# Quadrupole Effect of Polarized Nuclei on Elastic Scattering*

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It is shown that for $\eta \equiv Z_1Z_2e^2/\hbar v \gtrsim 4$, the deviation from Coulomb scattering is determined more or less uniquely by $\eta$ in the quadrupole elastic scattering of charged particles by polarized nuclei. The dependence on the mode of damping of the wave function for $\eta \leq 4$ is also discussed.

## INTRODUCTION

THE quadrupole elastic scattering of $\alpha$ particles by polarized nuclei at the incident energies below the Coulomb barrier has been recently investigated[1–3] in terms of the deviation from the Coulomb scattering cross section. The deviation would arise both from the nuclear shape as well as from the Coulomb excitations of target nuclei. Although it is difficult to estimate the ratio of these two effects in the general case, it has been argued[1] that certain highly deformed nuclei may exhibit a much larger shape effect than the Coulomb-excitation effect on elastic scattering even for a reasonable degree of polarization of target nuclei.

By introducing quadrupole interaction as a perturbation, the deviation from the Coulomb scattering cross section in the case of head-on collisions can be expressed by the following equation,[4] if the target nucleus is oriented along the nuclear axis and the latter is parallel to the direction of the relative motion before the collision:

$$\delta = -\frac{2m}{\hbar^2}\frac{Q_0 Z_1 e^2 k}{\eta} S(\eta)\,, \tag{1}$$

where

$$S(\eta) = \sum_{l=0} \cos(2\sigma_l - 4\sigma_0)\frac{1}{(2l-1)(2l+3)} \times \left\{2l - 2\eta^2 \sum_{m=1} \frac{1}{m^2+\eta^2}\right\}, \tag{2}$$

with $\eta = Z_1Z_2e^2/\hbar v$ and $\sigma_l = \arg\Gamma(l+1+i\eta)$. $Z_1$ and $Z_2$ are the charges of the projectile and the target, respectively; $m$ is the reduced mass, and $k$ is the magnitude of the wave vector of the projectile. $Q_0$ stands for the intrinsic quadrupole moment of the target nucleus.

It is necessary, therefore, to evaluate the sum of the series $S(\eta)$ defined in Eq. (2). However, as the orbital angular momentum $l$ becomes larger, the intervals of the values of $l$ over which $S(\eta)$ undergoes oscillation also become larger, and it is difficult to evaluate the sum in an unambiguous manner. This difficulty is somewhat relieved if we group the terms of the same sign and then evaluate the sum of the resulting alternating series.[1] In practice, however, this method is not so useful unless $\eta$ is very large because otherwise the convergence is too slow and the errors involved are too large even when a very large number of terms are computed. It also tends to give overestimation by disregarding the screening effect. These difficulties may be removed by employing a screened Coulomb potential such as

$$V(r) = \frac{Z_1Z_2e^2}{r}e^{-\lambda r}\,, \tag{3}$$

where $\lambda^{-1}$ is of the order of the radius of the target atom. However, an exact calculation using this potential would be very laborious.

## CALCULATION

We found by actual computation that $S(\eta)$ is insensitive to the mode of damping of the wave function if $\eta$ is large, as long as the screened potential diminishes the wave function smoothly. If we use the first Born approximation, although this would not be very good for large values of $Z_1$ and $Z_2$, the use of a screened Coulomb potential of the type given in Eq. (3) instead of the pure Coulomb potential is tantamount to replacing the integral[5]

$$M_{ll'}^{-3} = (k_1k_2)^{-1}\int_0^\infty F_{l'}(k_2r)r^{-3}F_l(k_1r)dr \tag{4}$$

by

$$M_{ll'}^{-3,\lambda} = (k_1k_2)^{-1}\int_0^\infty F_{l'}(k_2r)r^{-3}e^{-\lambda r}F_l(k_1r)dr\,, \tag{5}$$

where $k_1=|\mathbf{k}_1|$ and $k_2=|\mathbf{k}_2|$, $\mathbf{k}_1$ and $\mathbf{k}_2$ being the wave vectors of the Coulomb-distorted plane waves of the impinging particle before and after collision, respectively ($k_1=k_2=k$ for an elastic collision); $F_l$ is the regular

