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Landau Damping of Magnetoplasma Waves for General Closed Fermi Surfaces

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By using a guiding-center distribution function, an expression for Landau damping is derived for closed Fermi surfaces of arbitrary shape in the case where the cyclotron frequency is much greater than the collision frequency. It is shown that the strength of the damping is determined by the average over the cyclotron orbit of the power delivered by the wave to the resonant electrons. The resonant coupling can always be written as a sum of electric and magnetic contributions, but the damping in general cannot, because of interference effects.

I. INTRODUCTION

In this paper we shall present a general semiclassical treatment of Landau damping of magnetoplasma waves for closed Fermi surfaces in the case where the cyclotron frequency is much greater than the collision frequency. Landau damping results from a resonant transfer of energy from the wave to charge carriers whose average velocity along the direction of propagation equals the phase velocity of the wave.[1–4] Kaner and Skobov[5] first predicted this kind of collisionless damping for helicon and Alfven waves propagating at an angle to the magnetic field. Such effects have recently been observed for helicon propagation in alkali metals by Grimes[6] and by Houck and Bowers,[7] and for Alfven waves in bismuth by Khaikin and Edelman.[8]

Despite statements[5,8] to the contrary, Landau damping can also occur for propagation along the field if the Fermi surface is nonspherical. Quinn[9] has in fact shown that for ellipsoidal energy surfaces with propagation along the field but not parallel to a principal axis of the ellipsoid, intra-Landau level transitions are allowed quantum mechanically.[10] Landau damping in just this geometry has been found for low-field helicon propagation in PbTe[11,12] and appears to occur for Alfven waves in bismuth as well.[13] The reason for such damping can be understood quite simply in a semiclassical treatment when it is noted[11] that the transverse electric field can do net work on the resonant electrons if the plane of the electron's cyclotron orbit is tilted with respect to the magnetic field, as will be the case if the magnetic field is not parallel to a principal axis of the ellipsoid.[14,15] This type of argument forms the basis for the present paper, in which we derive expressions for the strength of the Landau damping for closed Fermi surfaces of arbitrary shape.

In order to treat the general case, we shall work with a distribution function that labels the particles according to the position of their guiding center rather than their actual position. The guiding center gives the average position of the particle during its cyclotron motion; in the absence of rf fields, the guiding center moves uniformly in the direction of the dc magnetic field with some average velocity $\mathbf{v}_{av}$. In the next section

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1 We shall use the term Landau damping here to include the effect of all field components in producing the resonant transfer of energy. For some purposes it is useful to have separate terms to distinguish the role of the different fields. As will be discussed in connection with Eq. (3.5), it is always possible for closed Fermi surfaces to divide the damping mechanism into a part due to the component of electric field along the dc magnetic field and a part due to the rf magnetic field. For the latter, the term magnetic Landau damping has recently been suggested by Stix (Ref. 2) and is used by Buchsbaum and Platzman (Ref. 3) and by Pearson (Ref. 4). However, it is important to realize that both the electric and magnetic fields affect the resonant electrons in the same way; consequently there are interference terms which generally prevent the total resonant damping from being written as a sum of electric and magnetic contributions. It should also be pointed out that attaching a key role to the magnetic field is essentially a matter of taste, since the total resonant damping can just as naturally be expressed in terms of the electric field only, as in Eq. (3.3).

2 T. H. Stix, as communicated via G. A. Pearson. Magnetic Landau damping is a substitute for the term transit-time damping previously used: T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Company, Inc., New York, 1962), p. 196.

3 S. J. Buchsbaum and P. M. Platzman, Phys. Rev. **154**, 395 (1967). These authors point out that magnetic Landau damping dominates for helicon waves in metals.

4 G. A. Pearson (to be published).

5 E. A. Kaner and V. G. Skobov, Zh. Eksperim. i Teor. Fiz. **45**, 610 (1963) [English transl.: Soviet Phys.—JETP **18**, 419 (1964)].

