Collective Excitations of a Granular Superconductor in the Presence of a Steady Uniform Current Flow

R. H. PARMENTER
Department of Physics, University of Arizona, Tucson, Arizona
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By a suitable linearization of the isospin equations of motion, a calculation is made of the collective excitations of a granular superconductor in the presence of a steady, uniform current flow. Aside from the case of vanishing frequency or wave vector, only collective excitations with wave vectors perpendicular to the direction of current flow can occur physically (all others violating the condition of electric charge neutrality in the superconductor). This insures that the uniform supercurrent is stable against decay by excitation of collective oscillations (excitations bounded in space and time). Those excitations which can occur in the presence of the steady, uniform current are appreciably modified by the latter.

I. INTRODUCTION

IN a recent paper[1] the writer introduced the idea of a granular superconductor[2] and investigated a number of properties of such a system with the aid of Anderson's isospin formulation[3] of the theory of superconductivity.[4] The Hamiltonian of this superconductor can be broken up into two parts: (1) the Bardeen–Cooper–Schrieffer (BCS) Hamiltonian associated with the individual homogeneous grains of superconductive material; (2) the tunneling Hamiltonian associated with the Josephson junctions separating adjacent grains. The latter Hamiltonian, when recast in the isospin formulation,[5] bears a strong formal analogy to the Heisenberg exchange Hamiltonian of the theory of magnetism. Invoking this analogy, the writer[1] passed to the limit of a continuum theory, in exactly the fashion pioneered by Landau and Lifshitz[6] in converting the Heisenberg theory into a continuum theory of magnetism, so-called micromagnetics.[7] With the aid of the commutation properties of the isospin operators, it is easy to obtain the equations of motion, which represent torque equations for each spin precessing in an effective pseudomagnetic field due to all the other spins. In contrast to the situation commonly encountered in magnetism, the pseudomagnetic field acting on a given spin results from a very large number of other spins. As was pointed out by Anderson,[3] this allows one to solve the equations of motion, with no loss of accuracy, by making the so-called semiclassical approximation (treating the isospins as classical quantities when solving the equations of motion, but not when setting up the equations of motion).

The equations of motion for these (now classical) isospins are a set of coupled, nonlinear integrodifferential equations. The two solutions that are independent of both time and position can be obtained almost by inspection; namely the time- and position-independent superconducting solution (BCS ground state) and the time- and position-independent normal-metal solution.[8] Time- and position-*dependent* solutions that differ only slightly from one of these two time- and position-*independent* solutions can be found by *linearizing* the equations of motion. This procedure, carried out in detail in A, enables one to determine the collective excitations associated with either the superconducting or the normal phase of the granular superconductor.

The purpose of the present paper is to lay to rest a rather disquieting possibility unearthed in the course of the previous computation of the collective excitations. In A-Sec. VI it was shown that, under certain conditions,[9] the superconducting phase has associated with it a collective oscillation[10] of vanishingly small phase velocity. But it is a well-known fact that the *minimum phase velocity* of any collective oscillation sets an *upper limit* to the drift velocity of the superfluid particles (here, Cooper pairs) in any superfluid system,[11] provided that there is no selection rule prohibiting the generation of the collective oscillations. In the absence of such a selection rule, this suggests that, under the conditions described in Ref. 9, our granular superconductor cannot superconduct in the literal sense of the word, despite the fact that each grain of the superconductor is in the thermodynamic

[1] R. H. Parmenter, Phys. Rev. **154**, 353 (1967). This paper will be referred to as A. Sections and equations from A will be identified by the prefix A; e.g., A-Sec. III, A-Eq. (4.7), etc.

[2] A granular superconductor is a system composed of many microscopic grains of homogeneous superconductor, with each grain boundary being an insulating layer (e.g., oxide) that is thin enough to allow appreciable tunneling by the Cooper pairs of the superconductor. In other words, the system is interlaced with Josephson junctions.

[3] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).

[4] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[5] The tunneling Hamiltonian of a Josephson junction was first reformulated in terms of isospins by P. R. Wallace and M. J Stavn, Can. J. Phys. **43**, 411 (1965).

[6] L. Landau and E. Lifshitz, Phys. Z. Sowjetunion **8**, 153 (1935).

[7] W. F. Brown, Jr., *Micromagnetics* (Interscience Publishers, Inc., New York, 1963); S. Shtrikman and D. Treves, in *Magnetism*, edited by G. T. Rado and H. Suhl (Academic Press Inc., New York, 1963), Vol. III, Chap. 8.

