Correlation Functions for the Two-Dimensional Ising Model*

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The energy-density–energy-density and energy-density–spin correlation functions are evaluated for the two-dimensional square Ising model. Results are obtained at zero magnetic field and near the critical point, i.e., for

$$\epsilon=(4J/kT)\mid(T-T_c/T_c)\mid\ll 1.$$

The correlation functions are

$$\langle E_RE_0\rangle-\langle E\rangle^2=(2J/\pi)^2\epsilon^2[K_1^2(\epsilon R)-K_0^2(\epsilon R)],$$

$$\langle E_R\sigma_0\rangle-\langle E\rangle\langle\sigma\rangle=(2J/\pi)\epsilon\langle\sigma\rangle\int_{2\epsilon R}^{\infty}ds\, s^{-2}e^{-s},$$

where R is the distance between the points in question and K_0 and K_1 are modified Bessel functions of the second kind. Very similar results hold if the coupling constants are different in the two directions. The above results agree in form with the structure predicted by the "scaling-law" approach.

I. INTRODUCTION

RECENT theoretical work on critical-point phenomena has emphasized a series of relations connecting various critical indices.[1] Kadanoff has introduced an Ising-type model from which he has shown how to deduce these relations, which he refers to as scaling laws.[2] A wide variety of experimental data confirms the approximate validity of these scaling laws for real physical systems.[3] The model also predicts the functional form of various spin-correlation functions. It is our purpose here to calculate two such functions, the energy-density–energy-density and the energy-density–spin-correlation functions, for the two-dimensional Ising model and compare with the functional forms predicted.

In Sec. II we derive the relevant scaling-law predictions. Sections III and IV present the actual calculations.

II. SCALING-LAW FORM FOR CORRELATION FUNCTIONS

The Ising-model Hamiltonian is

$$H=-\tfrac{1}{2}\sum_{r,r'}J(r,r')\sigma_r\sigma_{r'}-\sum_r h\sigma_r=\sum_r(E_r-h\sigma_r),\qquad(2.1)$$

where $J(r,r')$ vanishes except for nearest neighbors, and E_r is the energy density at point r. It is also convenient to introduce ϵ, a dimensionless measure of the interaction near T_c, as

$$\epsilon=(4J/kT)\mid T-T_c\mid/T_c.\qquad(2.2)$$

In Kadanoff's "derivation" of the scaling laws, the Ising model is discussed in terms of interacting cells of spins. The cell variables α, μ_α, $\tilde{h}$, $\tilde{J}$, and $\tilde{\epsilon}$ correspond to the lattice site variables r, σ_r, h, J, and ϵ. We let the number of nearest neighbors equal z, and let $\sigma_{r+1}(\mu_{\alpha+1})$ mean a nearest neighbor to $\sigma_r(\mu_\alpha)$.

A change in coupling constant and magnetic field will change the free energy by

$$\delta F=-z\delta J\sum_r\langle\sigma_r\sigma_{r+1}\rangle-\delta h\sum_r\langle\sigma_r\rangle$$
$$=-z\delta\tilde{J}\sum_\alpha\langle\mu_\alpha\mu_{\alpha+1}\rangle-\delta\tilde{h}\sum_\alpha\langle\mu_\alpha\rangle.\qquad(2.3)$$

The sum over r can be replaced by a sum over α times the number of sites per cell, L^d, where L is a cell length and d is the dimensionality of the lattice.

It is postulated that the site and cell problems scale according to

$$\tilde{h}=hL^x,\qquad\tilde{\epsilon}=\epsilon L^y,\qquad\tilde{J}=JL^y.\qquad(2.4)$$

For the purpose of calculating critical fluctuations, (2.3) and (2.4) lead to the following identifications:

$$\sigma_r=L^{x-d}\mu_\alpha,\qquad\sigma_r\sigma_{r+1}=L^{y-d}\mu_\alpha\mu_{\alpha+1}.\qquad(2.5)$$

