tion, which seems to cause some temperature dependence of ρ_T/ρ_0 in n-type Ge[2].

IV. CONCLUSION

From our data on samples of germanium doped with low concentrations of Ga, we conclude that a magnetic field affects phonon-assisted hop conduction in p-type Ge via its influence on the acceptor wave functions. Many of the features observed are similar to those occurring in n-type Ge, but some are not because of the difference between acceptor and donor impurity states.

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Path Variable Formulation of the Hot Carrier Problem

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A theoretical treatment of transport phenomena in strong electric fields is presented. Instead of the Legendre polynomial expansion of the distribution function usually employed in solving the transport equation, we transform the Boltzmann equation to a coordinate system determined by the collision-free trajectories of the particles and formulate an integral equation for the distribution function. This method is applied to hot carriers in nonpolar semiconductors, where the relevant transport equation is then reduced to a one-dimensional integral equation. This equation is solved numerically and energy distributions are calculated for n- and p-type germaniun. The calculations for heavy holes in germanium demonstrate the non-Maxwellian nature of the distribution function as well as its strong displacement in momentum space, and are in excellent agreement with experiment. The energy distributions for electrons show weaker deviations from Maxwellian and smaller ratios of drift to rms velocity, this being due to the weaker coupling to optical phonons for electrons as compared to holes.

I. INTRODUCTION

THE central theoretical problem associated with hot-carrier phenomena is the calculation of the steady-state particle distribution function in the presence of a strong electric field. Since the transport equation describing the distribution function is generally an integrodifferential equation, one is usually obliged to seek approximate solutions appropriate to specific physical situations.

The two major simplifications usually introduced in effecting such calculations are related to the role of carrier-carrier scattering and to the displacement of the distribution function in momentum space.

The role of carrier-carrier scattering in determining the form of the distribution function had first been discussed by Fröhlich[1] and has been examined in detail by Stratton.[2] The essential idea is that sufficiently strong carrier-carrier scattering results in a displaced Maxwellian distribution, characterized by an effective temperature greater than that of the lattice. Having fixed this form of the distribution function, it is then possible to calculate the displacement in momentum space and the effective temperature from momentum and energy balance equations, which are readily derived from the Boltzmann equation.

When carrier-carrier scattering is not sufficiently strong to justify this form of the distribution function, it is necessary to solve the Boltzmann equation with due regard to the detailed nature of the various scattering mechanisms. It is in carrying out this program that one usually introduces the assumption of a weakly displaced distribution function in momentum space, thus enabling one to represent the distribution function by only the two lowest-order terms in a Legendre polynomial expansion in the cosine of the angle between the electric field and momentum vector. This is roughly equivalent to assuming that the ratio of the averaged drift to root-mean-square velocity is small.

Recent experimental studies in p-type[3,4] germanium have demonstrated the inapplicability of this assumption in this case. In particular, the studies of Bray and co-workers[4] have emphasized the "streaming" character

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[1] H. Fröhlich, Proc. Roy. Soc. (London) **A188**, 521 (1947); **A188**, 532 (1947); H. Fröhlich and B. V. Paranjape, Proc. Phys. Soc. (London) **B69**, 21 (1956); **B69**, 866 (1956).

[2] R. Stratton, Proc. Roy. Soc. (London) **A242**, 355 (1957).

[3] A. C. Baynham and E. G. S. Paige, Phys. Letters **6**, 7 (1963).

[4] Ralph Bray and William E. Pinson, Phys. Rev. Letters **11**, 268 (1963); W. E. Pinson and Ralph Bray, Phys. Rev. **136**, A1449 (1964); R. Bray, W. E. Pinson, D. M. Brown, and C. S. Kumar, in *Proceedings of the International Conference on the Physics of Semiconductors, Paris, 1964* (Academic Press Inc., New York, 1965), p. 467.

of hole transport, characterized by a strong displacement of the distribution function in momentum space, as well as the non-Maxwellian nature of the heavy-hole distribution.

In this paper we present a formulation of transport phenomena in intense electric fields which specifically avoids the use of Legendre polynomial expansions and the infinite set of coupled integrodifferential equations which result from this technique. Instead, we use the collision-free trajectories of the particles to transform the Boltzmann equation into an integral equation. This transformation[5] is particularly useful in treating the hot-carrier phenomena in covalent semiconductors since in this case the resulting integral equation may be reduced to a one-dimensional equation.

