Transport Phenomena in the Simple Metals*

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The low-excitation-energy phenomena characteristic of the simple metals is discussed. The model used takes into account both the electron-phonon and electron-electron interactions. The result obtained is an extended form of Landau's Fermi-liquid theory. The parameters of the theory are related to the underlying interactions, and the relationships corresponding to the relation of the effective mass with the "scattering function" discovered by Landau are developed. The theory is used to classify the renormalization effects in the interacting electron-phonon model of the metal. The results are valid when excitation energies no greater in magnitude than the Debye energy are involved, with the exception that the usual differential form of the Landau-Boltzmann transport equation does not hold if time variations of frequency comparable to the Debye frequency are considered.

I. INTRODUCTION

WE discuss in this paper the system of electrons and phonons, taking into account the Coulomb interactions of the electrons. We can also include static lattice effects in a phenomenological way. This model of the simple metals should be, and has proved, remarkably successful in dealing with those phenomena involving excitation energies low in comparison with the atomic-binding energies (but perhaps comparable with lattice-vibration energies).

It has long been appreciated that the interactions undergone by the electrons with each other,[1,2] and with the phonons,[3] although quite strong in magnitude, have extraordinarily simple consequences, largely because the bulk of the electrons are frozen into place in momentum space by the low-excitation condition and the Pauli principle.

However, most of the theoretical treatments up to this time, while convincing enough, have either treated one aspect of the interaction problem to the exclusion of others, or have resorted to approximations now known to be unnecessary. We therefore feel that it is a worthwhile exercise to carry out a full formal treatment of the problem, making only the assumption that the system is "normal" and using the "weak-momentum-dependence approximation" of Migdal[3] which has proven validity.

There are a number of treatments of this problem in the literature. Abrikosov, Gorkov, and Dzyaloshinski[4] have discussed the electron-phonon problem and the Coulomb interaction problem separately, in their excellent book. Nozières[5] gives a more exhaustive treatment of the direct interaction problem, and Schrieffer[6] discusses extensively the electron-phonon problem, and includes some of the effects of Coulomb interaction. Many authors have recognized that somehow the phonon effects could be taken into account by building on the underlying Coulomb quasiparticle system[7–9]; generally, they were content to neglect all further Coulomb interaction effects. However, Leggett[10] treats superfluidity as a weak aftereffect in much the same spirit as we treat the phonons.

There is also a body of literature dealing specifically with both types of interactions. Among the earliest of these we mention that of Quinn and Ferrell.[11] Silverstein[12] and Simkin[13] also make important contributions. More recently, Batyev and Pokrovskii[14] analyzed the electron-phonon-Coulomb system graphically, but did not treat transport properties. Very recently, after the results we will present had been obtained, a paper of Heine, Nozières, and Wilkins[15] appeared which discusses in detail certain features of the electron-phonon-Coulomb system. We will accordingly lay more stress on those aspects of the problem which they did not treat.

The result of our investigation is of course that the electron-lattice system can be regarded as a set of quasi-

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[1] L. Landau, Zh. Eksperim. i Teor. Fiz. **30**, 1058 (1956); **32**, 59 (1957) [English transl.: Soviet Phys.—JETP **3**, 920 (1957); **5**, 101 (1957)].

[2] V. P. Silin, Zh. Eksperim. i Teor. Fiz. **33**, 495 (1957) [English transl.: Soviet Phys.—JETP **6**, 387 (1958)].

[3] A. B. Migdal, Zh. Eksperim. i Teor. Fiz. **34**, 1438 (1958) [English transl.: Soviet Phys.—JETP **7**, 996 (1958)].

[4] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood-Cliffs, New Jersey, 1963).

[5] P. Nozières, *Theory of Interacting Fermi Systems* (W. A. Benjamin and Company Inc., New York, 1963).

[6] J. R. Schrieffer, *Superconductivity* (W. A. Benjamin and Company Inc., New York, 1964).

[7] J. Bardeen, L. Cooper, and J. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[8] R. E. Prange and L. P. Kadanoff, Phys. Rev. **134**, A566 (1964)—referred to hereafter as PK.

[9] T. Holstein, Ann. Phys. (N. Y.) **29**, 410 (1964).

[10] A. J. Leggett, Phys. Rev. **140**, A1869 (1965).

[11] J. J. Quinn and R. A. Ferrell, Phys. Rev. **112**, 812 (1958).

[12] S. D. Silverstein, Phys. Rev. **130**, 1703 (1963); **128**, 631 (1962); Ph.D. thesis, University of Illinois (unpublished).

[13] D. Simkin, Ph.D. thesis, University of Illinois (unpublished).

[14] E. G. Batyev and V. L. Pokrovskii, Zh. Eksperim. i Teor. Fiz. **46**, 262 (1964) [English transl.: Soviet Phys.—JETP **19**, 181 (1964)].

[15] V. Heine, P. Nozières, and D. Wilkins, Phil. Mag. **13**, 741 (1966).

particles (electrons and phonons) whose interactions can be taken into account in the self-consistent-field approximation. In other words, the Landau-Fermi liquid theory applies in the generalized form discussed in PK.[8]

The novel aspect of the study lies in the details of the result which reveal exactly how the phonon and Coulomb contributions to the effective mass and quasiparticle interactions combine.

Unfortunately for the theorist, very little has been done experimentally to separate the contribution from phonon interaction and Coulomb interactions, and in fact, it conspires that little can be done by present methods to investigate experimentally the interaction of the quasiparticles. Even so, our theory can be regarded as replacing or justifying the theoretical treatments of a vast array of experimental results.

Aside from the complete classification of the correlation phenomena in the simple metals, the present contribution consists of two results.

The first is a classification of the renormalizations, both of matrix elements and mass. In particular, the mass renormalization is most naturally regarded as multiplicative, i.e.,

$$(m/m^*)=(m/m_c)Z. \tag{1}$$

Here m_c is the mass when all phonon effects are neglected, and Z is the electron-phonon-interaction (EPI) renormalization factor, taking precisely the same form as when Coulomb interactions (CI) are neglected, but with renormalized parameters.

The second result relates the quasiparticle interaction function $\Phi(kk')$ to its values in the absence of EPI, and in the absence of Coulomb forces but with (renormalized) phonons. This result is not so uncomplicated, generally involving an integral equation, but in the case most often encountered, all phonon effects can be shown to drop out of the result. It allows a knowledge of the phonon properties gained, say, from superconducting tunneling theory, to be combined with observed values of $\Phi(kk')$ to give the contribution of the Coulomb interaction alone.

The plan of the paper is as follows. In Sec. II we discuss the graphical analysis which allows us to discard all but a small subset of Feynman diagrams. The Feynman diagrams determine the structure of the self-energy functions. The basic result is that the electron self-energy can be split into two parts, one of which is the self-energy in the absence of phonons and the other of which has the form of the electron self-energy arising from EPI with additional renormalizations coming from the CI. The contribution which mixes the two turns out to be negligible. In Sec. III we derive the Landau-Boltzmann transport equation. In Sec. IV we discuss the scattering function which determines the dependence of quasiparticle energy on the state of the system and discuss the properties of this function corresponding to the Landau relationship between the scattering function and the effective mass. In Sec. V we work out the predictions of the theory in some cases and list the results in other cases. In the remainder of Sec. I we review the basic approximations which are made in the theory

The usual derivation of the Landau theory in the absence of phonons is based on the smallness of the rate of decay of a quasiparticle near the Fermi surface into quasiparticle-quasihole pairs. This smallness is a consequence of the small amount of phase space available for these pairs. It is known that the decay rate should be proportional to ω^2/μ where ω is the energy relative to the Fermi energy and μ is the chemical potential (or the Fermi energy). The theory is thus restricted to excitation energies, frequencies, and temperatures which are considerably less than 1 eV. In the justification of the Fermi liquid theory in the presence of phonons, the smallness of the decay rate cannot in general be invoked as relatively close to the Fermi surface the width of a quasiparticle state of definite momentum is comparable to its energy. It was shown, however, in PK, that it is possible to redefine a quasiparticle state as a state of definite energy but indefinite momentum. All of the functions of the theory are relatively insensitive to the momentum value and consequently the blurring of the momentum plays no important role. The difference between the two variables can be seen by noting that the typical phonon energies are of the order of the Debye energy ω_D, which is two to three orders of magnitude less than the Fermi energy μ, whereas typical phonon momenta are comparable with the electronic momentum. The analysis is based on an extension of Migdal's discussion which neglects contributions of the order of ω_D/μ and is presumably exact to that order. [Actually, the theory to be presented requires generalization if the external frequency is allowed to become comparable with the Debye frequency. We will omit a discussion of this generalization here, in view of the fact that no experiment has been done in this frequency range.]

