Three-Body Correlations in Nuclear Matter. II*

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The contribution of three-body correlations to the energy of nuclear matter is formally calculated. This is necessary to get numerical results from the three-body wave function of Day and Kirson. The potential energy for "particle" states is redefined. This definition, as well as other results, differ from previous ones.

1. INTRODUCTION

FOR the correct treatment of nuclear matter, it is essential that an expansion be made in the number of interacting particles, not in the number of successive interactions. Bethe[1] has given a treatment of the simultaneous interaction of three particles; this was subsequently improved by Day[2] and Kirson.[3]

The significant quantity in the theory is the three-body wave-function defect, i.e., the amount by which the actual wave function of three interacting bodies differs from the wave function without three-body correlations. Kirson[3] has shown that this defect function depends essentially only on the mutual distances of the three particles r_{12}, r_{23}, and r_{31}, not on the orientation of the triangle $\mathbf{r}_{12}$, $\mathbf{r}_{23}$, and $\mathbf{r}_{31}$ in space. This is due to the fact that $k_F c<1$, where k_F is the Fermi momentum and c the radius of the repulsive core. Day[2] has given explicit expressions for the defect function $Z^{(1)}$.

Less attention has been paid to the problem of evaluating the *energy* of the nucleus due to three-body correlations. The contribution of a given set of three nucleons is described by a diagram like Fig. 1 in which K_0, K_1, and K_2 denote simply the momenta of all three particles at the levels indicated. For reasons explained in B3, the interaction A and the preceding propagator are described by η_{12}, the two-body wave function defect for two particles *in* the Fermi sea, while B is described by g_{23}, the interaction function for two particles off the energy shell. The wave function arriving at B from below is

$$\Phi-Z^{(1)},$$

with Φ the unperturbed wave function of the three particles, and $Z^{(1)}$ the three-body wave-function defect discussed above. Our task is now to calculate the contribution of Fig. 1 to the energy.

2. THREE-BODY ENERGY

Figure 1 shows directly that the energy of a group of three nucleons, interacting by three-body correlations, is

$$W_3(K_0)=\int dK_1 dK_2 \langle K_0|\hat{\eta}_{12}|K_1\rangle \times\langle K_1|\hat{g}_{23}|K_2\rangle\langle K_2|\hat{Z}^{(1)}|K_0\rangle. \quad (2.1)$$

Here each K_i stands for the momenta of all three particles; specifically K_0 refers to the initial and the others to the two intermediate states. The final state is of course identical with the initial. The η, g, and Z are operators and have therefore been denoted by a caret. The first operator represents

$$\hat{\eta}_{12}=\hat{g}_{12}Q/e, \quad (2.2)$$

i.e., the last interaction and the preceding propagator. The last interaction is separated from the rest in this manner because it ends with particles *in* the Fermi sea, and is therefore different from the preceding interactions g_{23} in which the particles are outside the sea, and hence off the energy shell. The most important factor is of course the three-body wave-function operator $Z^{(1)}$; we assume this has been calculated using the methods of Day[2] and Kirson.[3] Following Kirson,[3] we shall assume that Z has the form

$$\hat{Z}^{(1)}|K_0\rangle=Z^{(1)}(r_{12},r_{23},r_{31})\Phi(K_0), \quad (2.3)$$

where Φ is the unperturbed wave function (product of three plane waves) and Z depends only on the distances between the three particles, not on the directions of the vectors $\mathbf{r}_{12}$, etc.

This leaves the problem of the matrix element of g_{23}. Particle 1 is just a spectator in this element, and we may consider the operator g_{23} as acting on the plane wave representing particles 2 and 3 in state K_1. Denoting the momenta of these particles by $\mathbf{b}$ and $\mathbf{p}_3$, respectively, we may write

$$\hat{g}_{23}\exp i(\mathbf{b}\cdot\mathbf{r}_2+\mathbf{p}_3\cdot\mathbf{r}_3)=e^{i\mathbf{P}\cdot(\mathbf{r}_2+\mathbf{r}_3)}v(\mathbf{r}_{23})\psi_{Pk}(\mathbf{r}_{23}), \quad (2.4)$$

where

$$\mathbf{P}=\tfrac{1}{2}(\mathbf{b}+\mathbf{p}_3), \quad \mathbf{k}=\tfrac{1}{2}(\mathbf{b}-\mathbf{p}_3) \quad (2.5)$$

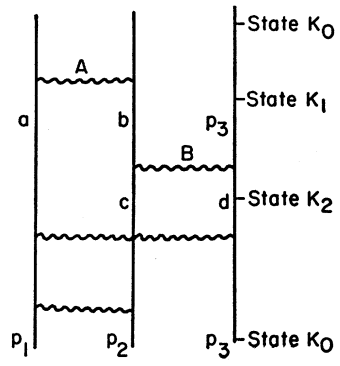

FIG. 1. Ladder diagram for three-body interaction. The initial state of the system is denoted by K_0, the state before the last interaction (A) by K_1, that before the preceding interaction (B) by K_2. There may be any number of interactions between B and the initial state. Momenta of particles at various stages are attached to the respective propagator lines.