In Sec. II of this paper we review the structure of the Legendre polynomial formulation of the hot-carrier problem. The transformation, or path variable method, is presented in Sec. III and applied to the hot-carrier problem in Sec. IV. Numerical results relevant to both n- and p-type germanium are presented in Sec. V and are shown to be in excellent agreement with experiment.

II. LEGENDRE POLYNOMIAL FORMULATION

The steady-state distribution function $f(\mathbf{p})$ for the case of a uniform electric field $\mathbf{E}$ is determined by the Boltzmann transport equation.

$$e\mathbf{E}\cdot\nabla_{\mathbf{p}}f=\hat{C}f\equiv\int d\mathbf{p}'[f(\mathbf{p}')T_{\mathbf{p}',\mathbf{p}}-f(\mathbf{p})T_{\mathbf{p},\mathbf{p}'}]. \quad (1)$$

The right-hand side of this equation represents the rate of change of f due to collisions, which are characterized by the transition rates $T_{\mathbf{p},\mathbf{p}'}$ between states of crystal momentum $\mathbf{p}$ and $\mathbf{p}'$.

We consider the case of spherical constant-energy surfaces for which the carrier energy is a quadratic function of p:

$$\epsilon=p^2/2m^*. \quad (2)$$

The ellipsoidal constant-energy surfaces encountered in n-type germanium and silicon may be treated in the same fashion. After transforming the ellipsoidal constant-energy surfaces into spheres, in the manner discussed by Herring and Vogt,[6] and introducing effective fields,[7] the problem becomes identical to that represented by Eqs. (1) and (2).

For isotropic scattering, then, the distribution function clearly depends only on the carrier energy and θ, the angle between the momentum vector $\mathbf{p}$ and the applied electric field $\mathbf{E}$. One may then expand the distribution function in Legendre polynomials in $\cos\theta$:

$$f(\mathbf{p})=\sum_{n=0}^{\infty}S_n(\epsilon)P_n(\cos\theta). \quad (3)$$

Inserting this expansion in Eq. (1) and making use of the orthogonality properties of the Legendre polynomials, one obtains the following infinite set of coupled integrodifferential equations:

$$\begin{aligned}
&[\quad\quad\quad\quad;S_1,\quad dS_1/d\epsilon\quad]=\hat{C}_0S_0,\\
&[S_0,\quad dS_0/d\epsilon\quad;S_2,\quad dS_2/d\epsilon\quad]=\hat{C}_1S_1,\\
&[S_{n-1},\,dS_{n-1}/d\epsilon;\,S_{n+1},\,dS_{n+1}/d\epsilon]=\hat{C}_nS_n.
\end{aligned} \quad (4)$$

The brackets appearing on the left-hand side of these equations represent a linear sum of the quantities indicated. The typical equation for $S_n(\epsilon)$ is thus coupled to $S_{n-1}(\epsilon)$ and $S_{n+1}(\epsilon)$, and in order to proceed further it is necessary to truncate this set. The usual assumption[8] at this point is to neglect all but the two lowest-order Legendre polynomials S_0 and S_1, and to seek approximate solutions of the resulting equations.

The physical basis for this procedure is the hope that the distribution function is only weakly anisotropic in momentum space and that therefore the higher-order Legendre polynomials are negligible. This is clearly not the case in p-type germanium where experimental studies of the distribution function have revealed large ratios of the drift to root-mean-square velocity and values of $S_2(\epsilon)$ comparable to those of $S_0(\epsilon)$.

III. THE PATH VARIABLE METHOD

In order to illustrate the transformation procedure to be employed in solving the transport Eq. (1), we first start with the Boltzmann equation in the relaxation time approximation:

$$\frac{\partial f}{\partial t}+\mathbf{F}\cdot\nabla_{\mathbf{p}}f+\mathbf{V}\cdot\nabla_{\mathbf{r}}f=\hat{C}f\approx-\left(\frac{f-f_0}{\tau(\mathbf{p})}\right), \quad (5)$$

where f_0 is the equilibrium distribution, $\mathbf{V}=\nabla_{\mathbf{p}}\epsilon$, and $\mathbf{F}$ is some arbitrary force field.