Thus, for $\epsilon\rightarrow 0$ and large spin separation $R=|r-r'|$, we can write the spin-spin, energy-density–energy-density, and energy-density–spin-correlation functions as

$$f_{\sigma\sigma}(\epsilon,h,R)=\langle\sigma_r\sigma_{r'}\rangle=L^{2(x-d)}\langle\mu_\alpha\mu_{\alpha'}\rangle=L^{2(x-d)}f_{\sigma\sigma}(\tilde{\epsilon},\tilde{h},R/L),$$
$$f_{EE}(\epsilon,h,R)=\langle\sigma_r\sigma_{r+1}\sigma_{r'}\sigma_{r'+1}\rangle=L^{2(y-d)}\langle\mu_\alpha\mu_{\alpha+1}\mu_{\alpha'}\mu_{\alpha'+1}\rangle=L^{2(y-d)}f_{EE}(\tilde{\epsilon},\tilde{h},R/L),$$
$$f_{E\sigma}(\epsilon,h,R)=\langle\sigma_r\sigma_{r+1}\sigma_{r'}\rangle=L^{x+y-2d}\langle\mu_\alpha\mu_{\alpha+1}\mu_{\alpha'}\rangle=L^{x+y-2d}f_{E\sigma}(\tilde{\epsilon},\tilde{h},R/L).\qquad(2.6)$$

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[1] For the definitions of the critical indices, see, e.g., M. E. Fisher, Natl. Bur. Std. (U.S.) Misc. Publ. **273**, 21 (1966).
[2] L. P. Kadanoff, Physics **2**, 263 (1966). Alternative theoretical arguments for the scaling laws have been advanced by M. E. Fisher, University of Kentucky Centennial Conference on Phase Transformations 1965, (to be published); B. Widom, J. Chem. Phys. **43**, 3898 (1965); A. Z. Patashinskii and V. L. Pokrovskii, Zh. Eksperim. i Teor. Fiz. **50**, 439 (1966) [English transl.: Soviet Phys.—JETP **23**, 292 (1966)].
[3] L. P. Kadanoff *et al.*, Rev. Mod. Phys. **39**, 2 (1967).

The last equality in (2.6) arises because the μ_α's describe essentially the same problem as the σ_r's, so the μ_α-correlation functions should have the same functional form as the σ_r-correlation functions, but with variables appropriate to the cell problem.

If we substitute (2.4) into (2.6) and furthermore demand that the σ_r-correlation functions be independent of the artificially introduced cell length L, we find that $f_{\sigma\sigma}$, f_{EE}, and $f_{E\sigma}$ should have the forms

$$f_{\sigma\sigma}=\epsilon^{2(d-x)/y}f_1(\epsilon^{1/y}R,\ \epsilon h^{-y/x}),$$
$$f_{EE}=\epsilon^{2(d-y)/y}f_2(\epsilon^{1/y}R,\ \epsilon h^{-y/x}),$$
$$f_{E\sigma}=\epsilon^{(2d-x-y)/y}f_3(\epsilon^{1/y}R,\ \epsilon h^{-y/x}). \tag{2.7}$$

For the two-dimensional Ising model, $x=15/8$ and $y=1$, so that (2.7) becomes, in zero magnetic field,

$$f_{\sigma\sigma}=\epsilon^{1/4}f_1(\epsilon R), \tag{2.8a}$$
$$f_{EE}=\epsilon^2 f_2(\epsilon R), \tag{2.8b}$$
$$f_{E\sigma}=\epsilon^{9/8}f_3(\epsilon R). \tag{2.8c}$$

Equation (2.8a) has been verified previously.[4] In Secs. III and IV of this paper we verify (2.8b) and (2.8c).

III. ENERGY-DENSITY–ENERGY-DENSITY CORRELATION FUNCTION

It is well known[5] that the two-dimensional Ising model can be reduced to a problem involving noninteracting fermions. We use this reduction in the particular form developed by Kadanoff.[6] He used a set of fermion variables b_{jk+} and b_{jk-} which refer in some sense to the lattice point (j, k). In terms of these variables, the energy density

$$E_{j,k}=J\sigma_{jk}\sigma_{j+1,k}+J'\sigma_{jk}\sigma_{j,k-1} \tag{3.1}$$

can be written as

$$E_{jk}=J[\cosh 2K^*+(2/i)\sinh 2K^* b_{j,k+1,+}b_{jk-}]+(2/i)J'b_{jk+}b_{jk-}. \tag{3.2}$$

For generality, we consider different horizontal and vertical coupling constants.