Another well-known idea which we shall utilize throughout is the causality relation according to which the self-energy obeys a dispersion relation,

$$\Sigma(kE)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dE'\frac{\Gamma(kE')}{E-E'}, \tag{2}$$

where Σ and $\Gamma/2$ are, respectively, the real and imaginary parts of the self-energy. This relation shows that the rapidly varying part of the self-energy as a function of its energy variable is associated with a rapidly varying imaginary part of the decay rate of the quasiparticles. This in turn means that the part of the self-energy coming from the Coulomb interactions is slowly varying as a function of the frequency variable, since its imaginary part close to the Fermi surface is small and weakly varying.

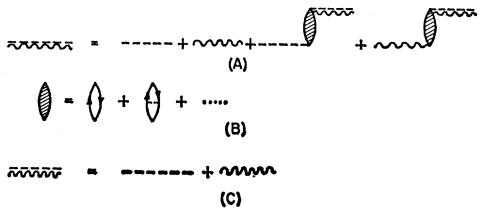

FIG. 1. Graphical representation of the integral equation for the polarization propagator. (A) Light dashed and wavy lines represent bare phonon and Coulomb interactions. (B) Shaded bubble represents the irreducible polarization part, the first terms of which are shown. (C) Heavy lines represent renormalized phonon and Coulomb propagators.

Unfortunately, this paper is by no means self-contained. It relies heavily on the work reported in Refs. 3, 8, and 16. A good knowledge of the field theoretic methods as discussed, for example, in Ref. 4 is also essential. We shall however, attempt to indicate by plausibility arguments enough to refresh the memory of those readers who are already familiar with the background material. Complete details can be found in the unpublished thesis of A. Sachs.

II. GRAPHICAL ANALYSIS

In this section, we discuss the graphical structure of the electron self-energy as well as the structure of the polarization or phonon self-energy. For this purpose we can use the graphical analysis for the equilibrium state as the numerical estimates are not affected by the small deviations from equilibrium considered.[8] Furthermore, the result of the analysis is to give the functional dependence of the self-energy on the distribution of quasiparticles. The distributions in turn are determined not by graphs but by solving a boundary condition equation which turns out to be the Landau-Boltzmann equation.[16] We take as the fundamental Hamiltonian

$$H=\sum_{\mathbf{k},\sigma}\epsilon_{\mathbf{k}}C_{\mathbf{k}\sigma}^{\dagger}C_{\mathbf{k}\sigma}+\sum_{\substack{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4\\ \sigma_1\sigma_2}}V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}C_{\mathbf{k}_1\sigma_1}^{\dagger}C_{\mathbf{k}_2\sigma_2}^{\dagger}C_{\mathbf{k}_3\sigma_2}C_{\mathbf{k}_4\sigma_1}$$
$$+\sum\omega_{\mathbf{q},\lambda}{}^{0}a_{\mathbf{q}\lambda}^{\dagger}a_{\mathbf{q}\lambda}+\sum\nu_{\mathbf{k}_1}{}^{0}{}_{\mathbf{k}_2}{}^{\lambda}[a_{\mathbf{q}(\mathbf{k}_1\mathbf{k}_2)\lambda}+a_{\mathbf{q}(\mathbf{k}_2\mathbf{k}_1)\lambda}^{\dagger}]$$
$$\times C_{\mathbf{k}_1\sigma}^{\dagger}C_{\mathbf{k}_2\sigma}, \quad (3)$$

which contains electron-annihilation and creation operators, bare phonon operators, and the Coulomb interaction. It is convenient to suppress the spin and polarization indices σ and λ, and to regard the unique phonon momentum, $\mathbf{q}=\mathbf{q}(\mathbf{k}_1\mathbf{k}_2)$ as given by $\mathbf{k}_1-\mathbf{k}_2$. This is done to avoid opening the Pandora's box of subscripts and does not represent an approximation.

According to the usual considerations, one calls the sum of all graphs beginning and ending with a bare phonon or unscreened Coulomb line the full polarization propagator or generalized dielectric function. The polarization propagator including its coupling parameters to electrons is thus

$$D(\mathbf{q}\omega)=[V(\mathbf{q})+D^0(\mathbf{q}\omega)][1+\Pi(\mathbf{q}\omega)D(\mathbf{q}\omega)]$$
$$=\{1-[V(\mathbf{q})+D^0(\mathbf{q},\omega)]\Pi(\mathbf{q}\omega)\}^{-1}$$
$$\times[V(\mathbf{q})+D^0(\mathbf{q}\omega)], \quad (4)$$

where Π is the irreducible polarization part and D^0 is the bare phonon propagator

$$D^0=\frac{2\omega_q{}^0|\nu^0(\mathbf{q})|^2}{\omega^2-\omega_q{}^{02}+i\eta}. \quad (5)$$

It is clear that the expression (4), when iterated, represents the sum of graphs beginning and ending with a bare phonon line $[D^0(\mathbf{q}\omega)]$ or bare Coulomb line $[V(\mathbf{q})]$, and containing an arbitrarily long chain of bubbles interspersed with D^0 and V's, each of which has a value given by the irreducible polarization part $\Pi(\mathbf{q}\omega)$. (See Fig. 1.) We have assumed for simplicity in notation, that the propagator is scalar, depending on the momentum and energy transferred by the polarization process. As discussed by Migdal,[3] the major contribution to the irreducible polarization part Π comes from electron states far from the Fermi surface (when appreciable momentum $\mathbf{q}$ is involved) and consequently, phonons play no part in determining it. For small frequencies Π will be nearly real.

The property of Π mentioned is discussed in detail in Ref. 3 and in PK when Coulomb forces are absent. It is based on the smallness of the phase space for possible decay products (mainly particle-hole pairs) of energy ω and momentum q, for small ω and appreciable q. The CI cannot increase this phase space unless some unforseen effect results in a low-frequency collective excitation strongly coupled to the polarization disturbances. The induced anharmonic coupling as well as the direct anharmonic terms omitted from the Hamiltonian are small and will be neglected, although lattice anharmonicity can play a significant role, and must be included if the theory is to account for experiments covering a wide temperature range.

In view of the near reality of Π, which by the dispersion relation implies that it is a smooth function of frequency, we can assert that there will be a pole in $D(\mathbf{q}\omega)$ at some frequency of ionic magnitude. This frequency is interpreted as the renormalized, observable phonon frequency, ω_q. The residue at the pole we denote by $2\omega_q\nu_1(\mathbf{q})^2$, defining the new electron-phonon coupling constant, (but still not the fully renormalized coupling parameter which includes all of the proper vertex corrections).