We now transform to a coordinate system determined by the collision-free trajectories of the particles

$$\begin{aligned}
&\frac{d\mathbf{r}^*(s)}{ds}=\mathbf{V}^*;\quad \mathbf{r}^*(t)=\mathbf{r},\\
&\frac{d\mathbf{p}^*(s)}{ds}=\mathbf{F};\quad \mathbf{p}^*(t)=\mathbf{p}.
\end{aligned} \quad (6)$$

These equations simply represent the collision-free trajectory of a particle which at time t has momentum $\mathbf{p}$ and position $\mathbf{r}$. Introducing the transformation (6)

[5] H. F. Budd, J. Phys. Soc. Japan **21**, 420 (1966).
[6] C. Herring and E. Vogt, Phys. Rev. **100**, 944 (1956).
[7] H. G. Reik and H. Riskin, Phys. Rev. **124**, 777 (1961).
[8] Recently Baraff has considered a "maximal anisotropy" truncation procedure. G. A. Baraff, Phys. Rev. **133**, A26 (1964).

into Eq. (5) one obtains a simple first-order differential equation

$$\frac{df}{ds}+\frac{f}{\tau(s)}=\frac{f_0}{\tau(s)}, \tag{7}$$

whose steady-state solution is

$$f(\mathbf{p},\mathbf{r},t)=\int_{-\infty}^{t}\frac{f_0(s)}{\tau(s)}\exp\left[-\int_s^t\frac{dy}{\tau(y)}\right]ds. \tag{8}$$

This method does not rely on any special properties of the applied forces, band structure, or relaxation time, and represents a simple generalization of the kinetic method first discussed by Chambers.[9,10]

We now apply the transformations (6) to Eq. (1) and obtain the following analogy to Eq. (7):

$$\frac{df}{ds}+\frac{f}{\tau(s)}=\int d\mathbf{p}'f(\mathbf{p}')T_{\mathbf{p}',\mathbf{p}^*(s)}, \tag{9}$$

where

$$\frac{1}{\tau}\equiv\int d\mathbf{p}'T_{\mathbf{p},\mathbf{p}'}.$$

Inverting this equation in the same manner as above one readily obtains the following integral equation for the distribution function:

$$f(\mathbf{p})=\int d\mathbf{p}'f(\mathbf{p}')K(\mathbf{p}',\mathbf{p}), \tag{10}$$

where

$$K(\mathbf{p}',\mathbf{p})=\int_0^\infty T_{\mathbf{p}',\mathbf{p}-\mathbf{F}s}\exp\left[-\int_0^s\frac{dy}{\tau(\mathbf{p}-\mathbf{F}y)}\right]ds.$$

We note that the kernel $K(\mathbf{p}',\mathbf{p})$ is essentially composed of two factors:

(1) $T_{\mathbf{p}',\mathbf{p}-\mathbf{F}s}ds$, the probability that a carrier is scattered from the initial state $\mathbf{p}'$ into the state $\mathbf{p}-\mathbf{F}s$ along the collision-free trajectory in a time interval ds near s.

(2)

$$\exp\left[-\int_0^s\frac{dy}{\tau(\mathbf{p}-\mathbf{F}y)}\right],$$

the probability that it suffers no further collisions in drifting under the influence of the field to the state $\mathbf{p}$. $K(\mathbf{p}',\mathbf{p})$ is simply the sum of all such events over all points of entry onto the collision-free trajectory. When this kernel is multiplied by $f(\mathbf{p}')$, the steady-state probability of finding a carrier in the state $\mathbf{p}'$, and integrated over all initial states $\mathbf{p}'$, one obtains $f(\mathbf{p})$ the steady-state probability of finding a carrier in state $\mathbf{p}$.

A similar result may be readily derived for the general case, the only essential difference being a different expression for the collision-free particle trajectories.

Alternate theoretical treatments of the hot-carrier problem, not employing the diffusion approximation, have been presented by Baraff[11] and Baraff and Buchsbaum.[12] In particular, Baraff's theory of ionization processes in semiconductors leads to an integral equation in energy space for the collision density function, which measures the number of collisions per second involving carriers of a given energy. The kernel of this equation is then interpreted in terms of the same sort of kinetic picture presented in our discussion following Eq. (10), although the method of derivation is quite different from the path variable treatment, which introduces the collision-free trajectories at the outset and may therefore be simply formulated for any set of applied force fields and band structure.

A simplified version of Eq. (10) has been studied by Stuart and Gerjuoy[13] in connection with ionization processes in gases. Using model scattering cross sections they recast their integral equation into a soluble form occurring in neutron transport theory.

IV. HOT CARRIERS

We now apply the preceding transformation method to the hot-carrier problem in nonpolar semiconductors. The dominant energy relaxation mechanism in this case results from optical-phonon interactions while the important momentum relaxation processes for hot carriers are due to both optical and acoustic phonons.