* Supported in part by the National Science Foundation.
[1] H. A. Bethe, Phys. Rev. **138**, B804 (1965); quoted hereafter as B3.
[2] B. Day, Phys. Rev. **151**, 826 (1966).
[3] M. W. Kirson, Nucl. Phys. (to be published).

are the average and the relative momentum of the two nucleons, and ψ_{Pk} is their wave function including interaction. It is convenient[4] to write

$$v(\mathbf{r}_{23})\psi_{Pk}(\mathbf{r}_{23})=g_{Pk}(\mathbf{r}_{23})\phi(\mathbf{r}_{23}),\qquad \phi=\exp i\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_3). \tag{2.6}$$

Now Kirson[3] has shown that the distribution of the momenta P, k is rather sharply peaked; therefore it is tempting to ignore the dependence of g on P and k. Then, including the spectator particle, we may put

$$\hat{g}_{23}|K_1\rangle=g(r_{23})\Phi(K_1), \tag{2.7}$$

where $\Phi(K_1')$ is the product of the three plane waves corresponding to state K_1, and $g(r_{23})$ is assumed to be independent of the momenta in K_1. If (2.7) is assumed, the integrations over K_1 and K_2 in (2.1) can be done immediately and give

$$W_3(K_0)=\langle K_0|\hat{\eta}_{12}g_{23}(r_{23})Z^{(1)}(r_{12},r_{23},r_{31})|K_0\rangle. \tag{2.8}$$

However, (2.7) is not correct. To see this, we write

$$\hat{g}_{23}|K_1\rangle=e\hat{\zeta}_{23}|K_1\rangle, \tag{2.9}$$

where ζ_{23} is the two-body defect function,[5] and the operator e may be written in the reference spectrum approximation[5]

$$e=\gamma^2-\nabla_{23}^2. \tag{2.10}$$

As is shown in BBP, γ increases with increasing excitation of the state K_1. Therefore, just *because* ζ is very insensitive to K_1 (see B3, p. 809), g is very sensitive: It increases rapidly with increasing energy. This is particularly true for the contribution from inside the core; we have

$$\hat{\zeta}_{23}|K_1'\rangle=|K_1\rangle, \qquad r_{23}<c \tag{2.11}$$

$$\hat{g}_{23}|K_1\rangle=(k_{23}^2+\gamma^2)|K_1\rangle, \qquad r_{23}<c \tag{2.12}$$

where k_{23} is the relative momentum of particles 2 and 3 in state K_1. The contribution from the core surface[6] is less sensitive to the energy of state K_1, and that from the long-range, attractive forces is insensitive, viz.,

$$\hat{g}_{23}|K_1\rangle\approx v(r_{23})|K_1\rangle, \qquad r_{23}\gg c. \tag{2.13}$$

Because g is sensitive to K_1, Kirson has evaluated it for an *average* momentum k_{23}. Since (2.12) is the most sensitive contribution, Kirson used the mean square of p_2', the momentum of nucleon 2 between the last two interactions (both γ and k_{23} depend chiefly on p_2'). Thus Kirson puts

$$\hat{g}_{23}|K_1\rangle=g_{23}(p_{2\,\mathrm{Av}}',r_{23})\Phi(K_1). \tag{2.14}$$

Then the integration still leads to (2.8). This is a reasonable procedure, but clearly an approximation. We shall develop a more systematic approach.

[4] This is actually convenient only for an ordinary, central force. For other types of forces, see Sec. 3.

[5] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963); quoted hereafter as BBP.

[6] BBP, Eq. (5.28).

First, we specify the momenta of the three particles involved in the states K_0, K_1, K_2. We denote them as follows:

$$\begin{aligned}&\text{State } K_0: \quad \mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\\ &\text{State } K_1: \quad \mathbf{a}\equiv\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-2\mathbf{k},\quad \mathbf{b}\equiv\mathbf{p}_3+2\mathbf{k},\ \mathbf{p}_3;\\ &\text{State } K_2: \quad \mathbf{a},\mathbf{c}\equiv\mathbf{p}_3+\mathbf{k}+\mathbf{k}',\quad \mathbf{d}\equiv\mathbf{p}_3+\mathbf{k}-\mathbf{k}'.\end{aligned} \tag{2.15}$$

Then the η matrix element is

$$\begin{aligned}\langle\mathbf{p}_1,\mathbf{p}_2|\eta_{12}|\mathbf{a},\mathbf{b}\rangle&=\int d^3r_{12}'\,\exp[-i(\mathbf{p}_1\cdot\mathbf{r}_1'+\mathbf{p}_2\cdot\mathbf{r}_2')]\\ &\quad\times\eta^*(\mathbf{p}_1,\mathbf{p}_2,r_{12}')\exp i(\mathbf{a}\cdot\mathbf{r}_1'+\mathbf{b}\cdot\mathbf{r}_2')\\ &=\int d^3r_{12}'\,\eta^*(r_{12}')\exp i(\mathbf{p}_3+2\mathbf{k}-\mathbf{p}_2)\cdot\mathbf{r}_{21}',\end{aligned} \tag{2.16}$$

and the g matrix element

$$\begin{aligned}\langle\mathbf{P},\mathbf{k}|g_{23}|\mathbf{P},\mathbf{k}'\rangle&=\int d^3r_{23}''\\ &\quad\times\exp[-i(\mathbf{b}\cdot\mathbf{r}_2''+\mathbf{p}_3\cdot\mathbf{r}_3'')]g_{Pk}^*(r_{23}'')\\ &\quad\times\exp i[\mathbf{k}'\cdot(\mathbf{r}_2''-\mathbf{r}_3'')+(\mathbf{p}_3+\mathbf{k})\cdot(\mathbf{r}_2''+\mathbf{r}_3'')],\end{aligned} \tag{2.17}$$