As an example, in the approximation that the ion-ion coupling is purely Coulombic in origin and with longitudinal phonons only, we have

$$|\nu^0(q)|^2=\left|\frac{4\pi Ze^2iq}{q^2}\left(\frac{n}{2M\omega_0}\right)^{1/2}\right|^2=\frac{4\pi e^2}{q^2}\frac{\omega_0}{2}, \quad (6)$$

[16] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin and Company, Inc., New York, 1963).

where ω_0 is the ionic plasma frequency, $(4\pi n(Ze)^2/M)^{1/2}$, and n, Z, and M are the ionic density, valence, and mass, respectively. This gives for the polarization propagator

$$D(\mathbf{q}\omega)=\frac{4\pi e^2}{q^2}\omega^2\Big/\left[\left(1-\frac{4\pi e^2}{q^2}\Pi\right)\omega^2-\omega_0^2\right]$$
$$=\frac{4\pi e^2}{q^2(1-(4\pi e^2/q^2)\Pi)}\left[1+\frac{\omega_q^2}{\omega^2-\omega_q^2}\right], \quad (7)$$

so that the new coupling is

$$2\omega_q\nu_1(\mathbf{q})^2=\frac{4\pi e^2}{q^2}\frac{\omega_q^2}{\epsilon(\mathbf{q}\omega_q)}, \quad (8)$$

with

$$\omega_q^2=\omega_0^2/\epsilon(\mathbf{q}\omega_q)=\omega_0^2\Big/\left[1-\frac{4\pi e^2}{q^2}\Pi(\mathbf{q},\omega_q)\right]. \quad (9)$$

We can identify $\epsilon(\mathbf{q}\omega)$ as the usual electronic dielectric constant. In the general case, one can include the transverse modes as well as the longitudinal and the polarization propagator becomes a matrix which has poles corresponding to the three or more types of phonons, as well as singularities associated with the propagation of light and plasma oscillations.[17] The pole terms can be separated out. The residues give the corresponding coupling constants. The residual term is the ordinary dielectric function (if we drop from consideration the transverse electromagnetic modes which contribute relativistic corrections except at long wavelengths). Actually one must be somewhat more careful and retain the imaginary part of the polarization part which is not usually very important, but which contains the phonon-drag effects. Thus we separate the total polarization propagator into a part, the phonon propagator, whose absorptive part is large only at low frequencies ($\lesssim\omega_D$) and a part which is large at higher frequencies ($\gg\omega_D$). The latter describes the propagation of electronic polarizations.

In the simple example worked out in the preceding paragraph, this separation corresponds to

$$D(\mathbf{q}\omega)=\frac{4\pi e^2}{q^2}\frac{\omega_q^2}{\epsilon(\mathbf{q}\omega_q)}\frac{1}{\omega^2-\omega_0^2/\epsilon(\mathbf{q}\omega)}+\frac{4\pi e^2}{q^2\epsilon(\mathbf{q}\omega)}. \quad (10)$$

The first term is the renormalized phonon propagation and the second term is the usual expression for the Coulomb interaction as modified by dielectric polarization. It is sometimes necessary to retain the frequency dependence and thus the damping in the phonon propagator, as we have done.[8]

We have neglected a term

$$\frac{4\pi e^2}{q^2}\omega_q^2\left[\frac{1}{\epsilon(q\omega)}-\frac{1}{\epsilon(q\omega_q)}\right]\frac{1}{\omega^2-\omega_q^2},$$

which could, if desired, be added to the high-frequency Coulomb contribution, but which in fact is very small in comparison with the terms retained, being a factor of order ω_q^2/μ^2 smaller.

We consequently can reinterpret the usual graphs in terms of the renormalized phonon propagators and electronic polarization propagation [Fig. 1(c)], provided we omit all graphs which would contribute to a polarization part. The latter must of course be omitted in order to avoid counting the polarization twice. Thus, in our example, a wavy line on a graph corresponds to the first term of (10) and a dotted line corresponds to the second term.

With this understanding we undertake an analysis of the electronic self-energy in the presence of the Coulomb forces and phonon interactions. We first describe the result. The result is that the first term in the real part of the self-energy which we denote Σ_c is just the self-energy in the absence of phonons. It has the properties used in the theories of Landau-Fermi liquid, namely, its corresponding imaginary part [see Eq. (2)] is small near the Fermi surface, i.e., for $|E-\mu|\ll\mu$. Indeed for practically all cases it is negligible in comparison with the imaginary part of the phonon-induced term and in fact, except possibly in some of the transition metals (to which our theory does not apply if the d bands are too narrow) this imaginary part has never been observed. No particular restriction is placed on the magnitude of the real part. Because of the smallness of the imaginary part at the Fermi energy, we may use the dispersion relation to prove that near the Fermi energy $\Sigma_c(\mathbf{k},E)$ can be expanded in powers of the energy, i.e.,

$$\Sigma_c(\mathbf{k}E)=\Sigma_c(\mathbf{k},0)+E\Sigma_c'(\mathbf{k},0), \quad (11)$$

where Σ_c' is negative and of order unity and can be taken as independent of the magnitude of the kinetic energy for the same reasons that the phonon contribution can be regarded as independent of this quantity.[3] [We measure all electron energies from the chemical potential μ. The second term of (11) is thus small for the important values of E, but its derivative with respect to energy is not.] The other contribution to the real part of the self-energy we denote by Σ_ϕ. This term has the attributes described in PK, that is, its corresponding imaginary part is relatively large and changes rapidly with energy near the Fermi energy thus giving rise, via the dispersion relation, to a sharp energy dependence of the real part. The magnitude of the real part is by comparison with Σ_c very small and it is only by virtue of its rapid variation with energy that it takes on importance. The momentum dependence is not pronounced on the other hand, so that the momentum can be evaluated at the Fermi surface. Thus, although formally the self-energy depends on four variables, it can be split into terms in which one of the dependences is extremely weak and can be omitted. Another im-

[17] See, J. W. Garland, Phys. Rev. Letters **11**, 1014 (1963); Phys. Rev. **153**, 460 (1967).

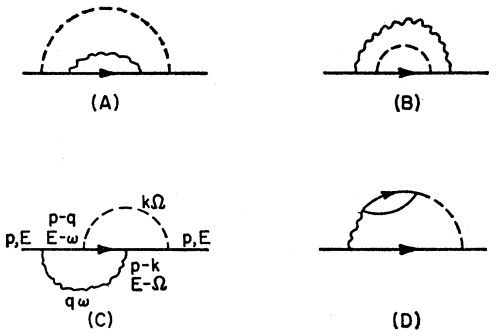

FIG. 2. Typical graphs contributing to the electron self-energy.

portant attribute of this result is that there is no term which deviates in form from a phonon-like term or a Coulomb-type term.

The proof of these assertions is based on assumptions about the self-energy which are true in all orders of perturbation theory and thus must be regarded as arguments in favor of the resulting theory rather than rigorous proofs. There seems to be no insurmountable difficulty in extending the argument to include the breakdown of perturbation theory associated with superconductivity. [This extension has been carried out by W. McMillan (private communication).]

We first consider a graph containing a (renormalized) phonon line. Because the electron-phonon coupling is very weak, special conditions are required to render a graph with phonon lines important. Usually it is stated that the dimensionless electron-phonon coupling parameter is comparable with unity and this is true as that parameter is conventionally defined. Actually of course the coupling which we introduced earlier has dimensions. In comparing the electron-phonon coupling in solids with the electron-electron interactions arising from the Coulomb forces, it may be helpful to normalize the two couplings in the same way.

It has long been recognized that the dimensionless parameter characterizing the strength of the Coulomb interactions is $r_s=(9\pi/4)^{1/3}me^2/k_F$, which is not small at metallic densities. Roughly speaking, each additional Coulomb line appearing in a graph carries with it a factor r_s.

In the most difficult situation to analyze, such as the graphs of Fig. 2(c) or Fig. 4(a), the addition of an extra CI line introduces a factor $e^2/q^2\epsilon(\mathbf{q}\omega)$, where $\mathbf{q}$ is the momentum carried by the line and ω is its energy. There will be two additional energy denominators and an integration over $\mathbf{q}$ and ω must be performed. The volume over which the $\mathbf{q}$ integration is important is comparable with the Fermi surface or Brillouin-zone volume, V_B. The energy range over which the frequency integration is important is comparable with the Fermi energy μ, and typically the energy denominators are of the order of Fermi energies. Thus we have the estimate for the contribution of the CI line, $(e^2/k_F^2)(V_B/\mu)\propto r_s$. Thus, by assuming that the scale of momenta is given by the Fermi momentum and of energy by the Fermi energy, we have arrived at the usual estimate.