A great simplification in the transport equation is possible in the case of nonpolar phonon interactions, since *all the anisotropies* of the distribution function relax with the *same* energy-dependent relaxation time. In particular, let us decompose the distribution function into its isotropic and anisotropic parts:

$$f=S_0+A, \tag{11}$$

where A is the sum of all the Legendre polynomials in expansion (3), except for the isotropic or $n=0$ term. The effect of collisions on f is similarly decomposed as follows:

$$\hat{C}f=\hat{C}S_0+\hat{C}A=\hat{C}S_0-\frac{A}{\tau_M(\epsilon)}, \tag{12}$$

where $\tau_M(\epsilon)$ is the momentum relaxation time.

We shall now show that the last form of Eq. (12) holds rigorously for optical-phonon scattering and is also valid in the limit of elastic-acoustic phonon scattering, which is the usual condition for the existence of a relaxation time in this case. This results from the fact the scattering is spherically symmetrical in both these cases.

[9] R. Chambers, Proc. Phys. Soc. (London) **A65**, 458 (1952).
[10] H. Budd, J. Phys. Soc. Japan **18**, 142 (1963).
[11] G. A. Baraff, Phys. Rev. **128**, 2507 (1962).
[12] G. A. Baraff and S. J. Buchsbaum, Phys. Rev. **130**, 1007 (1963).
[13] G. W. Stuart and E. Gerjuoy, Phys. Rev. **119**, 892 (1960).

Let us consider the collision-induced rate of change of a particular anisotropy:

$$\hat{C}S_n(\epsilon)P_n(\cos\theta)=\int d\mathbf{p}'S_n(\epsilon')P_n(\cos\theta')T_{\mathbf{p}',\mathbf{p}}-\frac{S_n(\epsilon)P_n(\cos\theta)}{\tau(\epsilon)},\quad(13)$$

where

$$\frac{1}{\tau(\epsilon)}=\int d\mathbf{p}'T_{\mathbf{p},\mathbf{p}'}.$$

If the integral term appearing in the right-hand side of Eq. (13) is zero for $n\geq 1$, our assertion is valid. In particular, this integral will vanish for $n\geq 1$ if $T_{\mathbf{p}'\mathbf{p}}$ is angularly independent.

The transition rates for phonon emission and absorption are in the notation of Fröhlich and Paranjape[1]

$$T_{\mathbf{p},\mathbf{p}'}=\frac{2\pi}{\hbar}B(q)\begin{bmatrix}1+N_q\\ N_q\end{bmatrix}\delta(\epsilon(\mathbf{p})-\epsilon(\mathbf{p}')\pm\hbar\omega_q);\quad\mathbf{p}'=\mathbf{p}\pm\mathbf{q}.\quad(14)$$

Here $\mathbf{q}$ and $\hbar\omega_q$ are the phonon crystal momentum and energy, respectively, $B(q)$ is the square of the interaction matrix element, and N_q is the equilibrium phonon distribution. $B(q)$ and $\hbar\omega_q$ are independent of q for nonpolar optical modes and thus $T_{\mathbf{p}',\mathbf{p}}$ is angularly independent and the integral term in Eq. (13) vanishes for $n\geq 1$.

For acoustic phonons, $B(q)\sim q$ and $\hbar\omega=qs$. Since the acoustic-phonon energy is small compared to electron energies one can neglect the phonon energy in the energy-conservation function and furthermore set $N_q\approx 1+N_q=kT/\hbar\omega_q$. Here again, $T_{\mathbf{p}',\mathbf{p}}$ is angularly independent and our assertion is proved.

Inserting the decomposition of Eq. (12) in Eq. (1), we obtain

$$\mathbf{F}\cdot\nabla_{\mathbf{p}}f=\hat{C}S_0-\left(\frac{f-S_0}{\tau_M(\epsilon)}\right).\quad(15)$$

Applying the transformation and procedure of Sec. III, we obtain

$$f(\mathbf{p})=\int_0^\infty\left[\hat{C}S_0+\frac{S_0}{\tau_M(\epsilon)}\right]_{\mathbf{p}-\mathbf{F}s}\times\exp\left[-\int_0^s\frac{dy}{\tau_M(\mathbf{p}-\mathbf{F}y)}\right]ds.\quad(16)$$

