of the Einstein field equations which appears to be the field exterior to some axially symmetric, rotating body. This is the only known exact solution of the field equations of its kind. In order to obtain more insight into the Kerr metric and also the more general problem of rotation in general relativity we have indicated here a simple method for constructing interior solutions which might serve as sources for the Kerr metric. These solutions are not obtained in the conventional manner of first choosing an equation of state and then solving the field equations. Instead the interior metrics are obtained by a method of guessing[2] (g method) which can be used for finding sources of any metric, which indicates that it possesses a positive mass source.[3] The essential idea of the g method is that one guesses an "interior" metric $g_{\mu\nu}$ and then calculates the resulting stress-energy tensor $T_{\mu\nu}$ using the Einstein field equations. Of course there are certain requirements imposed on this interior solution. They are that the energy density be non-negative, that the stresses be not too large compared to the energy density, and that $g_{\mu\nu}$ satisfy the Lichnerowicz boundary conditions[4] (that $g_{\mu\nu}$ be C^1 in some coordinate system) at the hypersurface separating the interior and exterior space times. One might also require that the nonshearing-type stresses be positive so that the body experiences essentially pressure-type forces. The key idea involved in obtaining solutions which satisfy the above conditions is the use of the similarity between the Schwarzschild spherically symmetric metric and the Kerr metric. We obtain Kerr source solutions by perturbing strongly the familiar Schwarzschild constant-density interior solution. Objections may be raised against this pro-procedure since it will probably lead to nonperfect-fluid–type solutions. In partial answer to this we present arguments in Sec. III which indicate that it may be impossible to obtain an exact solution of the field equations which represents a perfect fluid and which could also serve as a source of the Kerr metric.

Throughout this paper we shall use units such that $c=G=1$.

II. SOURCE CONSTRUCTION

The terms in the Kerr metric can be rearranged so that it has the form

$$ds^2=r^2(d\theta^2+\sin^2\theta d\phi^2)+2drdu-(1-2m/r)du^2+\left(\frac{a}{r_1}\right)^2 r_1^2\cos^2\theta d\theta^2+\left(\frac{a}{r_1}\right)^2\left(1+\frac{2mr\sin^2\theta}{r^2+a^2\cos^2\theta}\right)r_1^2\sin^2\theta d\phi^2$$
$$+\left(\frac{a}{r_1}\right)2r_1\sin^2\theta d\phi dr+\left(\frac{a}{r_1}\right)\frac{4mrr_1\sin^2\theta}{r^2+a^2\cos^2\theta}d\phi du-\left(\frac{a}{r_1}\right)^2\frac{2m}{r}\frac{r_1^2\cos^2\theta}{r^2+a^2\cos^2\theta}du^2. \quad (1)$$

The constant r_1 has been introduced in an obvious way and the reasons for its use will become apparent shortly. Next we notice that if $a=0$ we simply get the Schwarzschild exterior solution in Vaidya coordinates.[5] Thus the Kerr metric can be written as

$$g_{\mu\nu}=S_{\mu\nu}+(a/r_1)A_{\mu\nu}, \quad (2)$$

where $S_{\mu\nu}$ is the Schwarzschild solution and $A_{\mu\nu}$ is that part of the Kerr metric which contains the angular momentum parameter a. (Note that the factor a/r_1 is dimensionless.)

For simplicity we choose $r=r_1$ as the boundary separating the interior and exterior solutions. In order to guarantee that the Lichnerowicz boundary condition will hold in some coordinate system it is sufficient to require that the first fundamental form

$$\text{I}=(g_{\mu\nu}dx^\mu dx^\nu)_{r=r_1}, \quad (3)$$

and the second fundamental form

$$\text{II}=(-n_{\mu;\nu}dx^\mu dx^\nu)_{r=r_1}, \quad (4)$$

[1] R. P. Kerr, Phys. Rev. Letters **11**, 237 (1963).
[2] J. L. Synge, *Relativity, The General Theory* (North-Holland Publishing Company, Amsterdam, 1960), pp. 309–317.
[3] W. C. Hernandez, Jr., Phys. Rev. **153**, 1359 (1967).
[4] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Electromagnetisme* (Masson et Cie., Paris, 1955).
[5] P. C. Vaidya, Nature **171**, 260 (1953).

be continuous across the boundary.[6,7] Here n_μ is the unit vector normal to the boundary and the notation $r=r_1$ means that the r coordinate differential is to be eliminated using this equation of the surface. If we choose all the metric components so that they are continuous across the boundary, then the requirement on the first fundamental form is of course satisfied. It is easy to verify that the condition on the second fundamental form will also be satisfied if we then choose interior metric components such that $g_{\theta\theta}$, $g_{\phi\phi}$, $g_{\phi u}$, and g_{uu} have continuous first partial derivatives with respect to r at the boundary and *all* the metric components have continuous first partial derivatives with respect to θ at the boundary. These requirements are actually more stringent than is necessary.