[8] As was pointed out in A, the latter is unstable against decay into the former via collective excitations.

[9] The conditions are that the matrix element for phonon-induced electron-electron attraction is, in absolute value, bracketed by the effective matrix elements for electron-electron Coulomb repulsion in (1) the superconducting phase and (2) the normal phase. [It is a well-known consequence of antiparallel-spin electron-electron correlation in the superconducting phase that the former Coulomb repulsion is numerically smaller than the latter. See, e.g., P. Morel and P. W. Anderson, Phys. Rev. **125**, 1263 (1962).]

[10] A collective *oscillation* is a collective excitation bounded in space and time, i.e., one having a *real* (not complex) frequency and wave vector.

[11] See, e.g., I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965), p. 6.

superconducting phase. This conclusion is particularly disquieting, since our continuum model of a granular superconductor should, for the present purposes, be a reasonable model for a conventional dirty superconductor.[12] This would suggest that any superconducting metal satisfying the conditions of Ref. 9 would lose its superconductivity upon alloying.

As we shall presently show, this difficulty is resolved in the following fashion. The collective oscillations do not quench the supercurrent; rather, it is the other way around: The supercurrent quenches the collective oscillations (via Poisson's equation). In A-Sec. III, it was shown that there are two classes of collective excitations satisfying the linearized equations of motion, but only one of these two classes is physically realizable, since the other class violates the condition of electric charge neutrality (the total conduction-electron density being modified by the presence of the excitation). As we shall see, the presence of a steady, uniform supercurrent[13] causes these two classes of excitations to mix, so that, with certain exceptions, all classes of excitations now violate charge neutrality and are thus quenched. The exceptions are excitations with wave vectors (either real or complex) perpendicular to the direction of flow of the supercurrent (and also the special cases of vanishing wave vector or frequency). But such oscillations are unable to extract both energy and momentum from the supercurrent, and thus cannot quench the supercurrent.

The state of steady, uniform current flow was obtained in A-Sec. V by direct solution of the equations of motion. Because of the mathematical simplicity of this solution, it is possible to carry through the same kind of linearization procedure with respect to this state of finite current density that was carried out in A with respect to the superconducting state of zero current density (BCS solution). This procedure can be carried out using either of the two forms of net effective electron-electron interaction (considered in A-Secs. III and VI, respectively). Since the conditions mentioned in Ref. 9 are appropriate only for the more complicated form of interaction (A-Sec. VI), we should logically work with this form. In the interest of simplicity of exposition, we will go through the details of the linearization procedure only for simpler form of interaction potential. It can be seen from the structure of the linearized equations of motion that the same conclusion (quenching of all collective excitations of finite wave vector and frequency except those moving normal to the supercurrent) will be obtained for either form of interaction. It might be added that even those collective excitations which are not quenched will still suffer modification by the presence of the supercurrent.

[12] The continuum model of the granular superconductor differs from a dirty superconductor only when the former contains electric fields, as occurs, for example, with the electromagnetic modes calculated in A-Sec. V. *Note added in Proof.* The effective Ginzburg–Landau coherence distance ξ of both the granular and the dirty superconductor depend in identical fashion on the Pippard coherence distance of the corresponding bulk material and on the normal-state conductivity mean free path (ξ being proportional to the geometric mean of the two).

[13] The assumption of spatial uniformity of our supercurrent means that we are restricting the discussion to specimens with small enough cross-sectional dimensions (e.g., thin films) that the real magnetic field due to the total supercurrent is ignorable. The conclusions to be drawn, however, are probably valid for all sizes of specimens.

II. MATHEMATICAL ANALYSIS

The isospin equations of motion are

$$\hbar(d\mathbf{s}/dt)=\mathbf{s}\times\mathbf{H}, \tag{1}$$

where the isospin $\mathbf{s}=\mathbf{s}_k(\mathbf{R})$ and the effective pseudomagnetic field $\mathbf{H}=\mathbf{H}_k(\mathbf{R})$ are both functions of internal momentum $\hbar\mathbf{k}$ and center-of-mass position $\mathbf{R}$ of the Cooper pair. It is more convenient for our present purposes to think of $\mathbf{s}$ and $\mathbf{H}$ as functions of the one-electron energy $\epsilon=\epsilon_k$ corresponding to $\mathbf{k}$ (the zero of energy being taken at the Fermi level). The x, y, and z components of $\mathbf{s}$ are given by