TABLE I. Combinations of parameters.

| Case | $A1$ | $A2$ | $A3$ | $B1$ | $B2$ | $B3$ |
|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 1 | 2 | 3 |
| $\lambda/k$ | $10^{-3}$ | $10^{-3}$ | $10^{-3}$ | $10^{-4}$ | $10^{-4}$ | $10^{-4}$ |

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[1] Y. N. Kim and H. C. Thomas, Phys. Rev. 148, 1083 (1966).
[2] C. F. Clement, United Kingdom Atomic Energy Research Establishment Report No. T.P. 196, 1965 (unpublished).
[3] Y. N. Kim, in Proceedings of the International Conference on Nuclear Physics, Gatlinburg, 1966, p. 42 (unpublished).
[4] See Eq. (22) of Ref. 1.
[5] See Eq. (15) of Ref. 1.

TABLE II. Values of $S(\eta)_{\rm scr}$ for the $A2$ mode of damping when $\eta$ is large. All values should be multiplied by $10^{-2}$.

| $\eta$ | 4.5 | 5 | 5.5 | 6 | 6.5 | 7 | 7.5 | 8 | 8.5 | 9 | 9.5 | 10 | 10.5 | 11 | 11.5 | 12 | 12.5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S(\eta)_{\rm scr}$ | −0.9483 | −0.0470 | 0.6404 | −0.0940 | −0.4195 | 0.2743 | 0.1375 | −0.3052 | 0.1528 | 0.0890 | −0.2088 | 0.1623 | −0.0286 | −0.0926 | 0.1439 | −0.1230 | 0.0592 |

solution to the radial Coulomb wave equation for the orbital angular momentum $l$.

The integral in Eq. (5) can be explicitly evaluated in terms of the Appell function.[6] Actual computation, however, is difficult because two variables are involved in the series expansion of this function. The following crude consideration, however, may show some convergence characteristics of $M_{ll'}{}^{-3,\lambda}$. As was pointed out after Eq. (23) of Ref. 1, we need consider only the case of $l=l'$. We note that for large values of $l$, the magnitude of the Coulomb wave function $F_l(k,r)$ varies slowly as a function of $r$, and, therefore, as a very crude estimation one may write

$$C=\frac{M_{ll'}{}^{-3,\lambda}}{M_{ll'}{}^{-3}}\approx\int_0^\infty r^{-3}e^{-\lambda r}dr\bigg/\int_0^\infty r^{-3}dr$$
$$=e^{-\lambda r}(1-\lambda r)-\lambda^2r^2\,\mathrm{li}(e^{-\lambda r}),\quad(6)$$

where $\mathrm{li}(x)$ is the logarithmic integral. For $\lambda r\gg1$, we obtain

$$\mathrm{li}(e^{-\lambda r})\approx-(\lambda r)^{-1}e^{-\lambda r}\quad(7)$$

and, therefore,

$$C\approx e^{-\lambda r},\quad(8)$$

as could be seen easily from Eq. (6) without calculation. For $\lambda r\ll1$, we obtain

$$C\approx e^{-\lambda r}(1-\lambda r)\approx e^{-2\lambda r}.\quad(9)$$

Therefore, the effect of the screened Coulomb field in our problem is roughly equivalent to modifying the contribution to the deviation from Coulomb scattering by a factor $e^{-\lambda r}$ outside the atomic electron cloud and by $e^{-2\lambda r}$ inside the atom.