6 C. C. Grimes, Bull. Am. Phys. Soc. **11**, 570 (1966).

7 J. R. Houck and R. Bowers, Bull. Am. Phys. Soc. **11**, 256 (1966).

8 M. S. Khaikin and V. S. Edelman, Zh. Eksperim. i Teor. Fiz. **49**, 1695 (1965) [English transl.: Soviet Phys.—JETP **22**, 1159 (1966)].

9 J. J. Quinn, Phys. Rev. **135**, A181 (1964).

10 In quantum language, Landau damping is an intra-Landau level transition in the limit of small momentum transfer.

11 J. N. Walpole, Ph.D. thesis, Massachusetts Institute of Technology, 1966 (unpublished).

12 J. N. Walpole and A. L. McWhorter, preceding paper, Phys. Rev. **158**, 708 (1967).

13 W. G. May and A. L. McWhorter (to be published). Propagation was along the trigonal axis. In Khaikin and Edelman's (Ref. 8) case, where propagation was along a binary axis, the threshold Alfven velocity and hence the threshold field for Landau damping should have decreased sharply when the magnetic field was exactly along the binary axis, since only the two electron ellipsoids with low Fermi velocity in the binary direction could then contribute. Khaikin and Edelman's data seem to show such an effect.

14 Mathematically, the occurrence of Landau damping for propagation along the field but not parallel to a principal axis of the ellipsoid immediately follows from a transformation (Ref. 15) to the equivalent spherical energy surface problem, since the transformed wavevector and field are not parallel. However, this approach does not reveal the physics of the original problem.

15 See, for example, S. G. Eckstein, Phys. Rev. Letters **12**, 360 (1964).

an expression for the guiding-center distribution function is obtained for the case of interest, where the cyclotron frequency is much larger than the collision frequency. Then in Sec. III we show that the quantity determining the strength of the Landau damping is the average over the cyclotron orbit of the power delivered by the wave to the resonant electrons.

II. DISTRIBUTION FUNCTION IN GUIDING CENTER COORDINATES

For simplicity only the case of a single closed Fermi surface will be considered explicitly; the generalization of the results to several closed surfaces is straightforward. We start from the semiclassical Boltzmann equation in $(\mathbf{r},\mathbf{k})$ coordinates:

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}}f+\frac{1}{\hbar}\mathbf{F}\cdot\nabla_{\mathbf{k}}f=\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}, \tag{2.1}$$

where

$$\mathbf{F}=-e(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{2.2}$$

and

$$\mathbf{v}=\hbar^{-1}\nabla_{\mathbf{k}}E. \tag{2.3}$$

If we let f_0 be the thermal equilibrium distribution and subscripts 1 denote the perturbations, then in the presence of a dc magnetic field $\mathbf{B}_0$ the linearized form of (2.1) is

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}}f_1-\frac{e}{\hbar}(\mathbf{v}\times\mathbf{B}_0)\cdot\nabla_{\mathbf{k}}f_1 =e\mathbf{E}_1\cdot\mathbf{v}\frac{\partial f_0}{\partial E}+\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}. \tag{2.4}$$

The collision term will be approximated by

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}=-\frac{f-\bar{f}_0}{\tau}=-\frac{f_1}{\tau}+\frac{\bar{f}_0-f_0}{\tau}, \tag{2.5}$$

where $\bar{f}_0$ is the local equilibrium distribution corresponding to the local density.[16] To first order in the perturbed density n_1

$$\bar{f}_0=f_0+\frac{\partial f_0}{\partial E_F}\frac{\partial E_F}{\partial n}n_1 \tag{2.6}$$

in terms of the Fermi energy E_F.

It is convenient to choose as coordinates[17] in $\mathbf{k}$ space the component k_z in the direction of $\mathbf{B}_0$, the total energy E, and the time s of the electron along its cyclotron orbit measured from some arbitrary reference point on the orbit. In terms of these coordinates the differential volume element in $\mathbf{k}$ space becomes

$$d^3k=(eB_0/\hbar^2)dk_z dE ds \tag{2.7}$$

and the Boltzmann equation takes the form

$$\frac{\partial f_1}{\partial t}+\frac{f_1}{\tau}+\mathbf{v}\cdot\nabla_{\mathbf{r}}f_1+\frac{\partial f_1}{\partial s}=\left(e\mathbf{E}_1\cdot\mathbf{v}-\frac{1}{\tau}\frac{\partial E_F}{\partial n}n_1\right)\frac{\partial f_0}{\partial E}. \tag{2.8}$$