$$s_1=\tfrac{1}{2}\sin\theta\cos\phi,$$
$$s_2=\tfrac{1}{2}\sin\theta\sin\phi, \tag{2}$$

and

$$s_3=\tfrac{1}{2}\cos\theta,$$

respectively, where the angles θ and ϕ are functions of $\mathbf{R}$ and ϵ. The x, y, and z components of $\mathbf{H}$ are given by

$$H_1=2[1+\xi^2\nabla_R{}^2]\Delta_1,$$
$$H_2=2[1+\xi^2\nabla_R{}^2]\Delta_2,$$
$$H_3=2\epsilon, \tag{3}$$

respectively, where (for the simpler form of interaction potential) the order parameters Δ_1 and Δ_2 are defined as

$$\Delta_1=N(0)V\int_{-\hbar\omega}^{+\hbar\omega}s_1d\epsilon,$$

$$\Delta_2=N(0)V\int_{-\hbar\omega}^{+\hbar\omega}s_2d\epsilon. \tag{4}$$

The constant ξ, defined in A, is an effective coherence distance. Just as in the BCS theory, $\pm\hbar\omega$ is the range in ϵ over which the interaction potential V is finite (and constant). It can be seen that the x and y components of $\mathbf{H}$ are functions of $\mathbf{R}$ alone, the z component a function of ϵ alone.

We write

$$\theta=\theta_0+\delta\theta,$$

$$\phi=\phi_0+\delta\phi, \tag{5}$$

where both (θ,ϕ) and (θ_0,ϕ_0) are solutions to the equations of motion, and $\delta\theta$, $\delta\phi$ are small in the sense that we may ignore terms of higher order than linear in $\delta\theta$ or $\delta\phi$ when solving the equations of motion. This procedure represents a linearization of the equations of motion with respect to the solution (θ_0,ϕ_0), chosen to be the case of steady, uniform current flow obtained in A-Sec. V. Thus

$$\theta_0=\arctan(g/\epsilon), \tag{6}$$

$$\phi_0=\boldsymbol{\kappa}_0\cdot\mathbf{R}, \tag{7}$$

where g and $\boldsymbol{\kappa}_0$ are constants. The corresponding order parameters are

$$\Delta_{10}=\Delta_0\cos\phi_0,$$
$$\Delta_{20}=\Delta_0\sin\phi_0, \tag{8}$$

where

$$\Delta_0\equiv\tfrac{1}{2}N(0)V\int_{-\hbar\omega}^{+\hbar\omega}\sin\theta_0 d\epsilon$$
$$=N(0)Vg\int_0^{\hbar\omega}(\epsilon^2+g^2)^{-1/2}d\epsilon$$
$$\cong N(0)Vg\ln(2\hbar\omega/g). \tag{9}$$

In order to satisfy the equations of motion, $\boldsymbol{\kappa}_0$ and g must be related by the condition

$$[1-(\xi\kappa_0)^2]\Delta_0=g. \tag{10}$$

The dc current density is proportional to $\Delta_0^2\boldsymbol{\kappa}_0$, so that a given value of g uniquely specifies Δ_0, κ_0, and the magnitude of current density. As

$$\Delta_0\rightarrow g\rightarrow\epsilon_0\equiv 2\hbar\omega\exp[-1/N(0)V], \tag{11}$$

$\boldsymbol{\kappa}_0$ and the current density approach zero, and our unperturbed state approaches the BCS ground state at the absolute zero of temperature.

We expand the sines and cosines of θ and ϕ as power series in $\delta\theta$ and $\delta\phi$, getting

$$\cos\theta=\cos(\theta_0+\delta\theta)\cong(\epsilon^2+g^2)^{-1/2}(\epsilon-g\delta\theta),$$
$$\sin\theta=\sin(\theta_0+\delta\theta)\cong(\epsilon^2+g^2)^{-1/2}(g+\epsilon\delta\theta),$$
$$\cos\phi=\cos(\phi_0+\delta\phi)\cong\cos\phi_0-\delta\phi\sin\phi_0,$$
$$\sin\phi=\sin(\phi_0+\delta\phi)\cong\sin\phi_0+\delta\phi\cos\phi_0, \tag{12}$$

so that

$$\delta s_1\equiv s_1-s_{10}$$
$$\cong\tfrac{1}{2}(\epsilon^2+g^2)^{-1/2}(\epsilon\cos\phi_0\delta\theta-g\sin\phi_0\delta\phi),$$
$$\delta s_2\equiv s_2-s_{20}$$
$$\cong\tfrac{1}{2}(\epsilon^2+g^2)^{-1/2}(\epsilon\sin\phi_0\delta\theta+g\cos\phi_0\delta\phi),$$
$$\delta s_3\equiv s_3-s_{30}$$
$$\cong-\tfrac{1}{2}(\epsilon^2+g^2)^{-1/2}g\delta\theta. \tag{13}$$