The guessing of an interior solution is now easy. To the $S_{\mu\nu}$ part of the metric we can simply match the Schwarzschild interior solution which is Vaidya coordinates is given by

$$ds^2=r^2(d\theta^2+\sin^2\theta d\phi^2)+\left[3\left(1-\frac{r_1^2}{R^2}\right)^{1/2}\left(1-\frac{r^2}{R^2}\right)^{-1/2}-1\right]drdu-\left[\frac{3}{2}\left(1-\frac{r_1^2}{R^2}\right)^{1/2}-\frac{1}{2}\left(1-\frac{r^2}{R^2}\right)^{1/2}\right]^2du^2, \quad (5)$$

where $r_1^2/R^2=2m/r_1$ defines R^2. In order for the solution to be real and analytic and have finite pressure everywhere we must have

$$2m<(8/9)r_1. \quad (6)$$

This metric alone describes a constant-density incompressible fluid. To the $A_{\mu\nu}$ part of the metric we match any symmetric tensor which is analytic, satisfies the boundary conditions, and gives the interior metric $(g_{\mu\nu})_{\rm int}$ the proper signature $(-+++)$.

For example, we might choose the form

$$(A_{\theta\theta})_{\rm int}=(a/r_1)B(r)\cos^2\theta. \quad (7)$$

The choice for an $(A_{\phi\phi})_{\rm int}$ might appear to be a little more difficult since the exterior $A_{\phi\phi}$ has a true singularity at $r=0$, $\theta=\pi/2$. But a generalized form like

$$(A_{\phi\phi})_{\rm int}=\left(\frac{a}{r_1}\right)B(r)\left(\frac{1+2mb(r)\sin^2\theta}{c(r)+a^2\cos^2\theta}\right)\sin^2\theta \quad (8)$$

is all we need. We must simply pick the functions $B(r)$, $b(r)$, and $c(r)$ such that the boundary conditions at $r=r_1$ hold and such that the metric is analytic. Since the remainder of the discussion does not depend on the particular choice of $(A_{\mu\nu})_{\rm int}$ we shall not be any more specific in its determination.

Our interior metric "guess" can be written as

$$(g_{\mu\nu})_{\rm int}=(S_{\mu\nu})_{\rm int}+(a/r_1)(A_{\mu\nu})_{\rm int} \quad (9)$$

and is analytic everywhere. It is also analytic in the parameter a/r_1 at $a/r_1=0$. So for very small values of a/r_1 the interior and exterior geometries differ only slightly from the Schwarzschild interior and exterior geometrics. Thus we can write for sufficiently small values of $|a/r_1|$ the expressions for the energy density and stresses as

$$\begin{aligned}\epsilon&=\epsilon_s+(a/r_1)h^0,\\ T_i^i&=p_s+(a/r_1)h^i,\\ T_j^i&=(a/r_1)h_j^i,\end{aligned} \quad (10)$$

where ϵ_s and p_s are the Schwarzschild values of the energy density (a constant) and pressure, respectively. The quantities h^0, h^i, and h_j^i are analytic functions and bounded for sufficiently small values of $|a/r_1|$. Obviously for values of $|a/r_1|$ sufficiently small, ϵ and T_i^i will each be positive and satisfy all the criteria stipulated earlier. There may exist a finite upper limit M on the range of values allowed to $|a/r_1|$ such that the criteria are satisfied. Indeed we suspect that such a limit does exist, but we cannot say what M is until the functions h^0, h^i, and h_j^i are calculated explicitly. In any case, if M does exist then the interior solution corresponding to $|a/r_1|=M$ shall be very different from the Schwarzschild interior solution.

III. SOURCE PROPERTIES

The problem of matching an interior solution to the Kerr solution was investigated by Boyer[8] who, among other things, obtained a rather complicated equation for the boundary which must be satisfied if the interior is to be a uniformly rotating perfect fluid. Thus it turns out that our simple $r=r_1$ boundary could not possibly yield a uniformly rotating perfect-fluid solution no matter how wisely we chose $(g_{\mu\nu})_{\rm int}$. If we chose some particular $(A_{\mu\nu})_{\rm int}$ and calculated explicitly the energy density and stresses the source described would be a somewhat flattened rotating body with the topology of a sphere ($r=r_1$ describes the surface). Thus, this particular source proves that a source with the topology of a ring[9] is, at least, not necessary and that "spherical"-type sources[10] are possible. It, of course, also serves to verify the conjecture that the Kerr metric represents a field which could be produced by an axially symmetric, rotating body. Upon further investigation the constructed source would probably be rotating differentially, be made of a strange nonperfect-fluid material and even have some solid-type properties. For very small a/r_1, however, the material will only differ slightly

[6] E. Cartan, *Lecons sur la Geometrie des Espaces de Riemann* (Gauthier-Villars, Paris, 1951), Sec. 207.
[7] See Ref. 3.
[8] R. H. Boyer, Proc. Cambridge Phil. Soc. **61**, 527 (1965).
[9] E. T. Newman and A. I. Janis, J. Math. Phys. **6**, 915 (1965).
[10] R. H. Boyer and T. G. Price, Proc. Cambridge Phil. Soc. **61**, 531 (1965).

from the incompressible fluid of the Schwarzschild interior.

