Existence of Long-Range Order in One and Two Dimensions

P. C. HOHENBERG
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received 24 October 1966)

It is pointed out that a rigorous inequality first proved by Bogoliubov may be used to rule out the existence of quasi-averages (or long-range order) in Bose and Fermi systems for one and two dimensions and $T \neq 0$.

I. INTRODUCTION

SINCE the work of London,[1] superconductivity and superfluidity have generally been associated with a specific type of order which London called long-range order of the average momentum. The earliest general mathematical characterization of this order was given by Ginzburg and Landau[2] for superconductors and by Penrose[3] for superfluids. They defined a system to have this long-range order if a suitable density matrix did not vanish for infinite separation of its spatial arguments. In the microscopic theories[4,5] these finite limiting values are taken to be anomalous averages (or quasi-averages[6]) which are nonzero by virtue of a broken symmetry (the conservation of number). In certain simple cases,[6] the justification for this can be given in mathematical form in terms of discontinuous limiting behavior as an external source coupled to the field goes to zero, and the volume Ω of the system goes to infinity. Alternatively,[4,5,7] a more intuitive justification can be given in terms of a restricted ensemble[7] having a specified value of the condensate function (the order parameter) but no definite value of the particle number. In either case the anomalous averages (such as $\langle\psi\rangle$ or $\langle\psi\psi\rangle$) are treated like ordinary ensemble averages, and their existence is taken to be equivalent to long-range order. We shall also assume this equivalence and not inquire further into its rigorous justification.

In this paper we wish to explore the consequences of an exact inequality[6,8] due originally to Bogoliubov, which we apply to cases where there is a broken symmetry, or anomalous average. We shall use it to calculate the fluctuations of the order parameter in superfluids and superconductors, thus proving that the assumption of a broken symmetry (or long-range order) in Bose or Fermi liquids leads to a contradiction in one and two dimensions at finite temperature. Using the Ginzburg-Landau[2] theory, similar results have been obtained for superconductors by Rice[9] and de Gennes[10] and for superfluids by Emery.[11] They have also been derived on the basis of hydrodynamic arguments by Ferrell,[12] Halperin, and Martin,[13] Chester and Reatto,[14] and, using the two-fluid expressions of Ref. 7, by Kane and Kadanoff.[15] However, the validity of some of these arguments[9,12] has been questioned[16] and in any case they are only approximate, whereas the present derivation is exact, given the existence of quasi-averages. It depends only on the commutation relations and the f sum rule.

In Sec. II we repeat Bogoliubov's[6] derivation in the form written down by Wagner.[8] In Sec. III we apply it to the Bose liquid both at low temperatures and near the transition, and in Sec. IV we show the necessary modifications which must be supplied to prove a theorem for a superconducting or superfluid Fermi liquid.

II. PROOF OF BOGOLIUBOV'S THEOREM

For completeness we repeat Bogoliubov's proof,[6] in the form given by Wagner.[8] Let $\langle A\rangle$ denote the quasiaverage of the operator A, or, equivalently, the ensemble average in a restricted ensemble[7] appropriate for a system with a broken symmetry. The properties of this average are just those of the equilibrium grand-

[1] F. London, *Superfluids* (Dover Publications, Inc., New York, 1960).
[2] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).
[3] O. Penrose, Phil. Mag. **42**, 1373 (1951); O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956); see also C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).
[4] N. N. Bogoliubov, J. Phys. USSR **11**, 23 (1947); S. T. Beliaev, Zh. Eksperim. i Teor. Fiz. **34**, 417 (1958) [English transl.: Soviet Phys.—JETP **7**, 289 (1958)]; P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966).
[5] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957); L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **34**, 735 (1958) [English transl.: Soviet Phys.—JETP **7**, 505 (1958)].
[6] N. N. Bogoliubov, Dubna report, 1962 (unpublished) [German transl.: Phys. Abh. SU **6**, 1 (1962); **6**, 113 (1962); **6**, 229 (1962)]. See also N. N. Bogoliubov, Physica **26**, 51 (1960). After a preliminary version of our paper was completed, we were able to obtain the German translation of this work, and found that the discussion of Fermi systems was more complete than in Wagner's (Ref. 8) paper, and essentially the same as our Sec. IV. Since our proof is briefer, and since Bogoliubov's paper is not generally available, we have retained our discussion in a complete and self-contained form.
[7] P. C. Hohenberg and P. C. Martin, Ann. Phys. (N.Y.) **34**, 291 (1965).
[8] H. Wagner, Z. Physik. **195**, 273 (1966).
[9] T. M. Rice, Phys. Rev. **140**, A1889 (1965); and (to be published).
[10] P. G. De Gennes, in *1965 Tokyo Lectures in Theoretical Physics*, edited by R. Kubo (W. A. Benjamin, Inc., New York, 1966), Part I, p. 117.
[11] V. J. Emery (unpublished).
[12] R. Ferrell, Phys. Rev. Letters **14**, 330 (1964).
[13] B. I. Halperin and P. C. Martin (unpublished).
[14] G. V. Chester and L. Reatto, Phys. Letters **22**, 276 (1966); L. Reatto and G. V. Chester, Phys. Rev. **155**, 88 (1967).
[15] J. W. Kane and L. P. Kadanoff, Phys. Rev. **155**, 80 (1967).
[16] Yu. A. Bychkov, L. P. Gor'kov, and I. E. Dzialoshinskii, JETP Pis'ma v Redaktsiyn **2**, 146 (1965) [English transl: JETP Letters **2**, 92 (1965)]; Zh. Eksperim. i Teor. Fiz. **50**, 738 (1966) [English transl.: Soviet Phys.—JETP **23**, 489 (1966).