with

$$\mathbf{P}=\mathbf{p}_3+\mathbf{k}. \tag{2.17'}$$

g_{Pk} depends on the parameters P and k. Finally, the Z matrix element

$$\begin{aligned}&\langle\mathbf{a},\mathbf{c},\mathbf{d}|Z^{1)}|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\rangle\\ &=\int\exp[-i(\mathbf{a}\cdot\mathbf{r}_1+\mathbf{c}\cdot\mathbf{r}_2+\mathbf{d}\cdot\mathbf{r}_3)]Z^{(1)}(r_{12},r_{23},r_{31})\\ &\quad\times\exp i(\mathbf{p}_1\cdot\mathbf{r}_1+\mathbf{p}_2\cdot\mathbf{r}_2+\mathbf{p}_3\cdot\mathbf{r}_3)d^3r_2d^3r_3\\ &=\int Z^{(1)}\exp i[(\mathbf{p}_3-\mathbf{p}_2+2\mathbf{k})\cdot\mathbf{r}_1+(\mathbf{p}_2-\mathbf{p}_3-\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_2\\ &\qquad+(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_3]d^3r_2d^3r_3.\end{aligned} \tag{2.18}$$

The integrals in (2.1) are $\pi^{-3}d^3k(2\pi)^{-3}d^3k'$. The integral in (2.18) is over the spaces of r_2 and r_3 only, with $\mathbf{r}_1$ kept fixed; integration over r_1 would merely give a factor Ω (normalization volume) which cancels against normalization factors.

Now $\mathbf{k}'$ occurs only in (2.17) in the form $e^{i\mathbf{k}'\cdot\mathbf{r}_{23}''}$, and in (2.18) as $e^{-i\mathbf{k}\cdot\mathbf{r}_{23}}$. Integration over $(2\pi)^{-3}d^3k'$ therefore gives $\delta(\mathbf{r}_{23}''-\mathbf{r}_{23})$, with corresponding simplification of the integrals:

$$W_3(K_0)=\pi^{-3}\int d^3k\,\langle\mathbf{p}_1,\mathbf{p}_2|\eta_{12}|\mathbf{a},\mathbf{b}\rangle\times\langle K_1|g^*Z^{(1)}|K_0\rangle. \tag{2.19}$$

Here, after simplification of the exponentials,

$$\langle K_1|g^*Z^{(1)}|K_0\rangle=\int d^3r_{12}d^3r_{23}\, g_{Pk}^*(\mathbf{r}_{23})$$
$$\times Z^{(1)}(r_{12},r_{23},r_{31})\exp i(\mathbf{p}_3-\mathbf{p}_2+2\mathbf{k})\cdot\mathbf{r}_{12}. \quad (2.20)$$

It may be noted that the exponential no longer contains $\mathbf{r}_3$, and is the complex conjugate of that in (2.16), as it should be. Of the initial momenta, only $\mathbf{p}_3-\mathbf{p}_2$ enters here, and $\mathbf{p}_3$ in the (relatively unimportant) parameter P, (2.17′).

Equations (2.20), (2.19), (2.16) are the greatest simplification we have been able to achieve without making assumptions about the dependence of $g(r)$ on k. Now (2.12) suggests that g has a component which is roughly proportional to k^2, and (2.13) suggests another component independent of k. The term from the core surface can probably be represented as a sum $a+bk^2$. Thus we believe it is a good approximation to set

$$g_{Pk}(r_{23})=g_1(r_{23})+k^2g_2(r_{23}). \quad (2.21)$$

The dependence of g on P can adequately be described as a dependence on k because, from (2.17′)

$$P^2\approx k^2 \quad (2.21')$$

since $p_3<k_F\ll k$, in general. The functions g_1 and g_2 in (2.21) are supposed to be independent of k; clearly, g_1 dominates at large r, g_2 at small r.

When (2.21) is inserted in (2.20), (2.19), the integral over d^3k can be carried out by closure. This is obvious for g_1. To do it for g_2, we note that according to (2.16),

$$(\mathbf{p}_3-\mathbf{p}_2+2\mathbf{k})^2\langle\cdots|\eta|\cdots\rangle=-\int d^3r_{12}'$$
$$\times\nabla^2\eta^*(r_{12}')\exp i(\mathbf{p}_3-\mathbf{p}_2+2\mathbf{k})\cdot\mathbf{r}_{12}'. \quad (2.22)$$

Assuming again

$$|\mathbf{p}_3-\mathbf{p}_2|\ll 2k, \quad (2.22')$$

the left-hand side is $4k^2\langle\cdots|\eta|\cdots\rangle$ which is the quantity we wish to calculate. Using the right-hand side of (2.22), the integration over k reduces to

$$\pi^{-3}\int d^3k\, \exp 2i\mathbf{k}\cdot(\mathbf{r}_{12}-\mathbf{r}_{12}')=\delta(\mathbf{r}_{12}-\mathbf{r}_{12}'), \quad (2.22'')$$

and we obtain

$$W_3(K_0)=\int d^3r_{12}d^3r_{23}\, Z^{(1)}(r_{12},r_{23},r_{31})$$
$$\times\{\eta^*(r_{12})g_1^*(r_{23})-\tfrac{1}{4}[\nabla^2\eta^*(r_{12})]g_2^*(r_{23})\}. \quad (2.23)$$