We seek to evaluate the correlation function

$$f_{EE}(\epsilon, h=0, R)=\langle E_{jk}E_{00}\rangle-\langle E_{jk}\rangle\langle E_{00}\rangle, \tag{3.3}$$

where R is the distance between $(0,0)$ and (j,k). Clearly expression (3.3) involves products of four b operators. However, this is a problem of noninteracting fermions. In this kind of problem it is always true that for fermion operators with different indices,

$$\langle bb'b''b'''\rangle=\langle bb'\rangle\langle b''b'''\rangle-\langle bb''\rangle\langle b'b'''\rangle+\langle bb'''\rangle\langle b'b''\rangle. \tag{3.4}$$

We can use (3.4) to simplify (3.3). A further simplification in f_{EE} occurs for large R, for then the difference between k and $k+1$ in terms such as $\langle b_{jk+}b_{00-}\rangle$ and $\langle b_{j,k+1,+}b_{00-}\rangle$ is negligible. These simplifications enable us to write (3.3) as

$$f_{EE}(\epsilon,R)=4(J\sinh 2K^*+J')^2\times[\langle b_{jk+}b_{00+}\rangle\langle b_{jk-}b_{00-}\rangle-\langle b_{jk+}b_{00-}\rangle\langle b_{jk-}b_{00+}\rangle]. \tag{3.5}$$

In evaluating the right-hand side of (3.5) it is convenient to consider the four expectation values as the four components of a matrix:

$$g(jk)=\begin{pmatrix}\langle(b_{jk+}b_{00+})_+\rangle & \langle(b_{jk+}b_{00-})_+\rangle\\ \langle(b_{jk-}b_{00+})_+\rangle & \langle(b_{jk-}b_{00-})_+\rangle\end{pmatrix}. \tag{3.6}$$

Here the $(\ )_+$ means that the b operators are to be ordered with respect to j with the largest value of j on the right, and a minus sign is introduced if the b's have to be interchanged.

Thus, (3.5) involves the determinant of this matrix. The components of $g(jk)$ were evaluated in Ref. 4 where it was shown that

$$g(jk)=\int_{-\pi}^{\pi}\frac{dp_x}{2\pi}\int_{-\pi}^{\pi}\frac{dp_y}{2\pi}\exp(ijp_x)\exp(ikp_y)G(p_x,p_y), \tag{3.7}$$

with

$$G(p_x,p_y)=q(p_y)/[q(p_y)-\exp(ip_x)]. \tag{3.8}$$

It is convenient to separate the p_x dependence from the 2×2 matrix dependence. This is done in Appendix 1, where it is shown that

$$g(jk)=\int_{-\pi}^{\pi}\frac{dp}{2\pi}\exp(ipk)\exp[-\gamma(p)\,|j|]\bar{A}(p)\quad \text{for } j<0, \tag{3.9}$$

and $\gamma(p)$ and the 2×2 matrix $\bar{A}(p)$ are defined in Appendix 1. Since we are interested in $g(jk)$ for small ϵ and large spin separation, i.e., large $|j|$, the main contribution to $g(jk)$ will come when $\gamma(p)\ll 1$. Thus, we approximate $\gamma(p)\approx\sinh 2K'(\epsilon^2+p^2)^{1/2}$ and extend the limits of integration to $\pm\infty$. We then shift the contour of integration through the substitution

$$p=\epsilon\sinh[u+i\tan^{-1}(k/|j|\sinh 2K')], \tag{3.10}$$

and we find that the four components of $g(jk)$ may be expressed in terms of the modified Bessel functions of the second kind, K_0 and K_1, where

$$K_n(x)=\int_0^\infty \exp(-x\cosh u)\cosh nu\,du. \tag{3.11}$$

Specifically we obtain[7]

$$f_{EE}(\epsilon,h=0,R)=[(J\sinh 2K^*+J')^2/\pi^2]\epsilon^2[K_1^2(\epsilon R)-K_0^2(\epsilon R)], \tag{3.12}$$

[4] J. P. Kadanoff, Nuovo Cimento **44B**, 276 (1966).
[5] T. Schultz, D. Mattis, and E. Lieb, Rev. Mod. Phys. **36**, 856 (1964).
[6] See Ref. 4. Throughout the rest of this paper, we use the notation and many of the results of Ref. 4.
[7] J. Stephenson [J. Math. Phys. **7**, 1123 (1966)] has used Pfaffians to derive the limiting forms (3.14) along a row of the square lattice.

where

$$R^2 = j^2 \sinh^2 2K' + k^2. \quad (3.13)$$

We may replace K^* and K' in (3.12) by $K_c = J/kT_c$. When the vertical and horizontal coupling constants are equal, $\sinh 2K_c = 1$ and the complete circular symmetry of f_{EE} is apparent. Expression (3.12) verifies the scaling-law prediction, Eq. (2.8b).