canonical-ensemble average (in particular, the fluctuation-dissipation theorem holds[7]), except that certain quantities which would vanish in the latter remain finite here when the volume Ω goes to infinity (e.g., $\langle\psi\rangle$ or $\langle\psi\psi\rangle$). We shall introduce the Fourier transform of a function or operator Φ in a quantization volume Ω (with periodic boundary conditions) by the usual relations

$$\Phi(\mathbf{r})=\Omega^{-1}\sum_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})\phi_{\mathbf{k}},$$

$$\phi_{\mathbf{k}}=\int d\mathbf{r}\,\exp(-i\mathbf{k}\cdot\mathbf{r})\Phi(\mathbf{r}), \tag{1}$$

except that for the particle fields ψ and $\psi^\dagger$ we use the normalization

$$\psi(\mathbf{r})=\Omega^{-1/2}\sum_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})a_{\mathbf{k}};$$

$$\psi^\dagger(\mathbf{r})=\Omega^{-1/2}\sum_{\mathbf{k}}\exp(-i\mathbf{k}\cdot\mathbf{r})a_{\mathbf{k}}^\dagger. \tag{2}$$

We write in the volume Ω explicitly for convenience, even though it is always assumed to tend to infinity and no final results can depend on Ω. The spectral weight function is

$$\begin{aligned}\tau_{A,B}(\mathbf{k},\mathbf{k}';t-t')&=\Omega^{-1}\langle[A_{\mathbf{k}}(t),B_{\mathbf{k}'}(t')]\rangle\\&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tau_{A,B}(\mathbf{k},\mathbf{k}';\omega)\exp[-i\omega(t-t')],\end{aligned} \tag{3}$$

and the response function

$$\chi_{A,B}(\mathbf{k},\mathbf{k}';z)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\tau_{A,B}(\mathbf{k},\mathbf{k}';\omega)}{\omega-z}. \tag{4}$$

Furthermore, we define the static response function

$$\chi^s_{A,B}(\mathbf{k},\mathbf{k}')=P\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\tau_{A,B}(\mathbf{k},\mathbf{k}';\omega)}{\omega}. \tag{5}$$

(P is the principal part), and the equal-time correlation function

$$\begin{aligned}C_{A,B}(\mathbf{k},\mathbf{k}')&\equiv\Omega^{-1}\langle\{A_{\mathbf{k}}(t)-\langle A_{\mathbf{k}}(t)\rangle,B_{\mathbf{k}'}(t)-\langle B_{\mathbf{k}'}(t)\rangle\}\rangle\\&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tau_{A,B}(\mathbf{k},\mathbf{k}';\omega)\coth\tfrac{1}{2}(\beta\omega)\end{aligned} \tag{6}$$

($\beta=1/T$, $\hbar=k_B=1$); the last relation is the fluctuation-dissipation theorem. From the definition of χ^s we infer the following properties:

$$\begin{aligned}&\chi^s_{A,B}(\mathbf{k},\mathbf{k}') \text{ is a linear form in } A_{\mathbf{k}} \text{ and } B_{\mathbf{k}'},\\&[\chi^s_{A,B}(\mathbf{k},\mathbf{k}')]^*=\chi^s_{B^\dagger,A^\dagger}(\mathbf{k}',\mathbf{k}),\\&\chi^s_{A,A^\dagger}(\mathbf{k},\mathbf{k})\equiv\chi^s_{A,A^\dagger}(\mathbf{k})\geq 0.\end{aligned} \tag{7}$$