Since both η and $Z^{(1)}$ are essentially independent of the momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, this result may simply be multiplied by ρ^2 to give the energy per particle. The effect of the nonvanishing momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ can be treated approximately, using the method of Kirson.[3]

Equation (2.23) can be further simplified. Since g depends only on r_{23}, we can integrate over the position of particle 1, i.e., over $\mathbf{r}_{12}$,keeping r_{23} fixed. This yields

$$F_1(r_{23})=\int d^3r_{12}\, \eta^*(r_{12})Z^{(1)}(r_{12},r_{23},r_{31}), \quad (2.24)$$

$$F_2(r_{23})=-\frac{1}{4}\int d^3r_{12}\,[\nabla^2\eta^*(r_{12})]Z^{(1)}(r_{12},r_{23},r_{31}), \quad (2.25)$$

$$W_3(K_0)=\int d^3r_{23}\,[g_1^*(r_{23})F_1(r_{23})$$
$$+g_2^*(r_{23})F_2(r_{23})]. \quad (2.26)$$

The function F_1 is identical with the F introduced in B3, Eq. (5.1), which was there shown to be small if r_{23} is inside the core and to increase rapidly (by about a factor of 1.5) outside. The other correlation function F_2 is new.

Kirson's Approximation

As was mentioned above, Kirson replaces $g(k,r_{23})$ by $g(k_{\mathrm{Av}},r_{23})$. If g has the form (2.21), this amounts to the replacement

$$F_2(r_{23})\rightarrow k_{\mathrm{Av}}{}^2F_1(r_{23}). \quad (2.27)$$

No matter how $k_{\mathrm{Av}}{}^2$ is chosen, this replacement cannot hold for all values of r_{23}. Kirson's procedure, even for the simple form (2.21) of g, can therefore at best be approximate.

However, from (2.12) and (2.21) it follows that g_2 is mostly important for small r_{23}, viz., $r_{23}\leq c$. In this case, F_1 and F_2 are nearly independent of r_{23} because the *explicit* dependence of $Z^{(1)}$ on r_{23} is small, and $r_{13}\approx r_{12}$ so that also the implicit dependence is small. Sample calculations, such as B3, Fig. 11, bear this out. Then the replacement (2.26), and thereby the Kirson approximation, are justified.

The situation is not quite this simple, because of the tensor force. As is well-known, the tensor force between nucleons 2 and 3 gives rise, in second order, to an effective, attractive central force $g_\kappa(r_{23})$. This effective force becomes weaker with increasing k, the effect being roughly proportional to k^2. This gives an additional contribution to $g_2(r_{23})$ which has relatively longer range. Nemeth and Bethe[7] have found that this is about as important as the core contribution. For the tensor contribution, the Kirson substitution (2.27) is not accurate.

3. THE INTERACTION $g(r_{23})$

We shall now investigate the form of $g(r_{23})$ for a more realistic potential, and its relation to the approixmation (2.21). We proceed in stages.

Let us first assume that the potential is central and has Serber form, then

$$g\phi(\mathbf{r})=v\psi(\mathbf{r})=v(r)[\psi(\mathbf{r})+\psi(-\mathbf{r})]. \quad (3.1)$$

[7] J. Nemeth and H. A. Bethe (to be published).

The right-hand side contains only even angular-momentum components. However,

$$\phi(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}} \tag{3.2}$$

contains all angular momenta and so, therefore, does $g(r)$.

Great simplification of (3.1) is possible for the long-range interaction in the sense of Moszkowski and Scott (MS). Here, ψ may be replaced by ϕ and (3.1) and (3.2) give

$$g(\mathbf{r}_{23})=v(r_{23})(1+e^{-2i\mathbf{k}\cdot\mathbf{r}_{23}}). \tag{3.3}$$

This is not of the form (2.21). However, insertion into (2.20) gives, apart from unimportant factors, the simple result

$$g^*(\mathbf{r}_{23})e^{2i\mathbf{k}\cdot\mathbf{r}_{12}}=v(r_{23})(e^{2i\mathbf{k}\cdot\mathbf{r}_{12}}+e^{2i\mathbf{k}\cdot\mathbf{r}_{13}}). \tag{3.4}$$

Closure with (2.16) can still be achieved; integration over $\mathbf{k}$ gives, again apart from unimportant factors,

$$\delta(\mathbf{r}_{12}'-\mathbf{r}_{12})+\delta(\mathbf{r}_{12}'-\mathbf{r}_{13}); \tag{3.4'}$$

(2.23) is replaced by

$$W_3(K_0)=\int d^3r_{12}d^3r_{23}$$

$$\times[\eta^*(r_{12})+\eta^*(r_{13})]v(r_{23})Z^{(1)}(r_{12},r_{13},r_{23}). \tag{3.5}$$

$Z^{(1)}$ is clearly symmetric[2] in r_{12} and r_{13}; therefore the two terms in (3.5) are equal, and the exchange term in (3.1) has simply the effect of doubling the result. (This is due to the neglect of the momenta p_1, p_2, p_3 in the initial state.) Thus (2.26) is replaced by

$$W_3(K_0)=\int d^3r_{23}\, g_1^*(r_{23})F_1(r_{23}), \tag{3.6}$$

with F_1 given by (2.24).

