Generalized Unitarity for Pions*

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Generalized (or off-shell) unitarity is applied to the two-pion system below its inelastic threshold. The exact relativistic result reproduces the known result of potential-scattering theory for the partial-wave amplitude $\kappa e^{i\delta}$ (κ,δ real) in any given isospin–angular-momentum channel: If one incoming pion is off the mass shell, then δ, as a function of the center-of-mass energy, is correctly given by its mass-shell value.

1. STATEMENT OF GENERALIZED UNITARITY

THE concept of generalized (or off-shell) unitarity has occasionally been mentioned in relativistic particle theory for at least nine years,[1–5] and in nonrelativistic potential theory for at least three.[6–9] (Little contact has existed so far between these two formulations.) The purpose of this note is mainly to show how a simple graphical method is conveniently applied to obtain physical results from generalized unitarity. The example selected (for its simplicity) is the two-pion system, but more general applications will be evident. As a by-product, it is shown that a result, already known in potential scattering, is also valid relativistically for pions. A relativistic, field-theoretic point of view will be taken throughout.

Muraskin and Nishijima's statement of generalized unitarity involves the ω functions, defined by

$$\omega_0=1\,,$$
$$\omega_n(X_1,\cdots,X_n)=(\Box_1+m^2)\cdots(\Box_n+m^2)\times\langle 0|T(\phi(X_1)\cdots\phi(X_n))|0\rangle,\quad (n\geq 1),\quad (1.1)$$

where, extending the formalism to include isospin, we represent by X a set of variables x (space-time) and ξ (isospin index); we have $\Box=\partial_\mu\partial^\mu$, $m=$physical pion mass, $|0\rangle=$physical vacuum, $\phi=$renormalized[10] interacting pion field in the Heisenberg picture; T denotes time-ordering.

It is convenient to define products $\Omega_{n\alpha\beta}(X_1,\cdots,X_{\alpha+\beta};Y_1,\cdots,Y_n;Z_1,\cdots,Z_n)$ as follows:

$$\Omega_{0\alpha\beta}=\omega_\alpha{}^*(X_1,\cdots,X_\alpha)\omega_\beta(X_{\alpha+1},\cdots,X_{\alpha+\beta})\,,$$
$$\Omega_{n\alpha\beta}=\omega_{\alpha+n}{}^*(X_1,\cdots,X_\alpha,Y_1,\cdots,Y_n)\times\omega_{\beta+n}(X_{\alpha+1},\cdots,X_{\alpha+\beta},Z_1,\cdots,Z_n)\quad (n\geq 1).\quad (1.2)$$

(The asterisk denotes complex conjugation.) These are used to construct the functions

$$\mathcal{T}_{\alpha\beta}(X_1,\cdots,X_{\alpha+\beta})=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\left[\prod_{k=1}^{n}\int d^4y_k\int d^4z_k\delta_{\eta_k\xi_k}\times\Delta_+(y_k-z_k)\right]\Omega_{n\alpha\beta}\,,\quad (1.3)$$

where a summation over the η_k and ξ_k is understood. (These indices are associated with y_k and z_k, respectively; in the term with $n=0$, the quantity in square brackets is to be interpreted as the unit factor.)

Generalized unitarity may now be written as follows:

$$\mathcal{T}_{00}=1\,,$$
$$\sum_{\alpha+\beta=r}\sum_{\alpha|\beta}\mathcal{T}_{\alpha\beta}(X_A,\cdots,X_{A'},X_B,\cdots,X_{B'})=0\,,\quad (r\geq 1)\,,\quad (1.4)$$

where $X_A,\cdots,X_{B'}$ is an arrangement of $X_1,\cdots,X_r$ in two sets: α variables $X_A,\cdots,X_{A'}$ and β variables $X_B,\cdots,X_{B'}$. The notation $\sum_{\alpha|\beta}$ indicates a summation over all such arrangements for given α and β. As an example ($r=2$), Eq. (1.4) reads

$$\mathcal{T}_{02}(;X_1,X_2)+\mathcal{T}_{11}(X_1;X_2)+\mathcal{T}_{11}(X_2;X_1)+\mathcal{T}_{20}(X_1,X_2;)=0\,,\quad (1.5)$$

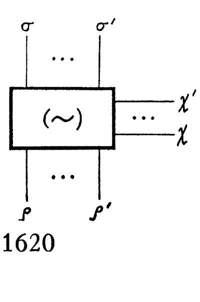

FIG. 1. Diagram representing the function $\lambda_n(P_\rho,\cdots,P_{\rho'},P_\sigma,\cdots,P_{\sigma'},P_\chi,\cdots,P_{\chi'})=$Fourier transform of ω_n. The momenta $p_\rho,\cdots,p_{\rho'}$ are on the lower mass shell, $p_\sigma,\cdots,p_{\sigma'}$ are on the upper mass shell, and $p_\chi,\cdots,p_{\chi'}$ are unrestricted. If the symbol $\sim$ is present, the diagram stands for $\lambda_n{}^\sim=$Fourier transform of $\omega_n{}^*$.

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[1] This concept occurs as early as 1957 in S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1957), in connection with pion production by a virtual photon. See also these authors's bibliographic references to the well-known, but less general theorem proved by Watson, Fermi, and Aizu.

[2] The unitarity statement used in this paper was derived on the basis of conventional field theory by M. Muraskin and K. Nishijima [Phys. Rev. **122**, 331 (1961)] directly from the asymptotic condition of Ref. 10.

[3] A corresponding statement of unitarity has also been obtained by considering the Heisenberg fields ϕ as secondary quantities, derived from the asymptotic fields. See J. C. Stoddart, Nuovo Cimento **34**, 1073 (1964).

[4] In the context of "pure" S-matrix calculations, see R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960); Phys. Rev. Letters **4**, 624 (1960).