Therefore, χ^s is a scalar product and satisfies a Schwarz inequality

$$|\chi^s_{A,B}(\mathbf{k},\mathbf{k}')|^2\leq\chi^s_{A,A^\dagger}(\mathbf{k})\chi^s_{B^\dagger,B}(\mathbf{k}'). \tag{8}$$

Furthermore, since $(\beta/2)\,|\coth(\beta\omega/2)|\geq|\omega|^{-1}$ and the integrands in (5)–(6) are positive, we have

$$\chi^s_{A,A^\dagger}(\mathbf{k})\leq(\beta/2)C_{A,A^\dagger}(\mathbf{k}). \tag{9}$$

III. THE BOSE LIQUID

We shall apply the two inequalities to a Bose superfluid in which, by assumption,

$$\begin{aligned}\Omega^{-1/2}\langle a_{\mathbf{k}}\rangle&=\Omega^{-1/2}\langle a_0\rangle\delta(\mathbf{k})\\&=\Omega^{-1/2}\langle a_{\mathbf{k}}^\dagger\rangle\equiv(\sqrt{n_0})\,\delta(\mathbf{k}),\end{aligned} \tag{10}$$

where $\delta(\mathbf{k})$ is a *Kronecker* delta. We take the operators A and B to be $(\mathbf{k}=-\mathbf{k}'\neq 0)$:

$$A_{\mathbf{k}'}(t)=i(\partial/\partial t)\rho_{-\mathbf{k}}(t);\qquad B_{\mathbf{k}}(t)=\Omega^{-1/2}a_{\mathbf{k}}(t). \tag{11}$$

The operator $\rho_{\mathbf{k}}$ is the Fourier transform of the density, and satisfies $\langle\rho_0\rangle=\int d\mathbf{r}\,\langle\rho(\mathbf{r})\rangle=N$. It is easy to verify the relations

$$\tau_{A,B}(\mathbf{k}',\mathbf{k};\omega)=\omega\tau_{\rho,B}(-\mathbf{k},\mathbf{k};\omega), \tag{12}$$

$$\tau_{A,A^\dagger}(-\mathbf{k};\omega)=\omega^2\tau_{\rho,\rho^\dagger}(-\mathbf{k};\omega), \tag{13}$$

$$\begin{aligned}\chi^s_{A,B}(-\mathbf{k},\mathbf{k})&=\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\tau_{\rho,B}(-\mathbf{k},\mathbf{k};\omega')\\&=\Omega^{-1/2}\langle[\rho_{-\mathbf{k}}(t),a_{\mathbf{k}}(t)]\rangle=-\sqrt{n_0}.\end{aligned} \tag{14}$$

The last equality depends only on the Bose commutation rules and leads to a nonzero answer if there is a broken symmetry (long-range order). As a consequence of the continuity equation $\partial\rho/\partial t+\nabla\cdot\mathbf{j}=0$, we can prove the f sum rule,[17] which implies

$$\begin{aligned}\frac{k^2n}{m}&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega\tau_{\rho,\rho^\dagger}(-\mathbf{k},\omega)\\&=P\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\tau_{A,A^\dagger}(-\mathbf{k};\omega)}{\omega}=\chi^s_{A,A^\dagger}(-\mathbf{k}).\end{aligned} \tag{15}$$

Finally, we have (since $k\neq 0$)

$$\begin{aligned}C_{B,B^\dagger}(\mathbf{k})&=\langle\{a_{\mathbf{k}}(t)-\langle a_{\mathbf{k}}\rangle,a_{\mathbf{k}}^\dagger(t)-\langle a_{\mathbf{k}}^\dagger\rangle\}\rangle\\&=2\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\rangle+1.\end{aligned} \tag{16}$$

Putting together the two basic inequalities (8) and (9), we get

$$C_{B,B^\dagger}(\mathbf{k})\geq 2T\chi^s_{B,B^\dagger}(\mathbf{k})\geq 2T\frac{|\chi^s_{A,B}(-\mathbf{k},\mathbf{k})|^2}{\chi^s_{A^\dagger,A}(-\mathbf{k})}, \tag{17}$$

which yields, using Eqs. (14)–(16),

$$\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\rangle\equiv n_{\mathbf{k}}\geq-\tfrac{1}{2}+\frac{T}{(k^2/m)}\frac{n_0}{n}. \tag{18}$$

[17] P. Nozières and D. Pines, *Quantum Liquids* (W. A. Benjamin, Inc., New York, 1966), Vol. I, p. 90.