We now change to the guiding center coordinates $\mathbf{r}_{\text{av}}$ by letting

$$\mathbf{r}_{\text{av}}=\mathbf{r}-\boldsymbol{\varrho}(k_z,E,s), \tag{2.9}$$

where, in notation simplified by suppressing k_z and E,

$$\boldsymbol{\varrho}(s)=\int_0^s ds'[\mathbf{v}(s')-\mathbf{v}_{\text{av}}] -\frac{1}{T}\int_0^T ds'\int_0^{s'} ds''[\mathbf{v}(s'')-\mathbf{v}_{\text{av}}]. \tag{2.10}$$

Here $T=2\pi/\omega_c$ is the cyclotron period and

$$\mathbf{v}_{\text{av}}=\frac{1}{T}\int_0^T \mathbf{v}(s)ds \tag{2.11}$$

is the mean velocity. The necessary property that

$$\int \boldsymbol{\varrho}\, ds=0 \tag{2.12}$$

is ensured by the fact that the second term of (2.10) is the average of the first. Since $\boldsymbol{\varrho}$ depends only on the $\mathbf{k}$ coordinates,

$$(\nabla_{\mathbf{r}}f_1)_{\mathbf{k}}=(\nabla_{\mathbf{r}_{\text{av}}}f_1)_{\mathbf{k}}, \tag{2.13}$$

while

$$\begin{aligned}\left(\frac{\partial f_1}{\partial s}\right)_{\mathbf{r}}&=\left(\frac{\partial f_1}{\partial s}\right)_{\mathbf{r}_{\text{av}}}+\nabla_{\mathbf{r}_{\text{av}}}f_1\cdot\left(\frac{\partial \mathbf{r}_{\text{av}}}{\partial s}\right)_{\mathbf{r}}\\ &=\left(\frac{\partial f_1}{\partial s}\right)_{\mathbf{r}_{\text{av}}}-\frac{\partial \boldsymbol{\varrho}}{\partial s}\cdot\nabla_{\mathbf{r}_{\text{av}}}f_1\\ &=\left(\frac{\partial f_1}{\partial s}\right)_{\mathbf{r}_{\text{av}}}-(\mathbf{v}-\mathbf{v}_{\text{av}})\cdot\nabla_{\mathbf{r}_{\text{av}}}f_1.\end{aligned} \tag{2.14}$$

Hence the Boltzmann equation becomes

$$\frac{\partial f_1}{\partial t}+\frac{f_1}{\tau}+\mathbf{v}_{\text{av}}\cdot\nabla_{\mathbf{r}_{\text{av}}}f_1+\left(\frac{\partial f_1}{\partial s}\right)_{\mathbf{r}_{\text{av}}} =\left[e\mathbf{E}_1(\mathbf{r},t)\cdot\mathbf{v}-\frac{1}{\tau}\frac{\partial E_F}{\partial n}n_1(\mathbf{r},t)\right]\frac{\partial f_0}{\partial E}, \tag{2.15}$$

a form which could have been deduced by direct physical argument.

If we assume an $\exp(i\omega t-i\mathbf{q}\cdot\mathbf{r})$ variation for $\mathbf{E}_1$ and

[16] Even when the total density fluctuation is negligible, it is necessary in general to include relaxation to the local equilibrium in a many-valley semiconductor, since there can be large fluctuations in the populations of the individual valleys (Ref. 11).