For very large R, i.e., $\epsilon R \gg 1$,

$$f_{EE} \approx \frac{1}{2\pi} (J \sinh 2K_c + J')^2 \frac{e^{-2\epsilon R}}{R^2}, \quad (3.14a)$$

whereas for very small ϵ, i.e., $\epsilon R \ll 1$,

$$f_{EE} \approx [(J \sinh 2K_c + J')^2/\pi^2](1/R^2). \quad (3.14b)$$

We may use (3.12) to calculate the specific heat near T_c. The specific heat per site is

$$C = (1/kT^2) \sum_{(j,k)} [\langle E_{jk} E_{00} \rangle - \langle E_{jk} \rangle \langle E_{00} \rangle]. \quad (3.15)$$

We expect that the main contribution to this summation over (j, k) will come from large R. Then we may use (3.12) and also replace the summation in (3.15) by an integration over R, where the lower limit is $R_{\min} \gtrsim 1$. Then the specific heat is

$$C = \frac{2\pi}{kT^2 \sinh 2K'} \int_{R_{\min}}^{\infty} R dR f_{EE}(\epsilon R)$$

$$= \frac{2\pi}{kT^2 \sinh 2K'} \frac{(J \sinh 2K^* + J')^2}{\pi^2} \int_{\epsilon R_{\min} << 1}^{\infty} x dx [K_1^2(x) - K_0^2(x)]$$

$$= \frac{-2(J \sinh 2K^* + J')^2}{\pi k T^2 \sinh 2K'} [(\epsilon R_{\min})^2 K_1^2(\epsilon R_{\min}) - (\epsilon R_{\min})^2 K_0^2(\epsilon R_{\min}) - \epsilon R_{\min} K_0(\epsilon R_{\min}) K_1(\epsilon R_{\min})]. \quad (3.16)$$

Since $\epsilon R_{\min} \ll 1$, we can approximate

$$K_1(\epsilon R_{\min}) \approx 1/\epsilon R_{\min},$$
$$K_0(\epsilon R_{\min}) \approx -\ln(\epsilon R_{\min}). \quad (3.17)$$

Then expression (3.16) simplifies to

$$C \approx \frac{-2(J \sinh 2K^* + J')^2}{\pi k T_c^2 \sinh 2K_c} \ln(\epsilon R_{\min})$$

$$\approx \frac{2(J \sinh 2K^* + J')^2}{\pi k T_c^2 \sinh 2K_c} \ln \epsilon^{-1}, \quad (3.18)$$

which is Onsager's famous result.[8]

IV. ENERGY-DENSITY–SPIN CORRELATION FUNCTION

This correlation function is

$$f_{E\sigma} = \langle E_{jk} \sigma_{00} \rangle - \langle E_{jk} \rangle \langle \sigma_{00} \rangle$$
$$= (2/i) J \sinh 2K^* \langle b_{j,k+1,+} b_{jk-} \sigma_{00} \rangle$$
$$+ (2/i) J' \langle b_{jk+} b_{jk-} \sigma_{00} \rangle$$
$$- (2/i) J \sinh 2K^* \langle b_{j,k+1,+} b_{jk-} \rangle \langle \sigma_{00} \rangle$$
$$- (2/i) J' \langle b_{jk+} b_{jk-} \rangle \langle \sigma_{00} \rangle. \quad (4.1)$$

As $\epsilon \to 0$ and $R \to \infty$, the difference between

$$[\langle b_{j,k+1,+} b_{jk-} \sigma_{00} \rangle - \langle b_{j,k+1,+} b_{jk-} \rangle \langle \sigma_{00} \rangle]$$

and

$$[\langle b_{jk+} b_{jk-} \sigma_{00} \rangle - \langle b_{jk+} b_{jk-} \rangle \langle \sigma_{00} \rangle] \quad (4.2a)$$

is negligible (see Appendix 2), so that (4.1) becomes

$$f_{E\sigma} = (2/i)(J \sinh 2K^* + J')$$
$$\times [\langle b_{jk+} b_{jk-} \sigma_{00} \rangle - \langle b_{jk+} b_{jk-} \rangle \langle \sigma_{00} \rangle]. \quad (4.2b)$$