Clearly the quantity

$$\Omega^{-1}\sum_{k\neq 0} n_{\mathbf{k}}=\int \frac{d^s k}{(2\pi)^s}\, n_{\mathbf{k}}=n-n_0, \qquad (19)$$

(where s is the dimensionality of the system) must be finite, and we see that this is incompatible with (18) in one and two dimensions for $T>0$, *unless* $n_0=0$. We conclude that there is no broken symmetry (long-range order) in one and two dimensions in a Bose liquid.

In three dimensions, very near the transition temperature T_λ, but for $T<T_\lambda$, one may wish to describe the correlation function in the form

$$n(\mathbf{k})\sim \text{const}/k^{2-\eta^*}$$

as $k\to 0$. From our inequality (18), we see that the exponent η^* turns out to be nonpositive.

[*Note added in proof.* In order to avoid confusion we wish to point out that η^* is not the same as the coefficient η introduced by Fisher.[18] The latter coefficient characterizes the behavior of the correlation function for fixed $\mathbf{k}$, in the limit $T\to T_\lambda$. The coefficient η^* on the other hand, applies for fixed $T<T_\lambda$, in the limit $\mathbf{k}\to 0$. This difference in behavior is well known,[18,19] and our theorem says nothing new about the parameter η. In terms of Josephson's[19] discussion, it merely states the obvious fact that $\rho_s\leq\rho$. The Bogoliubov inequality (18), does lead to the exact result $\eta^*\leq 0$, which is a necessary restriction on any accurate theory of the lambda transition in three dimensions. The possibility of a positive η^* was raised by Kane and Kadanoff[15] (they called it η) and by M. Fisher (private communication) and is ruled out by Eq. (18).]

IV. COOPER PAIRING IN FERMI SYSTEMS

For fermions, the analogous statement would be that it is inconsistent to assume that the quasiaverage $\langle\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rangle$ (the arrows denote spin states) is finite for infinite volume in one and two dimensions. However, the order-parameter correlation function is not as directly related to a thermodynamic quantity as $n_{\mathbf{k}}$, and we shall have to use a slightly more complicated argument.[20] We begin by assuming that the quasi-average $\langle a_{\uparrow\mathbf{q}}a_{\downarrow-\mathbf{q}}\rangle$ is nonzero for one or more regions of $\mathbf{q}$, and we shall show a contradiction in one and two dimensions. For this purpose we introduce the order parameter

$$\Delta\equiv\int d\bar{\mathbf{r}}\, s(\mathbf{r}-\bar{\mathbf{r}})\,\langle\psi_\uparrow(\mathbf{r})\psi_\downarrow(\bar{\mathbf{r}})\rangle$$
$$=\Omega^{-1}\sum_{\mathbf{q}} S(\mathbf{q})\,\langle a_{\uparrow\mathbf{q}}a_{\downarrow-\mathbf{q}}\rangle, \qquad (20)$$

where the "smearing function" $s(\mathbf{r})$ is arbitrary (a Gaussian, for instance) but has the properties

$$\int s(\mathbf{r})\,d\mathbf{r}=S(0)=1; \qquad s(0)=\Omega^{-1}\sum_{\mathbf{q}} S(\mathbf{q})<\infty. \qquad (21)$$

We apply the Bogoliubov inequality to the operators

$$A_{\mathbf{k}'}(t)=i(\partial/\partial t)\,\rho_{-k}(t); \qquad B_{\mathbf{k}}=\sum_{\mathbf{q}} S(\mathbf{q})a_{\downarrow\mathbf{k}-\mathbf{q}}a_{\downarrow\mathbf{q}}. \qquad (22)$$

The fermion commutation rules yield for $k\neq 0$

$$\frac{1}{\Omega}\langle[B_{\mathbf{k}},\rho_{-\mathbf{k}}]\rangle=\Omega^{-1}\sum_{\mathbf{q}}[S(\mathbf{q})+S(\mathbf{k}-\mathbf{q})]\langle a_{\uparrow\mathbf{q}}a_{\downarrow-\mathbf{q}}\rangle$$
$$\equiv\Delta+\eta(\mathbf{k}). \qquad (23)$$