The same type of argument can be made in analyzing the contribution of a phonon line. In this case there will be a factor, $|\nu(\mathbf{q})|^2\omega_q/(\omega^2-\omega_q^2)$; integrations over $\mathbf{q}$ and ω, and two additional energy denominators. The frequency integral contributes a factor ω_D. The energy denominators are of the form $[E\pm\omega-\mathcal{E}(\mathbf{p}-\mathbf{q})]$, where $\mathcal{E}(\mathbf{p}-\mathbf{q})$ is the energy of the rest of the graph. In the simplest cases $\mathcal{E}(\mathbf{p}-\mathbf{q})$ is just ϵ_{p-q}. Since ω and E are small ($\lesssim\omega_D$) the size of the energy denominator is determined by $\mathcal{E}(\mathbf{p}-\mathbf{q})$. For most of the $\mathbf{q}$'s in the Brillouin zone, $\mathcal{E}(\mathbf{p}-\mathbf{q})$ will be comparable with the Fermi energy. Thus we have the estimate for the over-all factor associated with the phonon line $|\nu(q)|^2V_B/\mu^2\simeq\omega_D/\mu\ll r_s$. If $\mathcal{E}(\mathbf{p}-\mathbf{q})$ is the energy of a more or less complicated array of particle-hole pairs and CI lines, it will typically be at least as large as μ. If it is simply ϵ_{p-q} (or a renormalized version thereof) there will be a thin shell of $\mathbf{q}$ values such that $\epsilon_{p-q}\lesssim\omega_D$. The volume of this shell is of order $V_B\omega_F/\mu$. Generally, only one of the two energy denominators can be made small in this way.

Thus a region of small volume in momentum space contributes equally with the bulk of the momentum values because of the small energy denominators. The small region is of primary importance however, because it has a rapid variation with energy. This can easily be seen by considering a derivative with respect to E. The derivative in effect squares the energy denominator. [It has to be remarked that it is not enough to have small energy denominators. They must also be predominantly of one sign. The Fermi factors which come into the graphs cause ϵ_{p-q} to be predominantly greater (or less) than zero. As a result for ϵ_{p-q} near the Fermi surface, this condition is fulfilled. If in more complicated situations $\mathcal{E}(\mathbf{p}-\mathbf{q})$ vanishes away from the Fermi surface, the positive and negative contributions from the small energy denominators cancel and no anomalously large contribution results.]

The rapid variation with energy compensates for the over-all smallness of the phonon contribution, since derivatives of the self-energy enter into the renormalization factors. Thus, only a very restricted type of graph containing phonon lines can be of importance. There is no such argument for the Coulomb contributions and in general a large class of graphs must be taken into account.

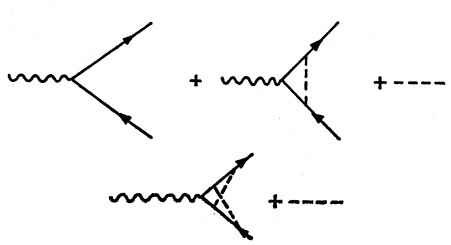

FIG. 3. Graphs contributing to irreducible vertex corrections.

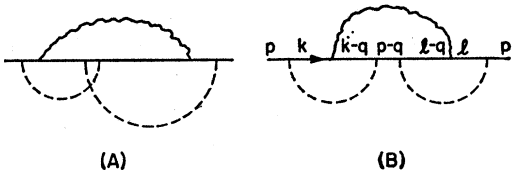

FIG. 4. Higher-order graphs for the electron self-energy.

Following Migdal we will study particular graphs, keeping in mind the possibility of making the energy denominator small and correspondingly restricting the region of the momentum sums. As only the graphs with both phonon and Coulomb lines offer any problem we begin with the simplest. These are of four types as seen in Fig. 2. In A, we have a phonon self-energy insertion into a graph which contributes to the Coulomb self-energy, Σ_C. In B, a Coulomb self-energy corrects the graph which is most important in the self-energy Σ_ϕ. Graph C can be regarded either as a phonon correction to a Coulomb vertex or as a Coulomb correction to a phonon vertex. Since the order of magnitude of the graph is at best ω_D, it must be regarded as a contribution to Σ_ϕ, and we must look for that part of the contribution which has a strong energy dependence. This can be achieved by considering that part of the momentum integrations with values of $\mathbf{p}-\mathbf{q}$ near the Fermi surface. Type D represents the conversion of a Coulomb line into a phonon line and must, by the considerations of the first part of this section, be omitted since it represents a correction to the polarization propagator.

In the calculation of the third contribution, the intermediate energy variable will be close to the external energy E because the phonon carries a low energy. Thus it is sufficient, in calculating the contribution, to obtain the vertex correction on-the-energy-shell and to replace the bare electron-phonon matrix element by a renormalized one. [Of course it is quite true that the Coulomb vertex corrections can contribute to the imaginary part of the self-energy. This contribution will be similar to other CI contributions, which are completely negligible in comparison with the phonon contributions, at low frequencies, and can in consequence be ignored.] Any higher-order Coulomb corrections to graphs with one phonon line fall into the categories: (a) The phonon line can be regarded as renormalizing the propagators in a graph of fundamentally Coulomb type; (b) the Coulomb corrections renormalize the propagators in a phonon type of graph; (c) they can renormalize the electron-phonon matrix element by replacing the electron-phonon vertex by the sum of all graphs of the type given in Fig. 3, or (d), the corrections are negligible.

It remains to show that graphs like Fig. 4(A) do not make a significant contribution. In this figure, it is clear that the contribution cannot be cast into one of the preceding permitted categories, (a)–(c). Because of its complexity, we do not carry out a formal analysis here. We content ourselves with pointing out that the Coulomb lines carry a high energy, and thus one cannot escape large-energy denominators without really serious limitation on the regions of integration. The presence of the phonon line thus renders the graph impotent. Alternatively, we may think of the Coulomb effect as being practically instantaneous. A "time-ordered" graph then shows that the decay through this process involves a particle-hole–pair creation, the very process which lacks the phase space to give a significant Coulomb damping at low energy. Thus, by the dispersion relation, there cannot be a strong energy dependence. Since the order of magnitude of the graph is small because of the phonon line, it can be neglected. The graph of Fig. 4(B) which can be relegated to the permitted category (c) is important because the momentum $p-q$ can be taken close to the Fermi surface. These points have been verified by explicit calculation of low-order graphs.

This completes our discussion of the graphs containing one phonon line. Graphs containing more than one phonon can be handled if the phonons just renormalize internal propagators. Even in this case, however, it is not necessary to keep such graphs.[3] The graphs containing "crossed" phonon lines can be ignored on the basis of the considerations of Migdal's paper,[14] or equivalently by an energy-denominator analysis. The point is that it is not possible to have enough small-energy denominators in the case of crossed phonon lines to compensate for the smallness of the $|\nu(q)|^2$. We thus have the result that the only graphs containing phonon lines which need be retained are those in which the phonon contribution to the fundamental electron-phonon self-energy is renormalized by proper Coulomb vertex corrections and internal self-energy corrections.