It is convenient to define

$$\delta s_\alpha\equiv\tfrac{1}{2}(\epsilon^2+g^2)^{-1/2}\epsilon\delta\theta=-(\epsilon/g)\delta s_3$$
$$=\cos\phi_0\delta s_1+\sin\phi_0\delta s_2,$$
$$\delta s_\beta\equiv\tfrac{1}{2}(\epsilon^2+g^2)^{-1/2}g\delta\phi$$
$$=-\sin\phi_0\delta s_1+\cos\phi_0\delta s_2 \tag{14}$$

and

$$\delta\Delta_\alpha\equiv N(0)V\int_{-\hbar\omega}^{+\hbar\omega}\delta s_\alpha d\epsilon,$$
$$\delta\Delta_\beta\equiv N(0)V\int_{-\hbar\omega}^{+\hbar\omega}\delta s_\beta d\epsilon. \tag{15}$$

Thus

$$\delta\Delta_1\equiv\Delta_1-\Delta_{10}=\cos\phi_0\delta\Delta_\alpha-\sin\phi_0\delta\Delta_\beta,$$
$$\delta\Delta_2\equiv\Delta_2-\Delta_{20}=\sin\phi_0\delta\Delta_\alpha+\cos\phi_0\delta\Delta_\beta, \tag{16}$$

and

$$\delta H_1\equiv H_1-H_{10}=2[1+\xi^2\nabla_R^2]\delta\Delta_1,$$
$$\delta H_2\equiv H_2-H_{20}=2[1+\xi^2\nabla_R^2]\delta\Delta_2,$$
$$\delta H_3\equiv H_3-H_{30}=0. \tag{17}$$

The linearized equations of motion are

$$\hbar(d/dt)\delta s_1=H_{30}\delta s_2-H_{20}\delta s_3-s_{30}\delta H_2,$$
$$\hbar(d/dt)\delta s_2=s_{30}\delta H_1+H_{10}\delta s_3-H_{30}\delta s_1,$$
$$\hbar(d/dt)\delta s_3=H_{20}\delta s_1-H_{10}\delta s_2+s_{10}\delta H_2-s_{20}\delta H_1. \tag{18}$$

These three equations are equivalent to two independent equations for $\hbar(d/dt)\delta s_\alpha$ and $\hbar(d/dt)\delta s_\beta$. The equation for $\hbar(d/dt)\delta s_\alpha$ is obtained by multiplying the equation for $\hbar(d/dt)\delta s_3$ by $-\epsilon/g$; alternatively, it is obtained by adding the equation for $\hbar(d/dt)\delta s_1$, multiplied by $\cos\phi_0$, to the equation for $\hbar(d/dt)\delta s_2$, multiplied by $\sin\phi_0$. The equation for $\hbar(d/dt)\delta s_\beta$ is obtained by adding the equation for $\hbar(d/dt)\delta s_1$, multiplied by $-\sin\phi_0$, to the equation for $\hbar(d/dt)\delta s_2$, multiplied by $+\cos\phi_0$. Thus

$$\hbar(d/dt)\delta s_\alpha=2\epsilon\delta s_\beta-\tfrac{1}{2}\epsilon(\epsilon^2+g^2)^{-1/2}(\cos\phi_0\delta H_2-\sin\phi_0\delta H_1),$$
$$\hbar(d/dt)\delta s_\beta=-(2/\epsilon)(\epsilon^2+g^2)\delta s_\alpha+\tfrac{1}{2}\epsilon(\epsilon^2+g^2)^{-1/2}(\sin\phi_0\delta H_2+\cos\phi_0\delta H_1). \tag{19}$$

The terms in (19) involving δH_1 and δH_2 can be rewritten in terms of $\delta\Delta_\alpha$ and $\delta\Delta_\beta$.