However, in view of the very crude nature of the above estimate, we assumed a damping factor of the form $\exp[-(\lambda r)^n]=\exp[-(\lambda/k)^n l^n]$ with two parameters $\lambda$ and $n$, where the classical relation $l=kr$ was used. Then each term of Eq. (14) of Ref. 1 was multiplied by this factor with the corresponding value of $l$ to obtain the modified value of $f_q(\theta)$, that part of the scattering amplitude which is due to the quadrupole interaction. This in turn modifies the value of $S(\eta)$ in Eq. (1). This modified value of $S(\eta)$ will be denoted by $S(\eta)_{\rm scr}$. The latter should replace the former in calculating the deviation from Rutherford scattering if the Coulomb field is screened. We have computed $S(\eta)_{\rm scr}$ for the combinations of parameters listed in Table I. The values of $k$ are determined by the incident energy and $\lambda^{-1}$ is of the order of the radius of the target atom.

[6] L. C. Biedenharn, J. L. McHale, and R. M. Thaler, Phys. Rev. **100**, 376 (1955).

$S(\eta)_{\rm scr}$ is insensitive to the choice of parameters if $\eta\geqslant4$. In this region, therefore, $S(\eta)_{\rm scr}$ is more or less uniquely determined by $\eta$ only, and the deviation from the Coulomb scattering can be calculated readily from Eq. (1). Table II shows the values of $S(\eta)_{\rm scr}$ for the $A2$ mode of damping for $12.5\geqslant\eta\geqslant4.5$. We have also computed $S(\eta)_{\rm scr}$ for the cases of $A1$ and $B2$. For $\eta\geqslant7$, the values obtained for these cases are equal to those for $A2$ listed in Table II to within $3\times10^{-6}$. For $4.5\leqslant\eta<7$, the differences are at most $\sim4\times10^{-4}$. These differences are too small to be shown in Fig. 1. We have not carried out computations for other modes of damping for $\eta\geqslant4.5$, because sample calculations show that again the differences from the values for $A2$ are very small, and detailed computation does not appear to deserve the labor involved. For $\eta\leqslant4$, $S(\eta)_{\rm scr}$ depends sensitively on the choice of $n$ and $\lambda/k$. Table III shows the results for various modes of damping. This is also shown in Fig. 1. By making use of these tables the deviation from the Coulomb scattering may be evaluated readily for listed values of $\eta$, and a few examples are given below in Table IV. In the case of the collision of $\alpha$ particles by heavy nuclei, $\eta$ becomes larger than those listed in the tables if the incident energy is well below the Coulomb barrier.

In this calculation, very large number of terms had to be computed in some cases to get the converging values of the series. In the case of $A3$, we need to take only 3000 terms, whereas for the case of $B1$, we had to compute as many as $10^5$ terms to secure sufficient convergence.

TABLE III. Values of $S(\eta)_{\rm scr}$ for various modes of damping when $\eta$ is small. All values should be multiplied by $10^{-2}$.

| $\eta$ | | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|---|
| $S(\eta)_{\rm scr}$ | $A1$ | −13.71 | −8.04 | −3.85 | −0.59 | 1.49 | 1.46 | −0.11 |
| | $B1$ | −12.70 | −8.33 | −3.65 | −0.58 | 1.49 | 1.46 | −0.11 |
| | $A2$ | −4.86 | −4.37 | −2.11 | 2.03 | 1.67 | 1.39 | −0.16 |
| | $B2$ | −9.75 | −7.01 | −5.57 | −0.24 | 1.81 | 1.49 | −0.15 |
| | $A3$ | −0.81 | −0.88 | 0.62 | 1.23 | 1.08 | 0.79 | 0.03 |
| | $B3$ | −7.21 | −5.25 | −8.21 | −1.00 | 2.50 | 2.16 | 0.17 |

TABLE IV. Deviation from the Coulomb scattering.

| Projectile | Target | Incident energy (MeV) | $\eta$ | Deviation |
|---|---|---|---|---|
| Proton | $Ta^{181}$ | 6.6 | 4.5 | 6% |
| Proton | $U^{238}$ | 10.2 | 4.5 | 11% |
| Muon | $U^{238}$ | 78.6 | 1 | 35% for $A1$<br>32% for $B1$ |