The Serber force, however, is not a sufficiently good representation of the realistic potential between nucleons, even apart from tensor forces. Considering just singlet even states, it has been shown[8,9] that the attraction in the ^1S state is greater than in the ^1D and ^1G states. The triplet even states behave similarly. A reasonable approximation is achieved if we choose $v(r)$ in (3.1) to represent the interaction in the D state (the G state does not matter), and then add a term for the *difference* between the S and D states, viz.,

$$g_s\phi(\mathbf{r})=v_s(\mathbf{r})\psi_0(r), \tag{3.7}$$

where ψ_0 is the $L=0$ component of ψ, and v_s is the extra S-state potential. If desired, this v_s may be chosen to include the effect of the repulsive core, or the entire "short-range potential" of Moszkowski and Scott, because these affect primarily the S-state wave function. It is obviously not convenient to calculate g_s from (3.7) because g_s will contain all angular-momentum components, due to $\phi(\mathbf{r})$. It is better to go back to (2.1) and evaluate this directly.

With the interaction (3.7), we have (omitting the spectator particle 1)

$$\hat{g}_{23}|K_1\rangle=v_s(r_{23}'')\psi_0(r_{23}'')e^{i(\mathbf{p}_3+\mathbf{k})\cdot(\mathbf{r}_2''+\mathbf{r}_3'')}, \tag{3.8}$$

$$\langle K_1|\hat{g}_{23}|K_2\rangle=\int d^3r_{23}''\, v_s(r_{23}'')\psi_0(r_{23}'')e^{i\mathbf{k}'\cdot(\mathbf{r}_2''-\mathbf{r}_3'')}. \tag{3.9}$$

In this matrix element, the $\exp i(\mathbf{p}_3+\mathbf{k})(\mathbf{r}_2''+\mathbf{r}_3'')$ has been canceled between initial and final state. We combine (3.8) and (2.18) and integrate over k'; this gives $\mathbf{r}_{23}''=\mathbf{r}_{23}$ and

$$\langle K_1|\hat{g}Z^{(1)}|K_0\rangle=\int d^3r_2d^3r_3\, v_s(r_{23})\psi_0(r_{23})Z^{(1)}$$

$$\times\exp i[(\mathbf{p}_3-\mathbf{p}_2+2\mathbf{k})\cdot(\mathbf{r}_1-\mathbf{r}_2)+\mathbf{k}\cdot\mathbf{r}_{23}]. \tag{3.10}$$

We now write[10]

$$v_s(r)\psi_0(r)\equiv g_s(k,r)\phi_0(r), \tag{3.11}$$

where ϕ_0 is the unperturbed S-state wave function,

$$\phi_0(r)=j_0(kr)=(4\pi)^{-1}\int d\Omega\, \exp i\mathbf{k}\cdot\mathbf{r}'. \tag{3.12}$$

Here $\mathbf{r}'$ is a vector of magnitude r_{23}, pointing in an arbitrary direction, and $\int d\Omega$ is an integral over all directions of $\mathbf{r}'$. In (3.11), g_s is close to v_s for $r>d$, with d the Moszkowski-Scott separation distance; this means that the zeros of ϕ_0 will not cause any singularities in g_s in this region; for $r<d$, g_s is more complicated, but ϕ_0 will in general have no zeros because kd is not large. It is likely that g_s (in contrast to the original g) *can* be represented, with good accuracy, by an expression like (2.21),

$$g_s(k,r)=g_{s1}(r)+k^2g_{s2}(r). \tag{3.13}$$

Inserting (3.11 and 3.12) into (3.10), the exponential will have a k-dependent term

$$\mathbf{k}\cdot(2\mathbf{r}_{12}+\mathbf{r}_{23}+\mathbf{r}'). \tag{3.14}$$

In addition there is k dependence in g_s. If we neglect this, i.e., write $g_s=g_{s1}$, we can effect closure with the η matrix element (2.16). Integration over k gives from the exponentials

$$\delta(\mathbf{r}_{12}+\tfrac{1}{2}\mathbf{r}_{23}+\tfrac{1}{2}\mathbf{r}'-\mathbf{r}_{12}')\exp\tfrac{1}{2}i(\mathbf{p}_2-\mathbf{p}_3)\cdot(\mathbf{r}_{23}+\mathbf{r}'), \tag{3.14'}$$

so that the g_1 part of (2.23) is replaced by

$$W_{3s}(K_0)=(4\pi)^{-1}\int d^3r_{12}d^3r_{23}\int d\Omega\, \eta^*(\mathbf{R}+\tfrac{1}{2}\mathbf{r}')g_{s1}(r_{23})$$

$$\times[\exp\tfrac{1}{2}i(\mathbf{p}_2-\mathbf{p}_3)\cdot(\mathbf{r}'+{}_{23}\mathbf{r})]Z^{(1)}(r_{12},r_{23},r_{31}), \tag{3.15}$$

[8] H. P. Noyes, in Proceedings of Conference on Nuclear Forces and the Few-Nucleon Problem, London, 1959 (unpublished); H. P. Noyes and T. Osborn (private communication).

[9] R. V. Reid, Jr., Cornell University thesis, 1967 (to be published).