III. THE LANDAU-BOLTZMANN EQUATIONS

In this section we concentrate on obtaining the transport equation in the form originally suggested by Landau. The procedure is very close to that used in PK. We focus attention on the electron-distribution function $g^<(\mathbf{k},E,\mathbf{r}t)$ and restrict consideration to the case in which the time variation is slow in comparison with typical lattice times. This restriction is necessary if we are to obtain the usual differential transport equations, but can be removed. The more general equations are a renormalized version of the differential-difference equations first obtained by Holstein.[9] According to Kadanoff and Baym,[17] the function $g^<$ which represents the distribution of electrons of momentum $\mathbf{k}$, energy E in the space-time region around $\mathbf{r}$, t satisfies the equation

$$[E-\epsilon_k-\Sigma(\mathbf{k},\mathbf{r},t),\, g^<]+[\Sigma^<,\,\mathrm{Re}g]=\Sigma^<g^>-\Sigma^>g^<. \quad (12)$$

The generalized Poisson brackets are defined and the functions $\Sigma^>$ and $\Sigma^<$ are related to the self-energy in the way discussed by Kadanoff and Baym.[16] As discussed in PK, $g^<$ in thermal equilibrium has the form $A(\mathbf{k}E)f(E)$ where A is strongly peaked in the variable ϵ_k. For the small deviations away from the thermal equilibrium

which we are considering, A remains large only when ϵ_k is near ϵ_F (for E small). The width of this peak is about Γ in magnitude which is no larger than the Debye frequency. As we consider only the usual case in which all the important state functions are slowly varying in the variable $\mathbf{k}$, the function A can be safely approximated by a δ function as far as the $\epsilon_\mathbf{k}$ integration is concerned. Consequently, we follow PK in integrating Eq. (12) with respect to ϵ_k while keeping E fixed. We deviate slightly from PK in defining the distribution of quasiparticles of energy E and momentum direction $\hat{k}$, (or of position on the Fermi surface) by

$$f(\hat{k},E,\mathbf{r}t)=\left(1+\frac{\partial\Sigma}{\partial\epsilon_k}\right)\int\frac{d\epsilon_k}{2\pi}g^<(\mathbf{k},E,\mathbf{r}t). \tag{13}$$

Unlike the simple case discussed in PK the integral $\int d\epsilon_\mathbf{k}A$ is not 2π but rather $2\pi[1+(\partial\Sigma/\partial\epsilon_\mathbf{k})]^{-1}$. This factor is in principle dependent on space and time to the extent that the electrons deviate from equilibrium. We shall see later that there is no need to take this dependence into account. The linearized equation for f becomes

$$(\Sigma_{c,t}+\Sigma_{\phi,t})\frac{\partial f^0}{\partial E}+(1-\Sigma_{c,E}{}^0-\Sigma_{\phi,E}{}^0)\frac{\partial f}{\partial t}+(\epsilon_{\mathbf{k},\mathbf{k}}+\Sigma_{c,\mathbf{k}}{}^0)\frac{\partial f}{\partial\mathbf{r}}$$
$$-(\Sigma_{\phi,\mathbf{r}}+\Sigma_{c,\mathbf{r}})\frac{\partial f^0}{\partial\mathbf{k}}=(1+\Sigma_{c,\epsilon_\mathbf{k}})\frac{\partial f}{\partial t}\bigg|_{\text{coll.}}. \tag{14}$$

We have introduced the notation $\Sigma_{,t}$, etc., to signify partial derivation with respect to t. We defer the discussion of the collision term. The crucial approximations made in obtaining this equation are, first, the self-energy functions have a weak dependence on k, second, the imaginary part of the self-energy remains small in comparison with the Fermi energy (although not necessarily small compared with the energy kT, ω_D, and E) and of course, the linearization approximation. We now introduce a fictitious kinetic energy variable $\epsilon_k{}^f$ in place of the true energy E by the implicit relation

$$\epsilon_\mathbf{k}{}^f=E-\Sigma_c(\epsilon_\mathbf{k}{}^f,\hat{k},E,\mathbf{r}t)-\Sigma_\phi(\hat{k},E,\mathbf{r},t), \tag{15}$$

and define the distribution in the new "momentum" space by $n(\mathbf{k}^f,\mathbf{r},t)$ where $\mathbf{k}^f$ has the same relationship to $\hat{k}$, $\epsilon_k{}^f$ as $\mathbf{k}$ does to $\hat{k}$, $\epsilon_\mathbf{k}$. Here

$$n(\mathbf{k}^f,\mathbf{r},t)=f(\mathbf{k}^f,E(\mathbf{k}^f,\mathbf{r},t)\mathbf{r},t) \tag{16}$$

and $E(\mathbf{k}^f,\mathbf{r},t)$ solves (15) for given $\epsilon_\mathbf{k}{}^f$. These momentum variables can be regarded simply as momenta even though they have partially lost the significance of being Fourier transforms of space variables. We will also in the future neglect to write the "f" on the momentum. The distribution function is accordingly governed by the Landau-Boltzmann equation

$$\frac{\partial n}{\partial t}+\nabla_\mathbf{k}E_\mathbf{k}\frac{\partial n}{\partial\mathbf{r}}-(\nabla_\mathbf{r}E_\mathbf{k})\frac{\partial n}{\partial\mathbf{k}}=\frac{\partial n}{\partial t}\bigg|_{\text{coll.}}. \tag{17}$$

Furthermore, n is related to the physical currents, charge, density, etc., in the standard way, for example

$$J(\mathbf{r},t)=-e\sum_\sigma\int\frac{d^3k}{(2\pi)^3}n(\mathbf{k},\mathbf{r},t)\nabla_\mathbf{k}E_\mathbf{k}. \tag{18}$$

A complete and formal proof of this latter point is quite nontrivial but we do not regard it as worthwhile to work out this exercise here in view of the arguments presented in PK. We shall not trouble ourselves with the presentation of the derivation of the phonon transport equation and merely remark that the same considerations encountered in PK ensue here. Thus, except for the collision terms which we still have to discuss, the transport equations have the announced form and nearly all of the lore connected with their solutions in particular cases of interest can be taken over verbatim.

IV. THE EFFECTIVE MASS RELATIONS

In this section we continue with the analysis of the dependence of the quasiparticle energy on space and time or more precisely on the distribution of quasiparticles. On the one hand, we can follow Landau[1] and assert that

$$E[\mathbf{k};n(\mathbf{r}t)]=E_\mathbf{k}{}^0+\int\Phi(\mathbf{k}\mathbf{k}')\delta n(\mathbf{k}',rt)\frac{d^3k'}{(2\pi)^3}, \tag{19}$$

which is the defining relation for the fundamental scattering function $\Phi(\mathbf{k}\mathbf{k}')$, where δn is the deviation of the distribution function from the equilibrium distribution $n_0=1/[e^{\beta E_\mathbf{k}{}^0}+1]$. On the other hand, we would like to know the relationship of this function to the corresponding functions which would be obtained in the absence of phonon interactions or in the absence of Coulomb interactions. These simpler functions have been studied and relationships are known for them, giving the effective mass of the electrons in terms of integrals over the scattering function. We can write the gradient of the energy in terms of the self-energy function as

$$\nabla_r E=\nabla_r(\Sigma_c+\Sigma_\phi)/[1-\Sigma_{c,E}-\Sigma_{\phi,E}], \tag{20}$$

where the notation $\Sigma_{c,x}$ signifies a partial derivative evaluated in the equilibrium situation and the relationship (14) is satisfied. We also note that the velocity is given by

$$\mathbf{V}_\mathbf{k}=\nabla_\mathbf{k}E_\mathbf{k}{}^0=(\epsilon_{,\mathbf{k}}+\Sigma_{c,\mathbf{k}})/(1-\Sigma_{c,E}-\Sigma_{\phi,E}). \tag{21}$$

Let us rewrite this as

$$\mathbf{V}_\mathbf{k}=\mathbf{V}_{c\mathbf{k}}Z(\mathbf{k}), \tag{22}$$

where the velocity $V_{c\mathbf{k}}$ is that in the absence of EPI

$$\mathbf{V}_{c\mathbf{k}}=(\epsilon_{,\mathbf{k}}+\Sigma_{c,\mathbf{k}})/(1-\Sigma_{c,E}), \tag{23}$$

and the electron-phonon renormalization factor is

defined by

$$Z(\mathbf{k})^{-1}=1-\Sigma_{\phi,E}/(1-\Sigma_{c,E}). \tag{24}$$

In the isotropic case we have in equilibrium that

$$\mathbf{V}_{c\mathbf{k}}=\epsilon_{,k}\frac{1+\Sigma_{c,\epsilon_\mathbf{k}}}{1-\Sigma_{c,E}}, \tag{25}$$

where $\epsilon_{,\mathbf{k}}$ is the velocity in the absence of any interactions. Thus the velocity or effective mass is renormalized in steps by multiplicative factors. Let us introduce the renormalized phonon self-energy by

$$\Sigma_p(\mathbf{k},E)=\frac{1}{1-\Sigma_{c,E}}\Sigma_\phi(\hat{k},E). \tag{26}$$