To evaluate these expectation values we borrow another trick from Ref. 4. Consider a generalization of the matrix in Eq. (3.6), so that, e.g., the upper left-hand corner is

$$g'(jk+j'k'+) = \langle (b_{jk+} b_{j'k'+} \sigma_{00})_+ \rangle / \langle \sigma_{00} \rangle \quad (4.3)$$

and similarly for the other three elements. Above T_c, expression (4.3) is 0/0 indeterminant. Below T_c, $\langle \sigma_{00} \rangle$ does not vanish, so that (4.3) is well defined then. In terms of g' and g, (4.2b) becomes

$$f_{E\sigma} = (2/i)(J \sinh 2K_c + J')$$
$$\times [g'(jk+jk-) - g(jk+jk-)] \langle \sigma_{00} \rangle. \quad (4.4)$$

Since the b's obey an equal j anticommutation relation, we can write

$$g'(jk+jk-) - g(jk+jk-)$$
$$= (-i/2) \operatorname{Tr}\tau_2 [g'(jkjk) - g(jkjk)], \quad (4.5)$$

where Tr means trace over the 2×2 matrix. Then (4.4) may be written as

$$f_{E\sigma} = -\langle \sigma \rangle (J \sinh 2K_c + J') \operatorname{Tr}\tau_2 [g'(jkjk) - g(jkjk)]. \quad (4.6)$$

We proceed to express g' in terms of g by writing an equation of motion for g'. This is done in a manner exactly analogous to Ref. 4, and we obtain

$$g'(j+1, k, j', k')$$
$$= \sum_{\bar{k}} q(k - \bar{k}) [1 - 2\eta_1(j\bar{k})] g'(j\bar{k}j'k')$$
$$- \delta_{jj'} q(k - k') [1 - 2\eta_1(jk')]. \quad (4.7)$$

[8] L. Onsager, Phys. Rev. **65**, 117 (1944).

Here

$$\eta_1(jk)=1 \quad \text{if } j=0 \text{ and } k\leq 0$$
$$=0 \quad \text{otherwise.} \qquad (4.8)$$

For the following calculations we want to pass to another matrix notation. Thus, we consider g' as a very large matrix whose rows are specified by (jk) and columns by $(j'k')$. Each element is actually a 2×2 matrix. If $a(jkj'k')$ is an element of another such matrix, then matrix multiplication is

$$\sum_{\bar{j}\bar{k}} g'(jk\bar{j}\bar{k})a(\bar{j}\bar{k}j'k')=(g'a)_{jkj'k'}. \qquad (4.9)$$

The matrices q and η have matrix elements

$$q(jkj'k')=\delta_{jj'}q(k-k'),$$
$$\eta(jkj'k')=\delta_{jj'}\delta_{kk'}\eta_1(jk). \qquad (4.10)$$

We also define a translation operator T such that

$$(Tg')_{jkj'k'}=g'(j+1,k,j',k'). \qquad (4.11)$$

Then Eq. (4.7) becomes

$$Tg'=q(1-2\eta)g'-q(1-2\eta). \qquad (4.12)$$

From Ref. 4, the same procedure for g yields

$$Tg=qg-q \quad \text{or} \quad g=-(T-q)^{-1}q. \qquad (4.13)$$

Rearranging (4.12) and multiplying on the left by $(T-q)^{-1}$ yields

$$g'=2g\eta g'+g-2g\eta$$

or

$$g'=(1-2g\eta)^{-1}(g-2g\eta). \qquad (4.14)$$

We can make a unitary transformation in (4.6) without changing the value of the trace.[9] Then we can use all the results already derived in Ref. 4, Sec. 3. The main idea is to factorize the matrix $\eta g\eta$ in terms of matrices c_0 and c_i which have certain important analytic properties. This factorization is the Wiener-Hopf technique for solving this problem. Here we simply list those results necessary for this calculation. They are