[17] I. M. Lifshitz, M. Ya. Azbel', and M. I. Kaganov, Zh. Eksperim. i Teor. Fiz. 31, 63 (1956) [English transl.: Soviet Phys.—JETP 4, 41 (1957)].

n_1, then we can set

$$f_1(\mathbf{r}_{av},\mathbf{k},t)=g_1(\mathbf{k})e^{i\omega t-i\mathbf{q}\cdot\mathbf{r}_{av}} \tag{2.16}$$

and obtain as the equation for g_1

$$(i\omega-i\mathbf{q}\cdot\mathbf{v}_{av}+1/\tau)g_1+\frac{\partial g_1}{\partial s}=\left(e\mathbf{E}_1\cdot\mathbf{v}-\frac{1}{\tau}\frac{\partial E_F}{\partial n}n_1\right)e^{-i\mathbf{q}\cdot\rho}\frac{\partial f_0}{\partial E}. \tag{2.17}$$

The formal solution to (2.17) is

$$g_1=\int_{-\infty}^{s}ds' e^{(i\omega-i\mathbf{q}\cdot\mathbf{v}_{av}+1/\tau)(s'-s)}\times\left(e\mathbf{E}_1\cdot\mathbf{v}-\frac{1}{\tau}\frac{\partial E_F}{\partial n}n_1\right)e^{-i\mathbf{q}\cdot\rho}\frac{\partial f_0}{\partial E}$$
$$=\frac{1}{1-\exp(-i\omega+i\mathbf{q}\cdot\mathbf{v}_{av}-1/\tau)T}\int_{s-T}^{s}ds' \times e^{(i\omega-i\mathbf{q}\cdot\mathbf{v}_{av}+1/\tau)(s'-s)}\times\left(e\mathbf{E}_1\cdot\mathbf{v}-\frac{1}{\tau}\frac{\partial E_F}{\partial n}n_1\right)e^{-i\mathbf{q}\cdot\rho}\frac{\partial f_0}{\partial E}, \tag{2.18}$$

the last expression being obtained by summing the geometric series which results from the contribution of each period of length T. In the limit where $\omega_c\gg\omega-\mathbf{q}\cdot\mathbf{v}_{av}$ and $\omega_c\tau\gg1$,

$$g_1\approx\frac{1}{i\omega-i\mathbf{q}\cdot\mathbf{v}_{av}+1/\tau}\frac{1}{T}\int_{s-T}^{s}\left(e\mathbf{E}_1\cdot\mathbf{v}-\frac{1}{\tau}\frac{\partial E_F}{\partial n}n_1\right)\times e^{-i\mathbf{q}\cdot\rho}\frac{\partial f_0}{\partial E}ds. \tag{2.19}$$

Substituting (2.19) into (2.16) and taking into account the periodic nature of the integrand, we finally obtain for f_1 an expression independent of s:

$$f_1(\mathbf{r}_{av},k_z,E,t)=\frac{e^{i\omega t-i\mathbf{q}\cdot\mathbf{r}_{av}}}{i\omega-i\mathbf{q}\cdot\mathbf{v}_{av}+1/\tau}\frac{\partial f_0}{\partial E}\times\left\langle e\mathbf{E}_1\cdot\mathbf{v}-\frac{1}{\tau}\frac{\partial E_F}{\partial n}n_1\right\rangle. \tag{2.20}$$

Here the angular brackets denote an average over the cyclotron orbit, i.e.,

$$\left\langle e\mathbf{E}_1\cdot\mathbf{v}-\frac{1}{\tau}\frac{\partial E_F}{\partial n}n_1\right\rangle=\frac{1}{T}\int_0^T\left(e\mathbf{E}_1\cdot\mathbf{v}-\frac{1}{\tau}\frac{\partial E_F}{\partial n}n_1\right)e^{-i\mathbf{q}\cdot\rho}ds. \tag{2.21}$$

The resonance which occurs when $\omega-\mathbf{q}\cdot\mathbf{v}_{av}=0$ is of course just the one that leads to Landau damping. As previously stated, Landau damping is produced by those electrons whose average velocity along the direction of propagation equals the phase velocity of the wave. Note that for the resonant particles the condition $\omega_c\gg\omega-\mathbf{q}\cdot\mathbf{v}_{av}$ is automatically satisfied, so that we need only $\omega_c\tau\gg1$ for (2.20) to be valid near resonance. It is also important for later use to observe that

$$\langle\mathbf{E}_1\cdot\mathbf{v}\rangle=\frac{1}{T}\int\mathbf{E}_1e^{-i\mathbf{q}\cdot\rho}\cdot\frac{\partial\mathbf{r}}{\partial s}ds=\frac{\omega_c}{2\pi}\mathbf{E}_1\cdot\int_C e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}_{av})}d\mathbf{r}, \tag{2.22}$$

where C is the trajectory in $\mathbf{r}$ space corresponding to the cyclotron orbit in $\mathbf{k}$ space.