$$\cos\phi_0\delta H_2-\sin\phi_0\delta H_1=2\cos\phi_0[1+\xi^2\nabla_R^2][\sin\phi_0\delta\Delta_\alpha+\cos\phi_0\delta\Delta_\beta]-2\sin\phi_0[1+\xi^2\nabla_R^2][\cos\phi_0\delta\Delta_\alpha-\sin\phi_0\delta\Delta_\beta]$$
$$=4\xi^2\boldsymbol{\kappa}_0\cdot\nabla_R\delta\Delta_\alpha+2[1+\xi^2(-\kappa_0^2+\nabla_R^2)]\delta\Delta_\beta, \tag{20}$$
$$\sin\phi_0\delta H_2+\cos\phi_0\delta H_1=2\sin\phi_0[1+\xi^2\nabla_R^2][\sin\phi_0\delta\Delta_\alpha+\cos\phi_0\delta\Delta_\beta]+2\cos\phi_0[1+\xi^2\nabla_R^2][\cos\phi_0\delta\Delta_\alpha-\sin\phi_0\delta\Delta_\beta]$$
$$=-4\xi^2\boldsymbol{\kappa}_0\cdot\nabla_R\delta\Delta_\beta+2[1+\xi^2(-\kappa_0^2+\nabla_R^2)]\delta\Delta_\alpha. \tag{21}$$

Thus the equations of motion become

$$+\tfrac{1}{2}\hbar(d/dt)\delta\theta=g\delta\phi-[1-\xi^2(\kappa_0^2-\nabla_R^2)]\delta\Delta_\beta-2\xi^2\boldsymbol{\kappa}_0\cdot\nabla_R\delta\Delta_\alpha, \tag{22}$$
$$-\tfrac{1}{2}\hbar g(d/dt)\delta\phi=(\epsilon^2+g^2)\delta\theta-\epsilon[1-\xi^2(\kappa_0^2-\nabla_R^2)]\delta\Delta_\alpha+2\epsilon\xi^2\boldsymbol{\kappa}_0\cdot\nabla_R\delta\Delta_\beta. \tag{23}$$

Let us assume solutions of the form

$$\delta\theta = [A_1(\epsilon) + \epsilon B_1(\epsilon)] \exp[i(\boldsymbol{\kappa}\cdot\mathbf{R} - \omega_0 t)], \tag{24}$$

$$\delta\phi = [A_2(\epsilon) + \epsilon B_2(\epsilon)] \exp[i(\boldsymbol{\kappa}\cdot\mathbf{R} - \omega_0 t)], \tag{25}$$

where $A_1(\epsilon)$, $A_2(\epsilon)$, $B_1(\epsilon)$, $B_2(\epsilon)$ are all *even* functions of ϵ. Define the two constants

$$B \equiv N(0)\, V \int_0^{\hbar\omega} \epsilon^2(\epsilon^2+g^2)^{-1/2} B_1(\epsilon)\, d\epsilon, \tag{26}$$

$$A \equiv N(0)\, Vg \int_0^{\hbar\omega} (\epsilon^2+g^2)^{-1/2} A_2(\epsilon)\, d\epsilon. \tag{27}$$

Thus we have

$$\delta\Delta_\alpha = B \exp[i(\boldsymbol{\kappa}\cdot\mathbf{R} - \omega_0 t)], \tag{28}$$

$$\delta\Delta_\beta = A \exp[i(\boldsymbol{\kappa}\cdot\mathbf{R} - \omega_0 t)]. \tag{29}$$

Substituting Eqs. (24), (25), (28), and (29) into (22) and (23), the equations of motion, and breaking each resultant equation into a part odd in ϵ and a part even in ϵ, we finally get

$$\begin{aligned}
+\tfrac{1}{2}i\hbar\omega_0 A_1(\epsilon) + gA_2(\epsilon) &= [1-\xi^2(\kappa_0^2+\kappa^2)]A + 2i\xi^2\boldsymbol{\kappa}_0\cdot\boldsymbol{\kappa}B,\\
(\epsilon^2+g^2)A_1(\epsilon) - \tfrac{1}{2}i\hbar\omega_0 g A_2(\epsilon) &= 0,\\
+\tfrac{1}{2}i\hbar\omega_0 B_1(\epsilon) + gB_2(\epsilon) &= 0,\\
(\epsilon^2+g^2)B_1(\epsilon) - \tfrac{1}{2}i\hbar\omega_0 g B_2(\epsilon) &= [1-\xi^2(\kappa_0^2+\kappa^2)]B - 2i\xi^2\boldsymbol{\kappa}_0\cdot\boldsymbol{\kappa}A.
\end{aligned} \tag{30}$$