$$g(jkj'k')=\int_{-\pi}^{\pi}\frac{dp}{2\pi}\exp[-\gamma\,|j-j'|+ip(k-k')]$$
$$\times\tfrac{1}{2}[1+c_0^{\tau_3}(p)c_i^{-\tau_3}(p)\tau_2] \qquad \text{for } j\leq j'$$
$$\times -\tfrac{1}{2}[1-c_0^{\tau_3}(p)c_i^{-\tau_3}(p)\tau_2] \qquad \text{for } j>j',$$
$$c_0(jkj'k')=\delta_{jj'}\int_{-\pi}^{\pi}\frac{dp}{2\pi}\exp[ip(k-k')]c_0(p),$$
$$c_i(jkj'k')=\delta_{jj'}\int_{-\pi}^{\pi}\frac{dp}{2\pi}\exp[ip(k-k')]c_i(p),$$
$$\eta c_0^{\pm\tau_3}\eta=c_0^{\pm\tau_3}\eta,$$
$$\eta c_i^{\pm\tau_3}\eta=\eta c_i^{\pm\tau_3},$$
$$2\eta g\eta=\eta+\eta c_0^{\tau_3}c_i^{-\tau_3}\tau_2\eta,$$
$$(1-2\eta g\eta)^{-1}=1-\eta-c_0^{\tau_3}\eta c_i^{-\tau_3}\tau_2. \qquad (4.15)$$

Thus

$$(1-2g\eta)^{-1}=1+2g\eta(1-2\eta g\eta)^{-1}=1-2gc_0^{\tau_3}\eta c_i^{-\tau_3}\tau_2. \qquad (4.16)$$

Equation (4.16) is used to evaluate (4.14) and then $(g'-g)_{jkjk}$ is found with the use of (4.15). Finally the trace indicated in (4.6) is taken, and the result is

$$f_{E\sigma}=\tfrac{1}{2}(J\sinh 2K^*+J')\langle\sigma\rangle\sum_{k^*}\eta(k^*)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dp}{2\pi}\frac{dp'}{2\pi}\exp[-|j|(\gamma+\gamma')]\exp[i(p-p')(k-k^*)]$$
$$\times\left[\frac{c_0(p)}{c_i(p')}+\frac{c_i(p')}{c_0(p)}-\frac{c_i(p)}{c_0(p')}-\frac{c_0(p')}{c_i(p)}\right]. \qquad (4.17)$$

We are interested in the case $\epsilon\ll 1$ and $|j|\gg 1$. Then only small p and p' values are important in (4.17). In this limit,

$$c_0(p)=\sinh K'(\epsilon-ip)^{1/2},$$
$$c_i(p)=\sinh K'(\epsilon+ip)^{1/2},$$
$$\gamma(p)=\sinh 2K'(\epsilon^2+p^2)^{1/2}. \qquad (4.18)$$

[9] In Ref. 4, a unitary transformation was made on g. The same transformation happens to convert τ_2 and $-\tau_2$. We have included this minus sign in expression (4.17).

The summation

$$\sum_{k^*=-\infty}^{\infty}\eta(k^*)\exp[i(p-p')(k-k^*)]=\frac{\exp[i(p-p')k]}{1-\exp[i(p-p')]}$$

and we may replace the denominator by $i(p'-p)$ inside the integrals. We also shift he contour of integration [see Eq. (3.10)], i.e., let

$$p=\epsilon\sinh(u+i\theta),$$
$$p'=\epsilon\sinh(u'-i\theta). \qquad (4.19)$$

When these substitutions are made in (4.17) we find that $f_{E\sigma}$ is independent of θ and is

$$f_{E\sigma}=\frac{2}{\pi^2}\,(J\sinh 2K^*+J')\,\langle\sigma\rangle\epsilon\int_0^\infty\int_0^\infty\frac{du\,du'\,\exp[-\epsilon R(\cosh u+\cosh u')]\cosh(u/2)\sinh(u'/2)\sinh u'}{\cosh u+\cosh u'}. \quad (4.20)$$

Further substitutions of

$$r\cos\phi=\sinh(u'/2),$$

$$r\sin\phi=\sinh(u/2) \quad (4.21a)$$

allow one to do the ϕ integration, and the final substitution

$$s=2\epsilon R(1+r^2) \quad (4.21b)$$

allows one to express the energy-density–spin-correlation function below T_c as

$$f_{E\sigma}=\frac{\epsilon}{\pi}\,\langle\sigma\rangle(J\sinh 2K_c+J')\int_{2\epsilon R}^\infty ds\,s^{-2}e^{-s}. \quad (4.22)$$

Since $\langle\sigma\rangle\sim\epsilon^{1/8}$, then $f_{E\sigma}\sim\epsilon^{9/8}f_3(\epsilon R)$, which explicitly verifies the scaling-law prediction, Eq. (2.8c).