This defines $\eta(\mathbf{k})$, which has the important property that

$$\lim_{\mathbf{k}\to 0}\eta(\mathbf{k})=\Delta,$$

since $S(\mathbf{q})$ is analytic and $\langle a_{\uparrow\mathbf{q}}a_{\downarrow-\mathbf{q}}\rangle$ is bounded. The f sum rule once again yields (15), which permits us to write the Bogoliubov inequalities (8) and (9) as

$$C_{B,B^\dagger}(\mathbf{k})\geq 2T\,\frac{|\Delta+\eta(\mathbf{k})|^2}{(n/m)k^2}. \qquad (24)$$

The small $\mathbf{r}$ behavior of the Fourier transform $c_{B,B^\dagger}(\mathbf{r})$ of $C_{B,B^\dagger}(\mathbf{k})$ is not simple, so that in order to prove our theorem we must rewrite $C_{B,B^\dagger}(\mathbf{k})$, using the commutation relations, in the form $(k\neq 0)$,

$$C_{B,B^\dagger}(\mathbf{k})=\Omega^{-1}\langle\{\sum_{\mathbf{q}} S(\mathbf{q})a_{\uparrow\mathbf{k}-\mathbf{q}}a_{\downarrow\mathbf{q}},\ \sum_{\mathbf{q}'} S(\mathbf{q}')a_{\downarrow\mathbf{q}'}^\dagger a_{\uparrow\mathbf{k}-\mathbf{q}'}^\dagger\}\rangle \qquad (25)$$
$$=(2/\Omega)\sum_{\mathbf{q},\mathbf{q}'} S(\mathbf{q})S(\mathbf{q}')\,\langle a_{\downarrow\mathbf{q}'}^\dagger a_{\uparrow\mathbf{k}-\mathbf{q}'}^\dagger a_{\uparrow\mathbf{k}-\mathbf{q}}a_{\downarrow\mathbf{q}}\rangle$$
$$-\Omega^{-1}\sum_{\mathbf{q}}[S^2(\mathbf{q})+S^2(\mathbf{k}-\mathbf{q})]\langle a_{\downarrow\mathbf{q}}^\dagger a_{\downarrow\mathbf{q}}+a_{\uparrow\mathbf{q}}^\dagger a_{\uparrow\mathbf{q}}\rangle+\Omega^{-1}\sum_{\mathbf{q}} S^2(\mathbf{q}). \qquad (26)$$

If $S(\mathbf{q})$ were a constant [$s(\mathbf{r})=\delta(\mathbf{r})$], then the last term of (26) would be infinite [$\delta(0)$], and $c_{B,B^\dagger}(\mathbf{r})$ would not be defined. With a Gaussian $s(\mathbf{q})$, this term is some finite constant (independent of $\mathbf{k}$); likewise, the next to the last term in (26) is an analytic function of $\mathbf{k}$ which is integrable. The first term, which we call $F(\mathbf{k})$, would be just the density-correlation function if $s(\mathbf{q})$ were a constant. Its Fourier transform $f(\mathbf{r})$ can be shown to be finite for $\mathbf{r}=0$ by the following argument:

$$f(\mathbf{r}_1-\mathbf{r}_2)\equiv\iint d\mathbf{r}'d\mathbf{r}''\, s(\mathbf{r}_1-\mathbf{r}')s(\mathbf{r}_2-\mathbf{r}'')\,\langle\psi_\downarrow^\dagger(\mathbf{r}')\psi_\uparrow^\dagger(\mathbf{r}_1)\psi_\uparrow(\mathbf{r}_2)\psi_\downarrow(\mathbf{r}'')\rangle. \qquad (27)$$

[18] M. E. Fisher, J. Math. Phys. **5**, 944 (1964).
[19] B. D. Josephson, Phys. Letters **21**, 608 (1966).
[20] A number of key points in the application of the theorem to fermions were suggested to me by Dr. B. I. Halperin.