This equation expresses the entire distribution function in terms of its isotropic part S_0, which remains to be calculated. Integrating Eq. (16) over a constant-energy surface yields

$$S_0(\epsilon)=\int_{-1}^1\frac{f(\mathbf{p})d(\cos\theta)}{2}=\int_{-1}^1\frac{d(\cos\theta)}{2}\int_0^\infty\left[\hat{C}S_0+\frac{S_0}{\tau_M(\epsilon)}\right]_{\mathbf{p}-\mathbf{F}s}\times\exp\left[-\int_0^s\frac{dy}{\tau_M(\mathbf{p}-\mathbf{F}y)}\right]ds.\quad(17)$$

After having solved Eq. (17) for S_0, one may calculate any of the Legendre polynomial expansion terms in Eq. (3) by integration:

$$S_n(\epsilon)=\frac{\int_{-1}^1P_n(\cos\theta)d(\cos\theta)\int_0^\infty ds\left[\hat{C}S_0+\frac{S_0}{\tau_M(\epsilon)}\right]_{\mathbf{p}-\mathbf{F}s}\exp\left[-\int_0^s\frac{dy}{\tau_M(\mathbf{p}-\mathbf{F}y)}\right]}{\int_{-1}^1P_n^2(\cos\theta)d(\cos\theta)}.\quad(18)$$

We now make use of the usual relaxation for phonon scattering and the transition rates for optical-phonon interactions:

$$T_{\mathbf{p}',\mathbf{p}-\mathbf{F}y}=\frac{2\pi}{\hbar}B_0N_0\left[\delta(\epsilon(\mathbf{p}-\mathbf{F}y)-\epsilon(\mathbf{p}')-\hbar\omega_0)+e^{\hbar\omega/kT}\delta(\epsilon(\mathbf{p}-\mathbf{F}y)-\epsilon(\mathbf{p}')+\hbar\omega_0)\right],$$
$$\frac{1}{\tau_M(\epsilon)}=\frac{1}{\tau_A(\epsilon)}+\frac{1}{\tau_0(\epsilon)}=\frac{1}{\tau_A(1)}\left[\left(\frac{\epsilon}{kT}\right)^{1/2}+\alpha\left(\left(\frac{\epsilon}{kT}+\frac{\hbar\omega_0}{kT}\right)^{1/2}+e^{\hbar\omega/kT}\left(\frac{\epsilon}{kT}-\frac{\hbar\omega_0}{kT}\right)^{1/2}\right)\right];\quad\alpha=N_0\left(\frac{\epsilon_0}{\epsilon_A}\right)^2\frac{\hbar\omega_0}{2kT},\quad(19)$$

where τ_A and τ_0 are the relaxation times for scattering acoustic and optical phonons, respectively, B_0 is the square of the optical-phonon interaction matrix element, N_0 is the equilibrium number of optical phonons, $(\epsilon_0/\epsilon_A)^2$ is the ratio of optical- and acoustic-phonon coupling constants defined by Brown and Bray,[14] and $\tau_A(1)$ is the acoustic-phonon relaxation time for a carrier of energy kT.

[14] D. M. Brown and R. Bray, Phys. Rev. 127, 1593 (1962).

Inserting these expressions in Eq. (17) results in

$$S_0(\epsilon)=\int_0^\infty S(\epsilon')K(\epsilon',\epsilon)d\epsilon',$$

where

$$K(\epsilon',\epsilon)=\int_0^\infty ds\int_{-1}^{1}\frac{d(\cos\theta)}{2}\left[2\pi(2m)^{3/2}(\sqrt{\epsilon'})T_{\mathbf{p}',\mathbf{p}-\mathbf{F}s}+\frac{\delta[\epsilon'-\epsilon(\mathbf{p}-\mathbf{F}s)]}{\tau_A(\epsilon')}\right]\left[\exp\left[-\int_0^s\frac{dy}{\tau_m(\mathbf{p}-\mathbf{F}y)}\right]\right]. \tag{20}$$

Scaling the time integrals by F and measuring all energies in units of kT, we obtain

$$S_0(x)=\int_0^\infty S_0(y)(\sqrt{y})K(x,y)dy;\quad x=\frac{\epsilon}{kT}, \tag{21}$$

$$K(x,y)=4\gamma\int_0^\infty ds\int_{-1}^{1}\frac{d(\cos\theta)}{2}\exp\left[-4\gamma\int_0^s dy\sum_{n=-1}^{+1}\alpha_n\left[x+y^2-2y(\sqrt{x})\cos\theta+\frac{n\hbar\omega_0}{kT}\right]^{1/2}\right]$$
$$\times\left[\sum_{m=-1}^{+1}\alpha_m\delta\left(x+s^2-2s(\sqrt{x})\cos\theta-y+m\frac{\hbar\omega_0}{kT}\right)\right]. \tag{22}$$