For very large R, i.e., $\epsilon R\gg 1$,

$$f_{E\sigma}=(\langle\sigma\rangle/2\pi)(J\sinh 2K_c+J')(e^{-2\epsilon R}/2\epsilon R^2). \quad (4.23a)$$

For very small ϵ, i.e., $\epsilon R\ll 1$,

$$f_{E\sigma}=(\langle\sigma\rangle/2\pi)[(J\sinh 2K_c+J')/R]. \quad (4.23b)$$

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APPENDIX 1

The eigenvalues of the 2×2 matrix $q(p)$ in Eq. (3.8) are $e^{\pm\gamma(p)}$, where $\gamma>0$ and satisfies

$$\cosh\gamma=\cosh 2K^*\cosh 2K'-\cos p\sinh 2K^*\sinh 2K'. \quad (A1.1)$$

We can write

$$q(p)=\bar{A}(p)e^{\gamma}+\bar{B}(p)e^{-\gamma}, \quad (A1.2)$$

where $\bar{A}$ and $\bar{B}$ are the projection operators onto the eigenstates $|e^{\pm\gamma}\rangle$.

Let

$$A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}. \quad (A1.3a)$$

Some algebra shows that

$$a_{11}=(1-a_{22})=\tfrac{1}{2}[1+(ia_3/2\sinh\gamma)],$$

$$a_{12}=a_{21}{}^*=(-a_1+ia_2)/2\sinh\gamma, \quad (A1.3b)$$

where

$$a_1=\sin p\sinh 2K^*\cosh 2K',$$

$$a_2=\cosh 2K^*\sinh 2K'-\sinh 2K^*\cosh 2K'\cos p,$$

$$a_3=\sin p\sinh 2K^*\sinh 2K', \quad (A1.3c)$$

and

$$\bar{A}+\bar{B}=1.$$

Then Eq. (3.8) can be written as

$$G(p_x,p_y)=\frac{\exp[\gamma(p_y)]\bar{A}(p_y)}{e^{\gamma}-\exp(ip_x)}+\frac{e^{-\gamma}\bar{B}(p_y)}{e^{-\gamma}-\exp(ip_x)}. \quad (A1.4)$$

Now the p_x dependence has been separated from the 2×2 matrix dependence, so that the p_x integration in Eq. (3.7) can be performed. We use contour integration and the fact that $\gamma>0$ to obtain

$$g(jk)=\int_{-\pi}^{\pi}\frac{dp}{2\pi}\,e^{ipk}e^{-\gamma|j|}\begin{matrix}\times\ \bar{A}(p) & \text{if } j\le 0\\ \times-\bar{B}(p) & \text{if } j>0\end{matrix}. \quad (A1.5)$$

As $T\to T_c$, $K'\to K^*$, and we let

$$\epsilon=2\,|\,K'-K^*\,|/\sinh 2K'. \quad (A1.6)$$

When the coupling constants are equal, ϵ reduces to Eq. (2.2).

APPENDIX 2

The physical content of the approximation in Eq. (4.2a) is that

$$\langle\sigma_{jk}\sigma_{j+1,k}\sigma_{00}\rangle\simeq\langle\sigma_{jk}\sigma_{j,k-1}\sigma_{00}\rangle, \quad (A2.1)$$

i.e , for large R it does not matter whether the two spins near (j,k) are along a row or a column. Relation (A2.1) is for a symmetric lattice ($J=J'$); when $J\neq J'$ an extra factor of $\sinh 2K^*$ adjusts for the asymmetry.

From Eqs. (3.1) and (3.2) we have that

$$\left(\frac{2}{i}\right)b_{j,k+1,+}b_{jk-}=\frac{\sigma_{jk}\sigma_{j+1,k}}{\sinh 2K^*}-\frac{\cosh 2K^*}{\sinh 2K^*}, \quad (A2.2a)$$

$$\left(\frac{2}{i}\right)b_{jk+}b_{jk-}=\sigma_{jk}\sigma_{j,k-1}. \quad (A2.2b)$$

Multiplication of Eqs. (A2.2) by σ_{00} shows that $\langle b_{j,k+1,+}b_{jk-}\sigma_{00}\rangle$ differs from $\langle b_{jk+}b_{jk-}\sigma_{00}\rangle$ by a constant. But if we subtract the uncorrelated parts, i.e., $\langle b_{j,k+1,+}b_{jk-}\rangle\langle\sigma_{00}\rangle$ and $\langle b_{jk+}b_{jk-}\rangle\langle\sigma_{00}\rangle$, as is done in Eq. (4.2a), the extra constant cancels.