Solving these four equations, we get

$$A_1 = (+\tfrac{1}{2}i\hbar\omega_0)\left\{\frac{[1-\xi^2(\kappa_0^2+\kappa^2)]A + 2i\xi^2\boldsymbol{\kappa}_0\cdot\boldsymbol{\kappa}B}{(\epsilon^2+g^2) - (\tfrac{1}{2}\hbar\omega_0)^2}\right\}, \tag{31}$$

$$A_2 = \left(\frac{\epsilon^2+g^2}{g}\right)\left\{\frac{[1-\xi^2(\kappa_0^2+\kappa^2)]A + 2i\xi^2\boldsymbol{\kappa}_0\cdot\boldsymbol{\kappa}B}{(\epsilon^2+g^2) - (\tfrac{1}{2}\hbar\omega_0)^2}\right\}, \tag{32}$$

$$B_1 = \left\{\frac{[1-\xi^2(\kappa_0^2+\kappa^2)]B - 2i\xi^2\boldsymbol{\kappa}_0\cdot\boldsymbol{\kappa}A}{(\epsilon^2+g^2) - (\tfrac{1}{2}\hbar\omega_0)^2}\right\}, \tag{33}$$

$$B_2 = \left(\frac{-\tfrac{1}{2}i\hbar\omega_0}{g}\right)\left\{\frac{[1-\xi^2(\kappa_0^2+\kappa^2)]B - 2i\xi^2\boldsymbol{\kappa}_0\cdot\boldsymbol{\kappa}A}{(\epsilon^2+g^2) - (\tfrac{1}{2}\hbar\omega_0)^2}\right\}. \tag{34}$$

The above solutions for A_1, A_2, B_1, and B_2 must be made consistent with the original definitions of the constants A and B. Thus, defining

$$I_1 \equiv N(0)\, V \int_0^{\hbar\omega} \left(\frac{\epsilon^2}{\epsilon^2+g^2}\right)\left(\frac{(\epsilon^2+g^2)^{1/2}}{(\epsilon^2+g^2) - (\tfrac{1}{2}\hbar\omega_0)^2}\right) d\epsilon, \tag{35}$$

$$I_2 \equiv N(0)\, V \int_0^{\hbar\omega} \left(\frac{(\epsilon^2+g^2)^{1/2}}{(\epsilon^2+g^2) - (\tfrac{1}{2}\hbar\omega_0)^2}\right) d\epsilon, \tag{36}$$

we can write the consistency relations as

$$[1-\xi^2(\kappa_0^2+\kappa^2)]B - 2i\xi^2\boldsymbol{\kappa}_0\cdot\boldsymbol{\kappa}A = B/I_1, \tag{37}$$

$$+2i\xi^2\boldsymbol{\kappa}_0\cdot\boldsymbol{\kappa}B + [1-\xi^2(\kappa_0^2+\kappa^2)]A = A/I_2. \tag{38}$$

This leads to the secular equation

$$[1-\xi^2(\kappa_0^2+\kappa^2) - 1/I_1][1-\xi^2(\kappa_0^2+\kappa^2) - 1/I_2] - [2\xi^2\boldsymbol{\kappa}_0\cdot\boldsymbol{\kappa}]^2 = 0, \tag{39}$$

the solutions to which give the characteristic frequency ω_0 versus the wave vector $\boldsymbol{\kappa}$ for the collective excitation.

Substituting (38) into (31) and (32), and (37) into (33) and (34), we get

$$A_1=(\tfrac{1}{2}i\hbar\omega_0/I_2)[(\epsilon^2+g^2)-(\tfrac{1}{2}\hbar\omega_0)^2]^{-1}A, \tag{40}$$

$$A_2=[(\epsilon^2+g^2)/gI_2][(\epsilon^2+g^2)-(\tfrac{1}{2}\hbar\omega_0)^2]^{-1}A, \tag{41}$$

$$B_1=(1/I_1)[(\epsilon^2+g^2)-(\tfrac{1}{2}\hbar\omega_0)^2]^{-1}B, \tag{42}$$

$$B_2=(-\tfrac{1}{2}i\hbar\omega_0/gI_1)[(\epsilon^2+g^2)-(\tfrac{1}{2}\hbar\omega_0)^2]^{-1}B. \tag{43}$$