Here the sums indicated are for $\frac{m}{n}=-1,\ 0,\ 1$, corresponding to optical-phonon emission, acoustic-phonon scattering, and optical-phonon absorption, respectively. The constants α_n and γ are given by

$$\alpha_0=1,\quad \alpha_{-1}=\alpha e^{\hbar\omega/kT},\quad \alpha_{+1}=\alpha,\quad \gamma=\frac{[2mkT]^{1/2}}{4F\tau_A(1)}. \tag{23}$$

The integral appearing in the exponential may be simply evaluated, thus leaving the double integral to be considered. We use the δ functions in the integrand in order to perform the integrals over $\cos\theta$, and obtain finally

$$N(x)=\int_0^\infty N(y)dy\sum_{m=-1}^{+1}\alpha_m K_0\left(x,\ y+\frac{m\hbar\omega_0}{kT}\right);\quad N(x)\equiv(\sqrt{x})S_0(x), \tag{24}$$

where

$$K_0(x,y)=\gamma\int_{|(\sqrt{x})-(\sqrt{y})|}^{(\sqrt{x})+(\sqrt{y})}(dt/t)\exp\left[-\gamma\sum_{n=-1}^{+1}\alpha_n g_n\right],$$

with

$$g_n=\mathrm{Re}\left\{\frac{a(t^2+b^2)}{t}-\frac{(t^2-a^2)(t^2-b^2)}{2t^2}\ln\left|\frac{t+a}{t-a}\right|\right\},$$

$$a=(y+n\delta)^{1/2}+(x+n\delta)^{1/2},\quad b=(y=n\delta)^{1/2}-(x+n\delta)^{1/2},\quad \delta=\hbar\omega_0/kT.$$

The present formulation of the hot carrier problem therefore results in a single one-dimensional integral equation, as compared to the usual infinite system of coupled integrodifferential equations.

The general method employed in solving Eq. (24) is one of iteration. The kernel of this equation was tabulated numerically and a trial function inserted in the integral, thus yielding a new distribution function which was reinserted in the integral, etc.

The tabulation of $K_0(x,y)$ which is a symmetric function of its arguments was carried out on an IBM 7044. It should be noted that $K_0(x,y)$ has a logarithmic singularity at the point $x=y$. This difficulty was bypassed by analytic integration of the right-hand side of Eq. (24) over a small range enclosing the singularity, the interval being chosen sufficiently small that $N(y)$ could be taken constant in the integration. The interval chosen was typically $10^{-4}kT$, which clearly introduces negligible error.

Approximately 10 iterations were necessary in order to determine a distribution function satisfying Eq. (24) to an accuracy greater than 0.01%.

V. NUMERICAL RESULTS AND DISCUSSION

We first consider the case of heavy holes in p-type germanium where the constant-energy surfaces are somewhat warped and consequently the effective mass

is slightly anisotropic. The effective masses ratios for the [100], [110], and [111] directions are 0.28, 0.35, and 0.37, respectively, while in our caculations we have taken a constant effective mass of $0.35m_0$. The optic- and acoustic-phonon coupling constants used in our calculations are those given by Brown and Bray.[14]

In Fig. 1 we present a typical energy distribution for heavy holes in p-type germanium at 77°K. The dashed curve is the calculated heavy-hole energy distribution, $N(\epsilon)$ for an electric field of 1370 V/cm. The upper solid curve is a Maxwellian distribution with the same average energy, and is merely presented for comparison. One readily sees the rather large deviations of the calculated distribution from the Maxwellian and in particular the strong depletion of the high-energy tail.