By consideration of the weak-field (Newtonian) limit of the Kerr metric we can say more about other possible sources. In his Eq. (5) Kerr[1] writes his metric in an asymptotically flat coordinate system as

$$ds^2=dx^2+dy^2+dz^2-dt^2+\frac{2mr^3}{r^4+a^2z^2}(k)^2, \tag{11}$$

$$(r^2+a^2)rk=r^2(xdx+ydy)+ar(xdy-ydx)+(r^2+a^2)(zdz+rdt),$$

with the function r defined by

$$r^4-(R^2-a^2)r^2-a^2z^2=0, \quad R^2\equiv x^2+y^2+z^2. \tag{12}$$

Next let the boundary of the source be given approximately by $R\simeq R_1$ and let $m/R_1=\epsilon^2$ (where $\epsilon\ll 1$). Calculating the spatial geodesics and retaining terms only up to order ϵ^2 (Newtonian limit), we finally get

$$\frac{d^2x^i}{dt^2}=-\frac{\partial}{\partial x^i}\left(-\frac{mr^3}{r^4+a^2z^2}\right). \tag{13}$$

Since the coordinates x^i are rectangular Cartesian coordinates up to order ϵ^2 in this approximation, we conclude that the exterior Newtonian gravitational potential up to order ϵ^2 is given by

$$\phi=\frac{-mr^3}{r^4+a^2z^2}. \tag{14}$$

With the use of Eq. (12) this can be expanded in multipoles as

$$\phi=-\frac{m}{R}+\frac{ma^2}{R^3}P_2(\cos\theta)-\frac{ma^4}{R^5}P_4(\cos\theta)+\cdots. \tag{15}$$

Letting $a=0$, the potential becomes the spherically symmetric one, $\phi=-m/R$. Now as the absolute value of a increases, the potential slowly distorts from spherical symmetry. As $|a|$ approaches values of the order $|a|\sim R_1$, the distortion becomes very large and thus quadrupole terms in the potential are of the order ϵ^2, the same magnitudes as the monopole term. The point of this discussion is the following. Suppose that there exists a *perfect-fluid* solution which can be matched to the exterior Kerr metric. Then a seeminly desirable characteristic of this solution would be that it have a reasonable Newtonian limit where (1) the rotation throughout the body is approximately uniform and has an average value ω and (2) the parameter a is allowed to vary continuously over the range of values $0\leq|a|\lesssim R_1$.

The value $a=0$ would correspond to zero distortion from spherical symmetry and hence a zero rotation rate. Values of $|a|$ on the order of R_1 would correspond to large distortions from spherical symmetry caused by a correspondingly large rotation rate. But now consider the specific value of the parameter $a=R_1\epsilon$. By expanding the metric of Eq. (11) in powers of ϵ and comparing it to the third-order Einstein-Infeld-Hoffmann approximation for a spinning particle, Kerr found that m is the Schwarzschild mass and ma the angular momentum about the z axis. Thus we have

$$ma\sim mR_1^2\omega. \tag{16}$$

Using the relationship

$$(a/R_1)^2=(m/R_1) \tag{17}$$

to eliminate a in Eq. (16) we get

$$m/R_1^2\sim R_1\omega^2. \tag{18}$$

This says that near the surface of the source there are regions where the centrifugal force is of the same order of magnitude as the gravitational force. This would imply a large rotational flattening and hence a mass quadrupole of the same order of magnitude as the mass monopole (ϵ^2). But substituting the value $a=R_1\epsilon$ into Eq. (15) we find that up to order ϵ^2 the mass quadrupole is zero. Thus the value $a=R_1\epsilon$ is not allowed. We conclude that fluid sources *having the Newtonian limit described* do *not* exist. Suppose the fluid rotates in a strongly nonuniform manner. The author contends that it appears likely that one could arrive at the same result through a more general line of reasoning. Letting $(E)_{\rm rot}$ be the rotational kinetic energy of the body, L the angular momentum, and I the moment of inertia, one can easily show that

$$(E)_{\rm rot}\geq L^2/2I. \tag{19}$$

(An equality sign is used when the rotation is uniform.) If we again estimate $I\sim mR_1^2$ and again choose $a=R_1\epsilon$, we find

$$(E)_{\rm rot}\gtrsim m^2/R_1. \tag{20}$$

This can be interpreted as saying that the kinetic energy of rotation is approximately the same magnitude or greater than the gravitational binding energy. It seems reasonable, though not proved, that one could argue that for a perfect fluid this would imply a large rotational flattening of the body and again a large quadrupole moment.

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