III. LANDAU DAMPING

Using the results of the preceding section, we can immediately obtain an expression for the strength of the Landau damping. The current $\mathbf{J}_1(\mathbf{r},t)$ is made up of contributions from all electrons whose trajectory passes through $\mathbf{r}$ at time t:

$$\mathbf{J}_1(\mathbf{r},t)=-e\int(eB_0/h^2)dk_zdEdsd\mathbf{r}_{av}\times f_1(\mathbf{r}_{av},k_z,E,t)\mathbf{v}(k_z,E,s)\delta(\mathbf{r}_{av}+\boldsymbol{\rho}-\mathbf{r})$$
$$=-(e^2B_0/h^2)\int dk_zdEf_1(\mathbf{r},k_z,E,t)\int e^{i\mathbf{q}\cdot\rho}\mathbf{v}ds. \tag{3.1}$$

Hence the power dissipated by the wave is

$$P=\tfrac{1}{2}\,\mathrm{Re}(\mathbf{E}_1\cdot\mathbf{J}_1^*)=-\tfrac{1}{2}\,\mathrm{Re}\left[(e^2B_0/h^2)\int dk_zdE\times f_1^*\int\mathbf{E}_1e^{-i\mathbf{q}\cdot\rho}\cdot\mathbf{v}ds\right]$$
$$=-\tfrac{1}{2}\,\mathrm{Re}\left[(2\pi em_c/h^2)\int dk_zdEf_1^*\langle\mathbf{E}_1\cdot\mathbf{v}\rangle\right], \tag{3.2}$$

where $m_c=(eB_0/\omega_c)$ is the cyclotron mass. For $q_zv_F\tau\gg1$ the value of P is independent[18] of the collision rate and at $T=0°$K becomes

$$P=\frac{1}{2\pi}\frac{e^2m_c}{h^2}\frac{1}{q_z(\partial v_{av}/\partial k_z)}|\langle\mathbf{E}_1\cdot\mathbf{v}\rangle|^2, \tag{3.3}$$

where $\langle\mathbf{E}_1\cdot\mathbf{v}\rangle$ is to be evaluated for the resonant electrons at the Fermi surface. It should be noted that there is a singularity in (3.3) if the Fermi surface is such that

[18] Relaxation to local equilibrium must be included for this to be true in general.

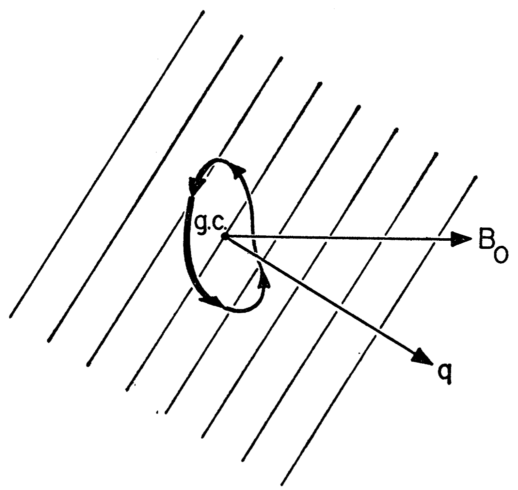

FIG. 1. Diagram illustrating for a general closed Fermi surface the motion of a resonant electron as viewed in the coordinate system moving along $\mathbf{B}_0$ at the average velocity of the electron. The planes of constant phase are indicated by the set of parallel lines. The guiding center of the electron's cyclotron motion is labeled g.c.

$\partial v_{\text{av}}/\partial k_z=0$. A similar singularity in Doppler-shifted cyclotron resonance has been discussed by Eckstein[19] in connection with magnetoacoustic attenuation.