To see that this is a natural definition we consider the expression for Σ_ϕ. From PK we have

$$\Sigma_p(E)=\frac{1}{1-\Sigma_{c,E}}\int\frac{dE'}{2\pi}\times[\Sigma_\phi{}^<(E')+\Sigma_\phi{}^>(E')]/(E-E'). \tag{27}$$

In this expression $\Sigma_\phi{}^<(E)$ is for example,

$$\Sigma_\phi{}^<(\mathbf{k}E)=2\pi\int\frac{dE'd^3k'}{(2\pi)^3}|\nu_2(\mathbf{k}\mathbf{k}')|^2g^<(\mathbf{k}',E',\mathbf{r}t)\times[\delta(E-E'+\omega_\mathbf{q})[1+N(\mathbf{q})]+\delta(E-E'-\omega_\mathbf{q})N(-\mathbf{q})]. \tag{28}$$

We have called $N(\mathbf{q})$ the distribution of phonons, $\omega_\mathbf{q}$ the renormalized phonon frequency, and $\nu_2(\mathbf{k}\mathbf{k}')$ the electron-phonon coupling including proper vertex corrections, so that the Coulomb corrections have as we have discussed earlier entered to the extent of renormalizing these functions. We may perform the $\epsilon_\mathbf{k}$ integration, and we find

$$\Sigma_p{}^<=\int\int\frac{dE'}{(2\pi)^3}\frac{d^2S'}{|\mathbf{V}_{c\mathbf{k}'F}|}|\nu_r(\mathbf{k}_F,\mathbf{k}_{F'})|^2f(\mathbf{k}_{F'}E',r,t)\times\{\delta(E-E'+\omega_\mathbf{q})[1+N(\mathbf{q})]+\delta(E-E'-\omega_\mathbf{q})N(-\mathbf{q})\}. \tag{29}$$

In this final expression we have the coupling constant completely renormalized by the Coulomb interactions

$$|\nu_r(\mathbf{k}_F\mathbf{k}_{F'})|^2=|\nu_2(\mathbf{k}_F\mathbf{k}_{F'})|^2[1-\Sigma_{c,E}(\mathbf{k}_F)]^{-1}\times[1-\Sigma_{c,E}(\mathbf{k}_{F'})]^{-1}. \tag{30}$$

An alternative way of writing the differential is

$$d^2S/|\mathbf{V}_{c\mathbf{k}_F}|=d\Omega_kN_c(\hat{k}_F)(2\pi)^3/4\pi, \tag{31}$$

where $N_c(\hat{k}_F)$ is the density of states in angle and energy at the Fermi surface of the electrons under the influence of the Coulomb interaction. Expression (29) is just the same as that given in PK with the density of states and coupling constants renormalized by the Coulomb interaction. Thus the real part of the self-energy is given by

$$\Sigma_p(\mathbf{k},E)=\int_{-\infty}^{\infty}\frac{dE'}{(2\pi)^3}\frac{d^2S'}{|\mathbf{V}_{c\mathbf{k}F'}|}|\nu_r(\mathbf{k}\mathbf{k}')|^2\times f(\mathbf{k}',E',\mathbf{r},t)\frac{2\omega_q}{(E-E')^2-\omega_q{}^2}. \tag{32}$$

[As discussed in PK, it is sometimes convenient to go a step farther and make the replacements $E'\rightarrow E(\mathbf{k}',\mathbf{r}t)$, $f\rightarrow n$, $Z\Sigma_p\rightarrow\Sigma_f$, $Z|\nu_r|^2Z'\rightarrow|\nu_f|^2$, $|ZV_{c\mathbf{k}_F}|\rightarrow|V_k|$, $d^2SdE/|V_k|\rightarrow d^3k$. Then Σ_f is expressed in terms of parameters renormalized by both Coulomb and phonon contributions. Since the collision terms are given in terms of $\Sigma^>$ and $\Sigma^<$, one is enabled to express $\partial n/\partial t|_{\text{coll}}$ exactly as it is expressed in PK with the modification that the EPI matrix elements, etc., are renormalized by both EPI and CI effects. The result can be described as having the form of the Born approximation for phonon emission and absorption with renormalized parameters. A similar result holds for scattering from static imperfections. The electron-electron (CI) scattering term is negligible.]

The Coulomb corrections to the quantities appearing in the expression for Σ_p can in principle depend upon the deviation of the distribution function from equilibrium but we can easily see that this dependence is much weaker than that given by the $\mathbf{r}$, t dependence of f. First of all, the space time dependence of the coupling $\nu_2(\mathbf{k}\mathbf{k}')$ is negligible since the importance intermediate states for the irreducible vertex corrections are far from the Fermi surface where the distribution function is frozen by the exclusion principle. Even beyond this we can see that the explicit dependences arising from the factors $1+\Sigma_{c,\epsilon_k}$ and $1-\Sigma_{c,E}$ can be neglected. This comes about because we can make the estimate $(\delta\Sigma_{c,\epsilon_k}/\delta n)\sim(1/n)$. Since Σ_p has the magnitude ω_D we find that this leads to a contribution to the scattering function of the order of ω_D/μ less than that given by varying $f(\mathbf{k},E)$. The variation with respect to $f(\mathbf{k},E)$ gives a relatively large contribution since it makes the energy denominator in (32) necessarily small.

We now return to an examination of the defining equation for the kinetic energy. Let us denote by $E_{c\mathbf{k}}$ the solution of the problem in the absence of phonon effects which we may obtain by dropping the term Σ_ϕ from Eq. (15). In other words

$$\epsilon_\mathbf{k}=E_{c\mathbf{k}}+\Sigma_c(\epsilon_\mathbf{k},0)+E_{c\mathbf{k}}\Sigma_{c,E}. \tag{33}$$

We may explicitly solve this to find

$$E_{ck}=\frac{\epsilon_k+\Sigma_c(\epsilon_k,0)}{1-\Sigma_{c,E}}. \tag{34}$$

Furthermore, the energy E_k is given by the solution of

$$E_k = E_{ck} + \Sigma_p(E_k), \tag{35}$$

as may be confirmed directly. Thus we have succeeded in expressing these relations as a result of two steps, the first being the solution of the problem in the absence of phonons and the second resulting from the additional phonon interactions. The remaining task is to find the dependence of the distribution function on phonons. Since we are considering the situation in which we integrate first over the kinetic energy variables, the natural dependence of the energy will be on the function f rather than on the momentum space dependent distribution function n. This means that we may expand the function E_c according to

$$\delta E_c = \int \Phi_c{}^0(\mathbf{k},E,\mathbf{k}',E')\delta f(\mathbf{k}'E')\frac{d^2S'}{|\mathbf{V}'_c|}\frac{dE'}{(2\pi)^3}. \tag{36}$$

The problem then is to relate the changes in the distribution function δf to the change in δn. In general this relation will depend on the effect of the phonons, since the fictitious momentum depends upon the true energy in a way which involves the phonon contribution. We obtain the desired relationship by taking the variational derivative of the defining equation

$$n_0 + \delta n = f_0(E(\mathbf{k},\epsilon_\mathbf{k},n_0)) + \delta f(\mathbf{k},E(\mathbf{k},\epsilon_\mathbf{k},n_0)\mathbf{r},t) + \frac{\partial f}{\partial E}\Big|_{E=E(n_0)} \int \frac{\delta E(k,\epsilon_\mathbf{k},n)}{\delta n(\mathbf{k}')}\delta n(\mathbf{k}')d\tau', \tag{37}$$

where

$$d\tau = d^3k/(2\pi)^3. \tag{38}$$

Thus, we find

$$\delta f(\mathbf{k},E(k\epsilon_\mathbf{k}n_0)\mathbf{r},t) = \int d\tau' \Big[(2\pi)^3\delta(\mathbf{k}-\mathbf{k}') - \frac{\partial f}{\partial E}\Big|_{E=E^0} \Phi(\mathbf{k}\mathbf{k}') \Big]\delta n(\mathbf{k}'). \tag{39}$$