The lower curve represents the magnitude of the natural logarithm of the probability distribution, $|\ln S_0|$. This curve is fairly well approximated by two straight lines intersecting in the neighborhood of the optical-phonon energy ($\hbar\omega_0=0.037$ eV); and thus the probability distribution is approximately two Maxwellian distributions with different "effective" temperatures above and below the optical-phonon energy. The "kink" in the $|\ln S_0|$ has been observed by Baynahm and Paige[3] and by Bray and Kumar[15] and the general features discussed above are in good agreement with the experimentally determined energy distributions, and with the recent Monte Carlo calculations of Kurosawa.[16]

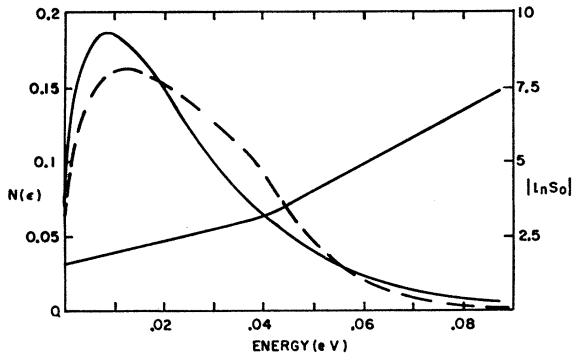

FIG. 1. Calculated heavy-hole distribution for p germanium at $E=1370$ V/cm (dashed curve): Maxwellian distribution with same average energy (upper solid curve); log of probability distribution (lower curve).

In Fig. 2 the solid curve represents the calculated average heavy-hole energy versus electric field for p-type germanium at 77°K. The average energy is seen to increase very slowly with electric field, varying almost linearly in the range $E=500$ V/cm to $E=2000$ V/cm. The circles and triangles in this figure correspond to the experimental results of Pinson and Bray.[4] Since the effective masses are slightly anisotropic there is roughly a 10–15% variation in the heavy-hole energy with orientation. The masses $m_{100}<m_{110}<m_{111}$, hence the average energies for these field directions are such that $\bar{\epsilon}_{100}>\bar{\epsilon}_{110}>\bar{\epsilon}_{111}$ and therefore our calculated energies using $m=0.35m_0$ should fall between the experimental values which were obtained for the [100] and [111] directions, which is indeed the case.

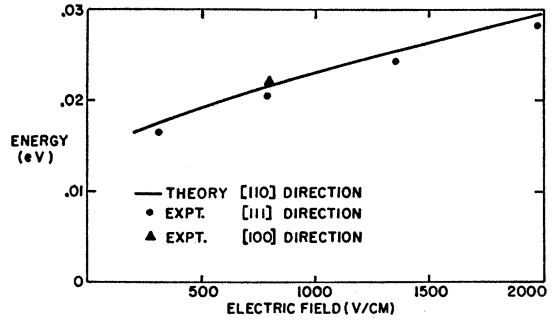

FIG. 2. Calculated average heavy-hole energy versus electric field for p-germanium (solid curve). Experimental results of Pinson and Bray are represented by circles and triangles for the applied field in the [111] and [100] directions, respectively.

In Fig. 3 the solid curve represents V_d/V_{rms}, the ratio of the heavy-hole drift velocity to rms velocity versus electric field for p-type germanium at 77°K. The large values of V_d/V_{rms} are striking and emphasize the strong displacement of the distribution function in momentum space and streaming nature of the carrier motion. Our results are compared with the measurements of Pinson and Bray[4] which are represented by the dashed curve in this figure. It is important to note that Pinson and Bray's results are for the ratio of the *total* drift velocity (heavy and light holes) to rms heavy-hole velocity, and therefore one would expect their results to be somewhat higher than ours.

Although the averages we have calculated agree very well with experiment, there are differences in the

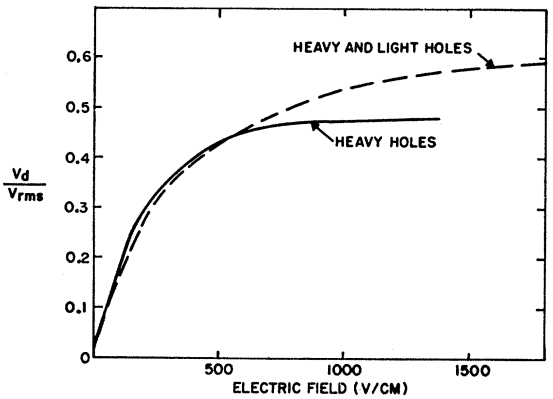

FIG. 3. Calculated ratio of drift to rms velocity versus electric field for heavy holes in p-germanium. Experimental results of Pinson and Bray for the ratio of the total, heavy+light hole, drift velocity to rms heavy-hole velocity (dashed curve).

[15] R. Bray and C. S. Kumar (to be published).
[16] Tatsumi Kurosawa, J. Phys. Soc. Japan, 21, 424 (1966).