From (3.3) we see that the strength of the Landau damping is determined by the average over a cyclotron orbit of the power delivered to the resonant electrons. If we let $\mathbf{u}=\mathbf{v}-v_{\text{av}}\hat{z}$ be the velocity of the electron relative to its guiding center, then we may write

$$\langle \mathbf{E}_1\cdot\mathbf{v}\rangle=\langle E_{1z}\rangle v_{\text{av}}+\langle \mathbf{E}_1\cdot\mathbf{u}\rangle. \tag{3.4}$$

In the coordinate system moving with the guiding center, the electron describes a stationary closed orbit (provided that the orbit in $\mathbf{k}$ space is closed) which in general is of complicated shape, as shown schematically in Fig. 1. For ellipsoidal energy surfaces with effective mass tensor $\mathbf{m}$, the path is an ellipse lying in a plane whose normal is the direction $\mathbf{m}\cdot\mathbf{B}_0$; for spherical energy surfaces the path is circular and lies in the plane normal to $\mathbf{B}_0$. In the case of helicon propagation along an axis of symmetry with $\mathbf{B}_0$ parallel to $\mathbf{q}$, E_{1z} will be zero. However, the second term of (3.4) will not in general be zero except for the special case of spherical energy surfaces. Hence, Landau damping can occur for helicons with $\mathbf{q}$ parallel to $\mathbf{B}_0$, as previously discussed.

It is possible to rewrite the $\langle \mathbf{E}_1\cdot\mathbf{u}\rangle$ term of (3.4) in terms of the rf magnetic field, since from Stokes' theorem and Maxwell's equations

$$\langle \mathbf{E}_1\cdot\mathbf{u}\rangle=(\omega_c/2\pi)\int \mathbf{E}_1\cdot d\boldsymbol{\varrho}$$

$$=-i\omega(\omega_c/2\pi)\int_A \mathbf{B}_1\cdot d\mathbf{a}. \tag{3.5}$$

The last integral is taken over the area A of the cyclotron orbit in the coordinate system moving with the guiding center. If the size of the orbit is sufficiently small compared to a wavelength that the variation of the fields over the orbit may be neglected, we then have

$$-e\langle \mathbf{E}_1\cdot\mathbf{v}\rangle=-ev_{\text{av}}E_{1z}-i\omega\boldsymbol{\mu}\cdot\mathbf{B}_1, \tag{3.6}$$

where

$$\boldsymbol{\mu}=-(e\omega_c/2\pi)\mathbf{A} \tag{3.7}$$

is the magnetic moment of the orbiting electron. The second term of (3.6) has the obvious interpretation as the time rate of change of the magnetic energy $-\boldsymbol{\mu}\cdot\mathbf{B}_1$.

Thus, as previously mentioned, the coupling $\langle \mathbf{E}\cdot\mathbf{v}\rangle$ between the resonant particles and the wave may always be written as a sum of electric and magnetic parts. However, since the damping of the wave is proportional to $|\langle \mathbf{E}\cdot\mathbf{v}\rangle|^2$, there in general will be interference effects between the electric and magnetic coupling.[20] This situation is in fact encountered in low-field helicon propagation in PbTe.[12]

As would be expected for an intra-Landau-level transition, the power absorbed by the resonant electrons produces an increase in their momentum along the magnetic field rather than an increase in the area of their cyclotron orbit in $\mathbf{k}$ space. This may easily be verified by multiplying (2.15) by the respective quantities and integrating over $\mathbf{k}$.

Finally, it should be remarked that (3.3) can be used for a perturbation calculation of the attenuation due to Landau damping. By taking the imaginary part in (3.2), one can obtain perturbatively the dispersive effect as well.

[19] S. G. Eckstein, Phys. Rev. Letters **16**, 611 (1966). See also E. A. Kaner, V. G. Peschanskii, and I. A. Privorotskii, Zh. Eksperim. i Teor. Fiz. **40**, 214 (1961) [English transl.: Soviet Phys.—JETP **13**, 147 (1961)].

[20] Similar results have been obtained by Buchsbaum and Platzman (Ref. 3) for the special case of spherical Fermi surfaces and wavelengths much larger than the cyclotron radius.