Consider the possibility of charge unbalance due to the collective excitation. The net charge density is proportional to

$$\begin{aligned} 2N(0)\int_{-\hbar\omega}^{+\hbar\omega} s_3 d\epsilon &= N(0)\int_{-\hbar\omega}^{+\hbar\omega}\cos\theta\, d\epsilon \\ &\cong -N(0)g\int_{-\hbar\omega}^{+\hbar\omega}(\epsilon^2+g^2)^{-1/2}\delta\theta\, d\epsilon \\ &\propto -2N(0)g\int_0^{\hbar\omega}(\epsilon^2+g^2)^{-1/2}A_1(\epsilon)\, d\epsilon \\ &= -i\hbar\omega_0 N(0)gAI_2^{-1}\int_0^{\hbar\omega}[(\epsilon^2+g^2)-(\tfrac{1}{2}\hbar\omega_0)^2]^{-1}(\epsilon^2+g^2)^{-1/2}\, d\epsilon. \end{aligned} \tag{44}$$

Under what conditions will (44) vanish? It cannot vanish from setting $g=0$, since $g>0$ here. (A vanishing g corresponds to setting our unperturbed state equal to the normal state of zero current at $T=0$.) It can vanish if $\hbar\omega_0=0$. However, here we are only really concerned with finite ω_0 (and finite $\boldsymbol{\kappa}$), since the supercurrent can lose both energy and momentum by emission of a quantum of collective oscillation only if $\hbar\omega_0$, $\hbar\kappa>0$. For finite $\hbar\omega_0$, (44) can vanish only if $A=0$. But A can be zero (with $B\neq0$) only if the nondiagonal terms in the secular determinant vanish, i.e., only if

$$\pm 2i\xi^2\boldsymbol{\kappa}_0\cdot\boldsymbol{\kappa}=0. \tag{45}$$

Thus, the only collective excitations which satisfy the condition of charge neutrality are those whose wave vector $\boldsymbol{\kappa}$ is perpendicular to the direction of dc current flow (in addition to those having vanishing $\boldsymbol{\kappa}$ or ω_0).

For the case of finite $\boldsymbol{\kappa}$ and ω_0, we could have inferred this result directly from an inspection of Eqs. (22) and (23). If it were not for the offending term $-2\xi^2\boldsymbol{\kappa}_0\cdot\nabla_R\delta\Delta_\alpha$ in (22), it would be possible to find solutions $\delta\theta$ and $\delta\phi$, both *odd* in ϵ, satisfying Eqs. (22) and (23). Similarly, if it were not for the offending term $+2\epsilon\xi^2\boldsymbol{\kappa}_0\cdot\nabla_R\delta\Delta_\beta$ in (23), it would be possible to find solutions $\delta\theta$ and $\delta\phi$, both *even* in ϵ, satisfying Eqs. (22) and (23). The presence of these additional terms insures that no solution $\delta\theta$ is purely odd in ϵ. But in order for the charge unbalance to vanish, it is necessary that $\delta\theta$ be odd in ϵ. Thus, unless the additional terms can be made to vanish (by setting $\boldsymbol{\kappa}_0\cdot\boldsymbol{\kappa}=0$), one cannot find collective excitations which satisfy charge neutrality. Exactly the same situation occurs when one sets up the equations of motion using the more complicated form of interaction (that of A-Sec. VI), this being the form appropriate to the case[9] where collective oscillations[10] may occur. Thus collective oscillations cannot cause the dc supercurrent to decay.

When $\boldsymbol{\kappa}$ is perpendicular to $\boldsymbol{\kappa}_0$, so that (45) is satisfied, then the secular equation [Eq. (39)] becomes

$$1=[1-\xi^2(\kappa_0^2+\kappa^2)]I_1. \tag{46}$$

This differs from the corresponding secular equation of A [A-Eq. (3.15)] only in that $(\kappa_0^2+\kappa^2)$ replaces κ^2, and g replaces ϵ_0 in I_1. This indicates that the collective excitations which still exist in the presence of a finite dc current have a modified dispersion relation (i.e., ω_0 is a function of $\boldsymbol{\kappa}_0$ as well as of $\boldsymbol{\kappa}$).