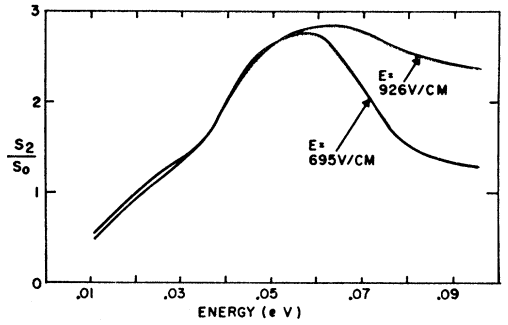

FIG. 4. Calculated ratio of the second to zeroth-order Legendre polynomial for heavy-hole distribution in p-type germanium.

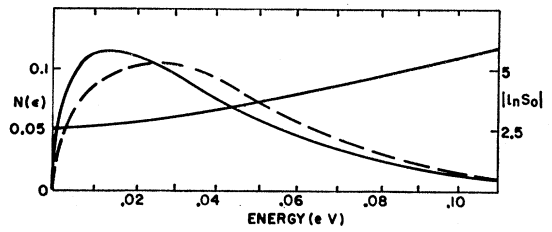

FIG. 5. Calculated electron-energy distribution for n-germanium at $E=1200$ V/cm (dashed curve). Maxwellian distribution with same average energy (upper solid curve). Log of probability distribution (lower curve).

detailed shape of the energy distribution. The calculated "kink" in the $|\ln S_0|$ curves is more pronounced than that observed by Bray and Kumar.[15] This is undoubtedly due to the effect of carrier-carrier scattering, which would tend to equalize the deviations in the high- and low-energy regions and thus reduce the sharpness of the "kinks."

A more drastic effect of carrier-carrier scattering is to be expected in the higher-order anisotropies of the distribution function. Since carrier-carrier scattering conserves both energy and momentum, one would not expect the *average* energy or velocity to be affected to lowest order. Of course there will be indirect changes in these quantities due to the energy dependence of the phonon scattering. A redistribution of carriers due to carrier-carrier scattering changes the phonon collision rates through the energy dependence of the relaxation times and consequently the average energy and drift velocity are indirectly affected. There is, however, no conservation rules which precludes a *direct* variation of a quantity such as $S_2(\epsilon)P_2(\cos\xi)$ due to carrier-carrier scattering.

The calculated ratios $S_2(\epsilon)/S_0(\epsilon)$ are represented by the solid curves in Fig. 4. The general features of these curves are similar to the experimental results of Bray, Pinson, Brown, and Kumar,[4] but the quantitative agreement is poor. The experimental results are roughly 2 to 3 times smaller than our calculated results and the peaks of experimental curves occur at roughly half the energy of the calculated peaks. We believe that these differences are largely due to the neglect of carrier-carrier scattering in our theory which would serve to diminish the calculated S_2/S_0 distribution. We know from the large ratio of V_d/V_{rms} that the heavy holes predominantly populate momentum states in the field direction. Carrier-carrier scattering would lead to an increased spread of this distribution around the field direction and consequently would diminish the ratio.

We have performed similar calculations in n-type germanium at 77°K with the electric field in the [100] direction. The many-valley theory is simpler in this direction since all the valleys are equivalently "heated" by the field. In this case, the transformation of the ellipsoidal constant energy surfaces into spheres leads to a problem essentially identical to that formulated above where now the relevant mass is simply the conductivity mass $m^*=0.12m_0$. The coupling constants used are those given by Jorgensen, Meyer, and Schmidt-Tiedemann.[17]

A significant difference between p- and n-type germanium arises, however, from the fact that the relative coupling to optical phonons is roughly an order of magnitude weaker for electrons than for holes.

The calculated energy distribution for n-type Ge at 77°K at a field of $E=1200$ V/cm is shown in Fig. 5. The average energy in this case is $\bar{\epsilon}=0.0416$ eV, which is considerably higher (see Fig. 2) than that in p-type germanium, even though the phonon-limited mobilities at 77°K are approximately equal for heavy holes and electrons. Similarly, we see from Fig. 5 that the deviations from Maxwellian are relatively small in this case, as compared to heavy holes, which again emphasizes the striking differences that result from the strong and weak coupling to optical phonons for heavy holes and electrons, respectively.

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[17] M. H. Jorgensen, N. I. Meyer, and K. J. Schmidt-Tiedemann, in *Proceedings of the International Conference on the Physics of Semiconductors, Paris, 1964* (Academic Press Inc., New York, 1965), p. 457.