[10] This g_s is of course different from that in (3.7).

where

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{r}_{12}+\mathbf{r}_{13}). \tag{3.15'}$$

As we have pointed out, $\frac{1}{2}|\mathbf{p}_2-\mathbf{p}_3|<k_F$ is small for our purposes, and will be neglected. Then only η depends on the direction of $\mathbf{r}'$. Let μ be the cosine of the angle between $\mathbf{R}$ and $\mathbf{r}'$, then

$$|\mathbf{R}+\tfrac{1}{2}\mathbf{r}'|^2=R^2+Rr_{23}\mu+\tfrac{1}{4}r_{23}^2, \tag{3.15''}$$

$$R^2=\tfrac{1}{4}|\mathbf{r}_{12}+\mathbf{r}_{13}|^2=\tfrac{1}{2}(r_{12}^2+r_{13}^2)-\tfrac{1}{4}r_{23}^2. \tag{3.15'''}$$

The integration of η over μ is then elementary. Denoting the argument of η by y, this gives

$$(4\pi)^{-1}\int d\Omega\ \eta(\mathbf{R}+\tfrac{1}{2}\mathbf{r}')=\int (ydy/Rr_{23})\eta(y)$$

$$=(Rr_{23})^{-1}[\chi(R-\tfrac{1}{2}r_{23})-\chi(R+\tfrac{1}{2}r_{23})], \tag{3.16}$$

$$\chi(x)=\int_x^\infty \eta(y)ydy. \tag{3.16'}$$

The upper limit ∞ of the integral in (3.16') is arbitrary, and chosen for convenience only. Equation (3.15) can then be written

$$W_{3s1}(K_0)=\int d^3r_{12}d^3r_{23}\ (Rr_{23})^{-1}g_{s1}(r_{23})Z^{(1)}(r_{12},r_{23},r_{31})$$
$$\times[\chi(R-\tfrac{1}{2}r_{23})-\chi(R+\tfrac{1}{2}r_{23})]. \tag{3.17}$$

The part g_{s2} of (3.13) can now be done similarly. We use (2.22) which introduces $\nabla^2\eta$, then (3.16) is replaced by

$$(4\pi)^{-1}\int d\Omega\ \nabla^2\eta(\mathbf{R}+\tfrac{1}{2}\mathbf{r}')=(Rr_{23})^{-1}\int_{R-\frac{1}{2}r_{23}}^{R+\frac{1}{2}r_{23}} ydy\ \nabla^2\eta(y)$$

$$=(Rr_{23})^{-1}[w(R+\tfrac{1}{2}r_{23})-\omega(R-\tfrac{1}{2}r_{23})], \tag{3.18}$$

with

$$w(x)=\eta+xd\eta/dx. \tag{3.18'}$$

The contribution of g_{s2} to W_3 is then

$$W_{3s2}(K_0)=\frac{1}{4}\int d^3r_{12}d^3r_{23}\ [w(R-\tfrac{1}{2}r_{23})-w(R+\tfrac{1}{2}r_{23})]$$
$$\times g_{s2}(r_{23})Z^{(1)}(r_{12},r_{23},r_{31}), \tag{3.19}$$

and $W_{3s}=W_{3s1}+W_{3s2}$. As in (2.24) and (2.25), we can leave the integration over r_{23} to the end, and define

$$F_{s1}(r_{23})=r_{23}^{-1}\int d^3r_{12}\ R^{-1}Z^{(1)}$$

$$\times[\chi(R-\tfrac{1}{2}r_{23})-\chi(R+\tfrac{1}{2}r_{23})], \tag{3.20}$$

$$F_{s2}(r_{23})=(4r_{23})^{-1}\int d^3r_{12}\ R^{-1}Z^{(1)}$$

$$\times[w(R-\tfrac{1}{2}r_{23})-w(R+\tfrac{1}{2}r_{23})], \tag{3.21}$$

$$W_{3s}(K_0)=\int d^3r_{23}$$

$$\times[g_{s1}(r_{23})F_{s1}(r_{23})+g_{s2}(r_{23})F_{s2}(r_{23})]. \tag{3.22}$$

The nuclear force can be quite adequately represented by the sum of an ordinary force (treated in Sec. 2), a a Serber force [treated in this section up to Eq. (3.6)], and a force acting in S states only [treated from Eq. (3.7) on]. Each force component may be taken to depend quadratically on k, as in (2.18).

The tensor force acts mainly in second order in which it is equivalent to a central force.[11] It should be considered, however, that this effective central force decreases with increasing k. This can again be done by assuming a form like (2.21) for the effective central g, with g_1 attractive and g_2 repulsive.

The method here outlined will probably give W_3 to about 5% if the various components of g are judiciously chosen. If greater accuracy is desired, g must be expanded in spherical harmonics and (2.20) used directly. However, then it is also necessary to improve the accuracy of $Z^{(1)}$ and abandon the assumption that it depends only on the three distances. Likewise, it would be necessary to take into account the finite momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ in initial and final state, in η and in such equations as (2.22).

4. ENERGY OF INTERMEDIATE STATES

In B3, I assumed that the three-body interaction can still be treated by assigning suitable potential energies $U(b)$ to intermediate (particle) states. In fact, I proposed to write

$$U(b)=\int d^3r_{23}\ g_{Pk}(r_{23})F_1(r_{23})/F_0, \tag{4.1}$$

where F_1 is the quantity defined in (2.24), g_{Pk} the effective potential function appropriate to particle state b, and F_0 the quantity defined in B3, Eq. (5.7),

$$F_0=\int d^3r_{12}\ \eta^2(r_{12}). \tag{4.2}$$

It is shown in B3, Eq. (5.8), that $F_1(r_{23}=\infty)=F_0$. We shall see that (4.1) is only a very crude approximation.