In this expression $\Phi(\mathbf{k}\mathbf{k}')$ is the complete scattering function of Landau. If phonons are completely ignored the relation is that for Coulomb interaction alone, and we can denote the scattering function by $\Phi_c(\mathbf{k}\mathbf{k}')$ which satisfies the equation

$$\Phi_c(\mathbf{k}\mathbf{k}') = \Phi_c{}^0(\mathbf{k}\mathbf{k}') - \int \Phi_c{}^0(\mathbf{k}\mathbf{k}'')\frac{\partial f_0}{\partial E''}\Phi_c(\mathbf{k}'',\mathbf{k}')d\tau''. \tag{40}$$

Since in the case of Coulomb interactions the scattering functions are slowly varying in their arguments we may replace $\partial f_0/\partial E$ by $-\delta(E_k-\mu)$ to sufficient approximation. In the isotropic situation the scattering function depends only on the difference of momentum and we may introduce the standard angular momentum decomposition.

$$\Phi(\mathbf{k}-\mathbf{k}') = \Sigma\Phi_l(|\mathbf{k}|,|\mathbf{k}'|)(2l+1)P_l(\cos\Theta_{\mathbf{k}\mathbf{k}'}). \tag{41}$$

Using (40), we find that the angular momentum components of the scattering function are

$$\Phi_{cl} = \Phi_{cl}{}^0/[1-N_c\Phi_{cl}{}^0]. \tag{42}$$

In these terms the effective mass relationship of Landau is

$$\frac{m_c}{m} = 1+N_c\Phi_{c1} = \frac{1}{1-N_c\Phi_{c1}{}^0}. \tag{43}$$

Turning now to the effect of phonons we must find the contribution of Σ_p to the scattering function. This can be done by taking the variation of equation (35). We thus reduce the problem to the computation of the variation $\delta\Sigma_p/\delta n$. Once again we see that according to (32) Σ_p is most easily expressed in terms of the f function and as we have noted before only the explicit dependence is of importance. We thus find that $\delta\Sigma_p/\delta f$ in this case is explicitly available in terms of phonon frequencies and coupling constants. Note that this function is rapidly varying in its energy variables and is independent of temperature (since we have ignored, for the sake of simplicity, lattice anharmonicity). Introducing the notation

$$\Phi_p{}^0(\mathbf{k}\mathbf{k}') = \frac{2\omega_q|\nu_r(\mathbf{k}\mathbf{k}')|^2}{(E_\mathbf{k}-E_{\mathbf{k}'})^2-\omega_q{}^2}, \tag{44}$$

and

$$\Phi_r(\mathbf{k}\mathbf{k}') = Z(\mathbf{k})[\Phi_c{}^0(\mathbf{k}\mathbf{k}')+\Phi_p{}^0(\mathbf{k}\mathbf{k}')]Z(\mathbf{k}'), \tag{45}$$

we find that the scattering function of Landau can be expressed as

$$\Phi(\mathbf{k}\mathbf{k}') = \Phi_r(\mathbf{k}\mathbf{k}') + \int \Phi_r(\mathbf{k}\mathbf{k}'')\Big(-\frac{\partial n_0}{\partial E_k''}\Big) \times \Phi(\mathbf{k}''\mathbf{k}')d\tau'', \tag{46}$$

where a factor of $Z(\mathbf{k}')$ appears in the conversion of energy variable to momentum variables in the integration. This is an integral equation which can be solved explicitly at sufficiently low temperatures in the isotropic case since the derivative may be approximated by a δ function. At higher temperatures, because of the strong energy variation of $Z(\mathbf{k})$, Φ_p and consequently of Φ_r, the equation is not solvable explicitly. Just as in PK, we can determine the renormalization factor $Z(\mathbf{k})$ which converts the intermediate mass m_c into the effective mass m^* from the scattering function $\Phi_p(\mathbf{k}\mathbf{k}')$. Since $Z(\mathbf{k})$ is equal to

$$Z(\mathbf{k}) = \frac{1}{1-\Sigma_{p,E}}, \tag{47}$$

we can use the explicit expression for Φ_p^0 to find

$$Z(\mathbf{k})^{-1}=1+\int \frac{d\Omega_{k'}}{4\pi}dE'N_c'\Phi_p^0(\mathbf{k}E,\mathbf{k}'E')\left(-\frac{\partial f_0}{\partial E'}\right). \quad (48)$$

An alternative expression which is sometimes useful is to write the equation for $Z(\mathbf{k})$ in terms of Φ_p, the solution of (45) with Φ_c^0 put to zero. The result is, in the isotropic case, at low temperature,

$$(m^*/m_c)=(1/Z)=1+N^*\Phi_{p0}=1+N_c\Phi_{p0}^0. \quad (49)$$

We have denoted by N^* the observable and fully renormalized density of electron states. Note that the isotropic average $l=0$ enters the expression, rather than the $l=1$ contribution. Thus we have exhibited the expression for the scattering function of Landau in terms of more elementary expressions and have found an integral equation which can be solved. The equations connecting the effective mass with the scattering function are replaced by Eqs. (43) and (49) which unfortunately are not in terms of the final scattering function Φ. To that extent the formalism has lost a simple relationship.

V. PROPERTIES OF THE METALLIC FERMI LIQUID

With the aid of the theory just developed it is possible to find systematically the CI and EPI corrections to any calculation of the low-frequency properties of a metallic electron-phonon system. In nearly all individual instances the results are well-known. We divide the corrections up into a number of categories.

A. Corrections to Fermi surface shape: No EPI effects, but possible CI effects.[15] No change of Fermi surface volume.[18] Same Fermi surface is determined by every experiment.

B. Corrections to phonon dispersion: No EPI effects (in the sense of Sec. II) but CI effects. Same phonon dispersion in every experiment.

C. Corrections to EPI matrix elements. These are fully corrected by CI with both wavefunction, $(1-\Sigma_{c,E})^{-1}$, and proper vertex renormalizations. There is also an EPI wave-function renormalization which generally cancels out in the static limit against the mass renormalization.[8]

D. Corrections to other electron matrix elements.[15] The same remarks apply here as in case C. It should be noted that certain relationships which hold in treating independent electrons in the Born approximation do not hold after CI renormalization. For example, the matrix elements do not cancel out of the Korringa ratio after renormalization. (However, estimates of the proper vertex correction[19] indicate it is approximately independent of momentum transfer; which result reintroduces the simplifications.)

E. Corrections to electron mass. CI introduces corrections (usually believed to be numerically small[19]) and EPI causes additional multiplicative effect. If the mass is a tensor, the EPI renormalization multiplies the whole tensor by a scalar (which can be dependent on position on Fermi surface, and distance from the surface). This EPI mass renormalization is equal to the EPI wave-function renormalization. This fact causes the EPI corrections to cancel from all mobilities,[8] i.e.,

$$\frac{\tau^*}{m^*}\propto\frac{1}{m^*\langle N^*(E_f)|M^*|^2\rangle}\propto\frac{1}{m_c\langle N_c|M_c|^2\rangle}=\frac{\tau_c}{m_c}.$$

A general argument showing that EPI effects disappear from quasi-steady-state transport coefficients was given in PK.

Note that $\omega\tau^*$ is affected by EPI, whereas $V_F^*\tau^*$ is not. Thus, mean-free-time effects have to be distinguished from mean-free-path effects in this sense.

The mass renormalization is also related to the scattering function, the results of which will be discussed below.