The matrix element in (27) is of the form

$$\langle\psi_\downarrow^\dagger(\mathbf{r}')\psi_\uparrow^\dagger(\mathbf{r}_1)\psi_\uparrow(\mathbf{r}_2)\psi_\downarrow(\mathbf{r}'')\rangle\equiv\langle a^\dagger b\rangle, \qquad (28)$$

which defines the operators a and b. The scalar product again satisfies a Schwarz inequality, so that

$$|\langle a^\dagger b\rangle|^2\leq\langle a^\dagger a\rangle\langle b^\dagger b\rangle$$
$$=\langle\rho_\downarrow(\mathbf{r}')\rho_\uparrow(\mathbf{r}_1)\rangle\langle\rho_\uparrow(\mathbf{r}_2)\rho_\downarrow(\mathbf{r}'')\rangle, \qquad (29)$$

where $\rho_{\uparrow\downarrow}(\mathbf{r})=\psi_{\uparrow\downarrow}^\dagger(\mathbf{r})\psi_{\uparrow\downarrow}(\mathbf{r})$. From Eqs. (27)–(29) we conclude that

$$f(0)\leq 2\left[\int d\mathbf{r}'\, s(\mathbf{r}_1-\mathbf{r}')\{\langle\rho_\uparrow(\mathbf{r}_1)\rho_\downarrow(\mathbf{r}')\rangle\}^{1/2}\right]^2, \qquad (30)$$

which is obviously finite since the density correlation function may not have any singularities which are not integrable. If we write (26) in the form

$$C_{B,B^\dagger}(\mathbf{k})=F(\mathbf{k})+R(\mathbf{k}), \qquad (31)$$

we see that $R(\mathbf{k})$ is regular at small $\mathbf{k}$,[21] and with the aid of Eqs. (24), (31), and (30) we have

$$F(\mathbf{k})\geq 2T\frac{|\Delta+\eta(\mathbf{k})|^2}{nk^2/m}-R(\mathbf{k}), \qquad (32)$$

$$\Omega^{-1}\sum_{\mathbf{k}\neq 0}F(\mathbf{k})<f(0)<\infty. \qquad (33)$$

Clearly, since $\eta(\mathbf{k})$ and $R(\mathbf{k})$ are regular at small $\mathbf{k}$, Eqs. (32) and (33) are in contradiction in one and two dimensions for $T\neq 0$ and for infinite volume, unless $\Delta=0$. Since $S(\mathbf{k})$ is arbitrary, subject to the conditions (21), we cannot have $\langle a_{\uparrow\mathbf{q}}a_{\downarrow-\mathbf{q}}\rangle\neq 0$ for any $\mathbf{q}$, and $\langle\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rangle=0$.

This result can be extended to cases where Δ is not the only anomalous average in the system.[16] Under certain circumstances it is conceivable that a breaking of translational invariance[16] might invalidate some of the arguments presented here, although a similar discussion could probably be used to exclude such a broken symmetry also.[22] A full discussion of the case treated by Gor'kov, Bychkov, and Dzialoshinskii[16] would depend on subtle details of their model and we shall not attempt it, but it seems to us extremely unlikely that their assumption of long-range order in one dimension ($\Delta\neq 0$) can be strictly valid, at least at finite temperature.

[21] For fixed $S(\mathbf{q})$, $R(\mathbf{k})$ is integrable.

V. CONCLUSION

We have shown that the long-range order generally associated with superconductivity and superfluidity[1–4] (i.e., the existence of anomalous averages $\langle\psi\rangle$ or $\langle\psi\psi\rangle$) is not consistent with exact sum rules in one and two dimensions. Our arguments do not depend on the range of the forces so long as these keep the f sum rule intact.[23] This work supplements approximate arguments previously given.[9–15] We do not make any statement about the existence of a phase transition, or even of supercurrents or flux quantization, except to say that the usual models (involving long-range order) are not valid. Needless to say, we can shed even less light on the interesting question of the existence of *approximate* long-range order, or of *essentially* persistent currents which might live for macroscopic times. However, the arguments presented here seem to us quite rigorous and simple. They have a sufficiently general character to be applicable to other cases of long-range order, such as the Heisenberg ferro- and antiferromagnet, and the crystal. A discussion of the extension of these ideas to other cases has recently been given by Mermin and Wagner.[22]

ACKNOWLEDGMENTS

I have benefited greatly from discussions with M. Fisher, N. D. Mermin, T. M. Rice, and especially G. V. Chester and B. I. Halperin.

[22] N. D. Mermin and H. Wagner, Phys. Rev. Letters **17**, 1133 (1966), and (to be published).

[23] The BCS *reduced* Hamiltonian (Ref. 5) does lead to superconductivity in one and two dimensions, but this is because its nonlocal interaction violates the f sum rule. See P. W. Anderson, Phys. Rev. **110**, 827 (1958); and N. N. Bogoliubov, D. N. Zubarev, and Yu. A. Tserkovnikov, Zh. Eksperim. i Teor. Fiz. **39**, 120 (1960) [English transl.: Soviet Phys.—JETP **12**, 88 (1961)].