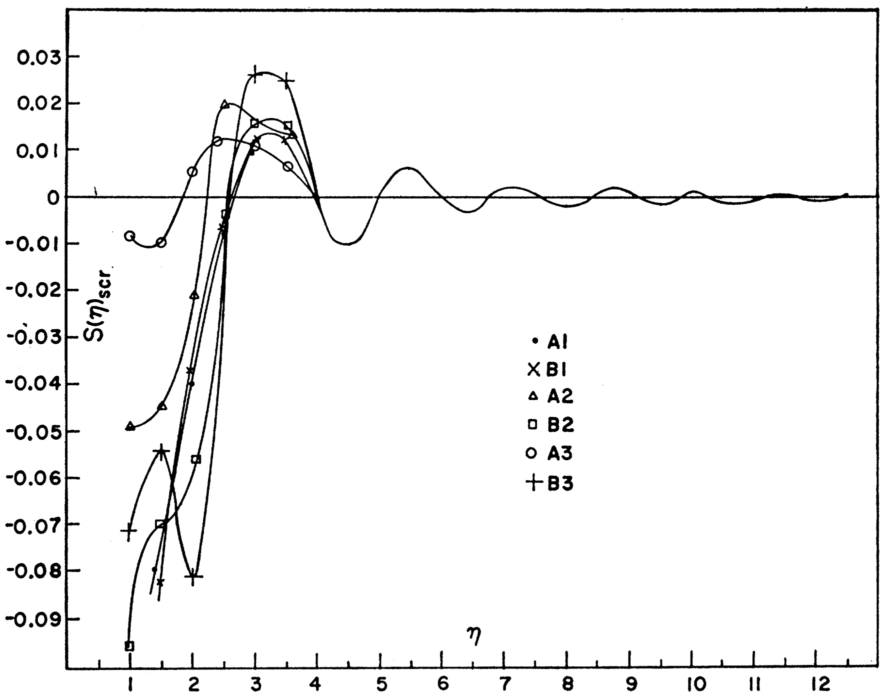


FIG. 1. Values of $S(\eta)$ for a screened Coulomb potential as a function of $\eta$. For the notations $A1$, etc., see the text.

The calculations were performed by means of an IBM electronic computer 7040 of the Texas Tech Computer Center. To assure the correctness of the machine program, sample values of $S(\eta)_{\rm scr}$ were checked by direct calculation by means of a desk calculator. Since the sum of a very large number of terms are computed, special care was taken about the possibility of large cumulative errors. Ten digits were kept in the computations, and the estimations based on the rapid decrease of the value of the individual term and the gradual increase of the number of terms with the same sign in the oscillatory series show that our results are reliable.

## DISCUSSION

Although $S(\eta)_{\rm scr}$ becomes very large for small values of $\eta$, the incident energy becomes larger than the Coulomb barrier in this case, and the specifically nuclear interaction becomes dominant. If we use the electrons as projectiles to avoid this difficulty, the deviation from Coulomb scattering becomes very small because of the small mass of electrons. On the other hand, the muons may exhibit, for $\eta<1$, a very large deviation from Coulomb scattering depending sensitively on the mode of damping. However, it must be stressed that the modes of damping introduced in the calculations are rather arbitrary, and a more accurate treatment of the problem is desirable.

The angular distribution of the deviation due to the shape effect of polarized target nuclei over a wide range of scattering angles should be compared with those arising from virtual excitations and from the resonance in $E2$ excitations. Although both effects are likely to be small compared to the shape effect,[1] a more detailed study would be desirable. We also note that the ratio of cross sections for elastic and $E2$ Coulomb-excitation scattering for a fixed projectile decreases rapidly as the incident energy increases. Therefore, a careful examination of the resolution of elastic and inelastic scattering may become important. For very small $\eta$, the problem should be treated relativistically. We are presently studying the problems along these lines.

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