Finally, by raising the temperature,[20,21] the frequency,[22] or the magnetic field,[23] it is in principle possible to observe changes in the EPI contribution to the mass. So far such experiments have not been found feasible, and so all mass determinations lead to the same m^* (in which the phonon corrections are usually quite significant). In these proposed experiments, there is no dependence on Φ.

F. Corrections arising from the Landau scattering function, Φ. These corrections are by far the most evasive. Azbel[24,25] has given a discussion analysing the difficulties encountered in attempting an experimental determination of the function. The main problem is that at low frequencies there is no correction, whereas at higher frequency the skin depth becomes so small that the distribution function is changed only for a small fraction of the electrons in an "effective band," which limits the contribution of the scattering function.

We discuss several cases. The first is that of the susceptibility or $\partial n/\partial\mu$ which determines the screening length. In this case CI effects remain, but EPI effects cancel out. Although quite simple arguments suffice to obtain this result,[26] it may be of interest to demonstrate the calculation within the framework of our formalism.

[18] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960).
[19] T. M. Rice, Ann. Phys. (N. Y.) **31**, 100 (1965).
[20] T. Krebs, Phys. Letters **6**, 31 (1963).
[21] R. E. Prange and S. Maitra, University of Maryland Technical Report, 1965 (unpublished).
[22] H. Scher and T. Holstein, Phys. Rev. **148**, 598 (1966).
[23] M. Fowler and R. E. Prange, Physics **1**, 3151 (1965).
[24] M. Ya. Azbel, Zh. Eksperim. i Teor. Fiz. **39**, 1138 (1960) [English transl.: Soviet Phys.—JETP **12**, 793 (1961)].
[25] A. Ya. Blank and V. L. Falko, Zh. Eksperim. i Teor. Fiz. **48**, 742 (1965) [English transl.: Soviet Phys.—JETP **21**, 490 (1965)].
[26] C. Herring, in *Magnetism*, edited by G. T. Rado and H. Suhl (Academic Press Inc., New York, 1966), Vol. IIb.

We calculate here the quantity $\partial n/\partial\mu$ which gives the change of density with chemical potential. The case of the spin susceptibility is very similar. According to Landau,[1] $\partial n/\partial\mu$ is, in the isotropic case,

$$(\partial n/\partial\mu)=2N^*/[1+N^*\Phi_0]. \tag{50}$$

Inserting the solutions for the scattering function we obtain

$$(\partial n/\partial\mu)=2N_c/(1+N_c\Phi_{c0}), \tag{51}$$

which has been simplified by using expressions (45), (46), and (49). Thus it is seen that $\partial n/\partial\mu$ is independent of phonon effects. It is possible to extend this calculation to the anisotropic case. Let us assume that the chemical potential changes and the Fermi surface is shifted at each point by $\delta\mathbf{k}_F$ normal to the surface. Then we have

$$\delta n=\frac{2}{(2\pi)^3}\int d^2S\,|\delta\mathbf{k}_F|\,. \tag{52}$$

Thus we find that the shift in chemical potential is

$$\delta\mu=\nabla E_{\mathbf{k}}{}^0\cdot\delta\mathbf{k}_F+\int\Phi(\mathbf{k}_F\mathbf{k}_F')\,|\delta\mathbf{k}_F'|\frac{d^2S}{(2\pi)^3}. \tag{53}$$

We obtain the equation

$$\frac{|\delta\mathbf{k}_F|}{\delta\mu}=\frac{1}{|\mathbf{V}_k|}-\frac{1}{|\mathbf{V}_k|}\int\Phi(\mathbf{k}_F\mathbf{k}_{F'})\frac{|\delta\mathbf{k}_{F'}|}{\delta\mu}\frac{d^2S'}{(2\pi)^3}. \tag{54}$$

We now notice that the solution to this equation is

$$\frac{|\delta\mathbf{k}_F|}{\delta\mu}=\frac{Z(\mathbf{k})}{|\mathbf{V}_k|}\left(1-\int\Phi_c{}^0(\mathbf{k}\mathbf{k}')\frac{Z(\mathbf{k}')}{|\mathbf{V}_{k'}|}\frac{d^2S'}{(2\pi)^3}\right), \tag{55}$$

which follows by substitution and use of Eq. (46). Therefore we see that $\delta k_F/\delta\mu$ is given by an expression independent of the phonon interactions and $\delta n/\delta\mu$ is given by the symmetrical expression

$$\frac{1}{2}\frac{\partial n}{\partial\mu}=\int\frac{1}{(2\pi)^3}\frac{d^2S}{|\mathbf{V}_{kc}|}-\int\frac{d^2Sd^2S'}{(2\pi)^6}\frac{1}{|\mathbf{V}_{kc}|}\times\Phi_c{}^0(\mathbf{k}\mathbf{k}')\frac{1}{|\mathbf{V}_{k'c}|}, \tag{56}$$

where the first term is the density of states in the absence of phonons.

We note that the similar argument for the spin susceptibility depends on the EPI being diagonal in spin, which may not be an adequate approximation in the heavier metals.

Other methods which may lead to information about the Φ function require somehow getting electromagnetic signals to penetrate at high frequencies to avoid the "effective zone" result mentioned earlier. Let us consider the *Gedanken* experiment of cyclotron resonance in a spatially constant field, which is of interest because it presents a case in which CI effects ought to cancel out in the absence of lattice effects. A simple analysis shows that the resonance frequency is $\omega_c{}^*[1+N^*\Phi_1]$ where $\omega_c{}^*$ is eH/m^*c as would be measured in the Azbel-Kaner[27] situation. The expression can be rewritten as

$$\omega_{\rm res}=(eH/mc)[1+(m_cN_c/m)(\Phi_{p0}{}^0-\Phi_{p1}{}^0)]^{-1}. \tag{57}$$

We see that it is by virtue of the EPI interaction that there remain some Coulomb effects.

The more realistic case of cyclotron resonance in the Galt[28] geometry has been considered by Platzman and Jacobs[29] who find that no very simple result is to be expected.

Recently, Walsh and Platzman[30] have found a region of magnetic field and frequency for which a metal is partially transparent, thus allowing a new type of mode to propagate. The dispersion relation of this mode depends on Φ. The analysis of the mode may lead to the partial determination of Φ in certain cases. Even more recently, Schultz and Dunifer[31] have observed penetrating waves which are believed to be spin waves, existing by virtue of a spin-dependent part of Φ. This experiment provides the best evidence so far available for the existence of Φ.

Thus in summary, we have found that the Landau-Fermi liquid theory holds for the electron phonon system in the presence of Coulomb interactions. Certain relations between the scattering function, effective mass, and wave-function renormalizations exist which allow simplifications in special cases. The effect of the phonons can be explicitly disentangled if the renormalized phonon spectrum and EPI couplings are known. The whole theory remains valid even at temperatures comparable with the Debye temperature, but in this case the Landau function Φ has significant dependence on the magnitudes of its arguments, as well as on their directions. Phonon renormalization effects cancel out completely in many instances, including nearly all experiments feasible near the Debye temperature. The renormalizations arising from the CI are not so prone to cancellation, so that nearly all conceivable experiments at frequencies below the optical, depend on the same Coulomb renormalized matrix elements, which can then be regarded as the fundamental parameters of the material.

[27] M. Ya. Azbel and E. A. Kaner, Zh. Eksperim. i Teor. Fiz. **30**, 811 (1956) [English transl.: Soviet Phys.—JETP **3**, 772 (1956)].

[28] J. K. Galt, W. A. Yager, F. R. Merritt, B. B. Cetlin, and A. D. Brailsford, Phys. Rev. **114**, 1396 (1959).

[29] P. M. Platzman and K. C. Jacobs, Phys. Rev. **134**, A974 (1964).

[30] W. M. Walsh, Jr. and P. M. Platzman, Phys. Rev. Letters **15**, 784 (1965).

[31] S. Schultz and G. Dunifer, Phys. Rev. Letters **18**, 283 (1967); P. M. Platzman and P. A. Wolff, Phys. Rev. Letters **18**, 280 (1967).