It is, of course, desirable to define a particle energy $U(b)$. It should be defined in such a way that

$$W_3(K_0)=(2\pi)^{-3}\int d^3b\ y^2(b)U(b), \tag{4.3}$$

where

$$y(b)=\int \eta(r)e^{-i\mathbf{b}\cdot\mathbf{r}}d^3r \tag{4.4}$$

[11] G. Brown and T. Kuo, Nucl. Phys. 85, 40 (1966).

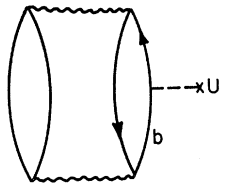

FIG. 2. A third-order Goldstone diagram in which the middle interaction is with the particle potential $U(b)$.

is the Fourier transform of η, denoted by F_{lj} in BBP.[12] Then the "insert" diagrams of the Goldstone theory, Fig. 2, will give exactly minus the right-hand side of (4.3). Adding this to W_3 gives zero; in other words, the three-body correlation energy is compensated by the potential insert diagrams of Fig. 2. The energy is then given simply by the first-order two-body interactions $\langle mn|G|mn\rangle$.

To this must be added the four-body (and higher) correlations, and diagrams with more than one potential insert. In addition to the expectation that these will be small, Rajaraman[13] has shown that the most important more-body correlations probably involve the long-range force in all interactions except the first and last, and this force is well represented by the potential $U(b)$. Therefore, choosing $U(b)$ to satisfy (4.3), we shall probably also minimize the net effect of the correlations of more than three bodies. The claim is then that at present it is best to find a $U(b)$ compatible with (4.3), insert it into the energy denominators in the Brueckner integral equation for the two-body G matrix, and calculate the nuclear matter energy by summing the two-body G's. Then no further corrections are needed.

To calculate $U(b)$, we go back to (2.19) and (2.20). We assume now

$$\mathbf{p}_3-\mathbf{p}_2=0; \tag{4.5}$$

then (2.15) gives

$$2\mathbf{k}=\mathbf{b}. \tag{4.6}$$

Equation (2.20) may be written[14]

$$\langle K_1|g^*Z^{(1)}|K_0\rangle=\int d^3r_{23}\, g_{Pk}{}^*(\mathbf{r}_{23})Y_b(r_{23}), \tag{4.7}$$

$$Y_b(r_{23})=\int d^3r_{12}\, j_0(br_{12})Z^{(1)}(r_{12},r_{23},r_{31}), \tag{4.8}$$

and (2.16)

$$\langle \mathbf{p}_1,\mathbf{p}_2|\eta_{12}|\mathbf{a},\mathbf{b}\rangle=y^*(b). \tag{4.9}$$

Then (2.19) takes the form (4.2), with

$$U(b)=\int d^3r_{23}\, g_{Pk}{}^*(r_{23})Y_b(r_{23})/y(b). \tag{4.10}$$

It is clear that by construction (4.10) satisfies (4.3). Equation (4.10) must be modified for various types of interaction, similar to Sec. 3.

Equation (4.10) is a reasonable definition of $U(b)$. To obtain Y_b from (4.8), we must integrate over the two variables r_{12} and r_{13} for every value of b and of r_{23}. This is evidently quite laborious, so an approximate discussion is in order.

$U(b)$ is most important [cf. (4.3)] for those states for which $y(b)$ is large, i.e., for[4] $b\approx 4$ F^{-1}. In this case, $j_0(br_{12})$ is similar in shape to $\eta(r_{12})$, in the sense that both rj_0 and $r\eta$ have peaks at $r=c$ (the detailed shapes are different). Then, from (2.24),

$$Y_b(r_{23})\sim F_1(r_{23}). \tag{4.11}$$

This justifies, for this most important range of b, the use of (4.1); it is easy to see that also the normalization is correct.

On the other hand, for small b, of order k_F, $j_0(br_{12})\approx 1$ for the important values of r_{12}; then Y_b will not depend very much on r_{23}. In this case, small distances r_{23} will be weighted in (4.10) about as much as large r_{23}, i.e., the influence of the repulsive core will not be cut down appreciably by the three-body correlations. This does not do any harm because the core is not very important for small b anyway; see BBP, Fig. 11. Indeed, it is desirable that the core not be suppressed, because otherwise $U(b)$ for b slightly above k_F can easily become less than $U(m)$ for an occupied state m slightly below k_F. Such a "negative energy gap" was found by Sprung and Bhargava[15] using (4.1), and was arbitrarily removed by them. Our consideration here justifies this removal; further justification will be given below.

A difficulty with (4.10) is the denominator $y(b)$. According to BBP, Fig. 14, this is zero just about at $b=k_F$. This would cause a singularity in $U(b)$. We shall now show that this singularity is removed by including the tensor force.

Tensor Forces

If the first and last interaction[16] is by tensor forces, Dahlblom[17] has shown that the function $Z^{(1)}$ must be modified to read

$$\begin{aligned} Z_T{}^{(1)}=&\eta_{12}{}^T[1-\zeta_{13}+\tfrac{1}{2}\zeta_{13}\zeta_{23}]\\ &-\tfrac{1}{2}P_2(\mathbf{r}_{12}\cdot\mathbf{r}_{13})\eta_{13}{}^T[1-\zeta_{12}+\tfrac{1}{2}\zeta_{12}\zeta_{23}]\\ &-\tfrac{1}{2}P_2(\mathbf{r}_{12}\cdot\mathbf{r}_{23})\eta_{23}{}^T[\zeta_{12}\zeta_{13}-\zeta_{12}-\zeta_{13}], \end{aligned} \tag{4.12}$$

where η^T is the radial part of the ^3D wave function which

[12] See BBP (Ref. 5), Fig. 14.
[13] R. Rajaraman, Phys. Rev. **155**, 1105 (1967).
[14] In (4.8) we have tacitly assumed that there is no correlation between the directions of $\mathbf{k}$ and $\mathbf{r}_{23}$; then $\exp i\mathbf{b}\cdot\mathbf{r}_{12}$ in (2.20) can be averaged over angle. This is, of course, not necessary.
[15] D. Sprung and P. Bhargava, Nucl. Phys. (to be published).
[16] If the first interaction is tensor, it introduces the spin-dependent operator S_{12} which must be compensated by the last interaction, which therefore also must be tensor. Intermediate interactions are assumed to be central, except that they may include the effective central force generated by the tensor force in second order, as at the end of Sec. 3. T. Dahlblom, K. G. Fogel, B. Quist, and A. Törn [Nucl. Phys. **56**, 191 (1965)] have calculated the effect of three successive tensor forces and found it to be small; this will not be treated here.
[17] T. Dahlblom (to be published).

is associated with an unperturbed ^3S wave by the tensor force [the function $\chi_{21}{}^{(0)}=-u_{21}{}^{(0)}$ in the notation of BBP, (6.29), and Fig. 13]. The reason for the difference between this result and Day's[2] is the summation over the spins. Dahlblom now points out that the tensor wave-function modification η^T is zero for $r<c$, and then rises gradually, but that ζ (which comes from the repulsive core) drops rapidly for $r>c$. Therefore the terms involving products $\eta\zeta$ are apt to be small, and the only important terms are

$$Z_{T3}{}^{(1)}=4\eta_{12}{}^T-2\eta_{13}{}^T. \tag{4.13}$$

This, however, is exactly the third-order contribution, without corrections for three-body correlations. Dahlblom has calculated F_1, Eq. (2.24) inserting (4.12), and has confirmed that it depends very little on r_{23}. The same will undoubtedly be true of $Y_{bT}(r_{23})$, Eq. (4.8), when (4.12) is used for $Z^{(1)}$, especially if b is not too large.

This means that for (initial and final) tensor interaction, $U(b)$ in (4.10) reduces, apart from a constant factor, to

$$U_T(b)\approx\int d^3r_{23}\; g_{PkT}{}^*(r_{23}), \tag{4.14}$$

which is just the elementary "bubble" interaction. Dahlblom has shown, using (4.12), that (again apart from a constant factor)

$$g_T=g(\text{even } L)+3g(\text{odd } L), \tag{4.15}$$

in contrast to the case of an initial *central* interaction for which $g(r_{23})$ is given by the even L states alone.

Generalizing (4.3), it can easily be shown that the correct three-body energy is obtained if we define $U(b)$ as follows:

$$U(b)=\frac{y_c{}^2(b)U_c(b)+y_T{}^2(b)U_T(b)}{y_c{}^2(b)+y_T{}^2(b)}, \tag{4.16}$$

where $U_c(b)$ is defined by (4.10) and (4.8) with $Z^{(1)}$ being Day's wave function with an initial central interaction, while $U_T(b)$ is obtained similarly with $Z^{(1)}$ being Dahlblom's expression (4.13). Similarly, y_c and y_T are the Fourier transforms (4.4) of η_c, η_T.

Equation (4.16) has the following advantages:

1. For small b, the tensor term dominates because $y_T\gg y_c$ (BBP, Fig. 14). This makes $U(b)$ essentially that of the simple "bubble" with even greater assurance than the argument below (4.11). It eliminates the danger of a "negative energy gap" between hole and particle states, especially because the high weight of the odd-state interaction (4.15) introduces a repulsion for the particle states. The dominance of the tensor term is also satisfactory because $U(b)$ for states of low b is chiefly needed to evaluate the two-body tensor G matrix.

2. There is no longer any singularity for $y_c(b)=0$.

3. For large b, above about 3 F^{-1} (BBP, Fig. 14), y_c dominates and our previous estimate of $U(b)$ remains correct.

Brandow[18] has recently constructed a somewhat different theory of nuclear matter, which however is also based on the idea that the energy should be expanded in the number of interacting particles. The difference lies mainly in the choice of the intermediate-state potentials $U(b)$. In Brandow's theory these are essentially zero, but it is then necessary to do a separate calculation of the three-body cluster energy, as we do in Sec. 2.

The discrepancy between these viewpoints is not as great as it may appear. Brandow states[19] that our present treatment of $U(b)$ is essentially equivalent to a further rearrangement of his expansion. His argument lends support to the Sprung-Bhargava[15] prescription whereby the potential-energy function $U(k)$ is forced to be continuous at k_F. We believe that the present approach will give a more accurate treatment of the energy contributions from the states just above k_F.

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[18] B. Brandow, Phys. Rev. **152**, 863 (1966).
[19] B. Brandow (private communication).