[5] For a recent application to the pion-nucleon system, see M. L. Thiebaux, Jr., Phys. Rev. **144**, 1224 (1966); **155**, 1707 (1967).

[6] C. Lovelace, in *Lectures at the 1963 Edinburgh Summer School*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964), and Phys. Rev. **135**, B1225 (1964), Sec. 2b.

[7] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965).

[8] K. L. Kowalski and D. Feldman, J. Math. Phys. **4**, 507 (1963).

[9] K. L. Kowalski, Phys. Rev. Letters **15**, 798 (1965); **15**, 908 (1965); Phys. Rev. **144**, 1239 (1966).

[10] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 205 (1955).

the semicolons being added to emphasize the partition.

For practical applications, we switch to momentum space and to the graphical notation of Fig. 1. Let us define Fourier transforms λ_n by

$$\omega_n(X_1,\cdots,X_n) = (2\pi)^{-4n}\left(\int d^4p_1 e^{-ip_1\cdot x_1}\cdots\int d^4p_n e^{-ip_n\cdot x_n}\right) \times\lambda_n(P_1,\cdots,P_n)\,, \quad (1.6)$$

where P_j stands for p_j (four-momentum) and ξ_j (isospin index). We then represent λ_n by the box diagram of Fig. 1, where the upper (lower) vertical lines correspond to momenta restricted to the upper (lower) mass shell $[p^2=m^2,\ p^0>0\ (p^0<0)]$, while the horizontal lines mean unrestricted momenta. In practice, the λ_n are physical transition amplitudes if all the momenta are on the mass shell. A corresponding diagram stands

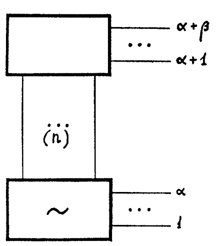

FIG. 2. Diagram for the Fourier transform of a typical term in Eq. (3). This diagram stands for the integral

$$(2\pi)^{-3n}\int\frac{d^3q_1}{2q_1^0}\cdots\int\frac{d^3q_n}{2q_n^0}\lambda_{\alpha+n}^{\sim}(P_1,\cdots,P_\alpha,q_1,\eta_1,\cdots,q_n,\eta_n) \times\lambda_{\beta+n}(P_{\alpha+1},\cdots,P_{\alpha+\beta},-q_1,\eta_1,\cdots,-q_n,\eta_n),$$

where $q_1,\cdots,q_n$ are on the upper mass shell; summation over $\eta_1,\cdots,\eta_n$ is understood. We note that labels may be omitted in a diagram when no ambiguity results.

for $\lambda_n^{\sim}$, the Fourier transform of ω_n^*. We have, suppressing isospin,

$$\lambda_n^{\sim}(p_1,\cdots,p_n)=\lambda_n^*(-p_1,\cdots,-p_n)\,. \quad (1.7)$$

The Fourier transform of a typical term in Eq. (1.3) is shown in Fig. 2, with accompanying instructions on how to evaluate it. As an example, the diagrammatic Fourier transform of $\mathcal{T}_{13}(X_1;X_2,X_3,X_4)$ with, say, p_2 on the upper mass shell and p_3 on the lower mass shell is shown in Fig. 3. We note that $\lambda_0=1$.

2. APPLICATION TO PION-PION SCATTERING

We now illustrate how Eq. (1.4) applies to the two-pion system. We assume that all the pions are real, except for one of the incoming ones, which may be virtual. The interest of this case arises from its relevance to the peripheral model of pion production.[11] We select

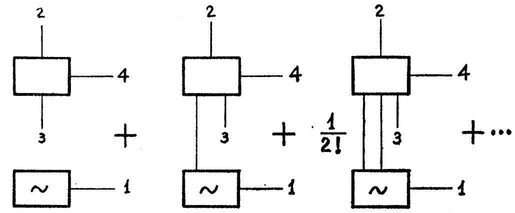

FIG. 3. Example showing the Fourier transform of $\mathcal{T}_{13}(X_1;X_2,X_3,X_4)$ with p_2 and p_3 on the upper and lower mass shell, respectively. The first term, which involves no internal lines, is the ordinary product of the two subdiagrams.

that special case of Eq. (1.4) which is represented by Fig. 4. Diagrams which, for reasons of kinematics or G-parity conservation, do not contribute are not shown. (In this connection we note that the two-line diagram is zero if one of the lines is a mass-shell momentum.) If we restrict the center-of-mass energy of the system to less than four pion masses, only the first three diagrams of Fig. 4 contribute. From conservation of energy, momentum, and isospin we have

$$\lambda_4(P_1,\cdots,P_4)=-i(\Lambda_0+\Lambda_1+\Lambda_2)(2\pi)^4 \times\delta(p_1+\cdots+p_4)\,, \quad (2.1)$$

where

$$\begin{aligned}\Lambda_0&=\tfrac{1}{3}\delta_{12}\delta_{34}\gamma_0\,,\\ \Lambda_1&=\tfrac{1}{2}(\delta_{13}\delta_{24}-\delta_{14}\delta_{23})\gamma_1\,,\\ \Lambda_2&=\tfrac{1}{2}(\delta_{13}\delta_{24}+\delta_{14}\delta_{23}-\tfrac{2}{3}\delta_{12}\delta_{34})\gamma_2\,.\end{aligned} \quad (2.2)$$

(The Kronecker deltas δ_{12}, etc., stand for $\delta_{\xi_1\xi_2}$, etc.) Assuming that p_1 and p_2 correspond to outgoing physical pions, while p_3 belongs to an incoming physical pion (i.e., with $-p_3$ as energy-momentum vector) and p_4 to a virtual one, we can write, in the center-of-mass system,

$$\gamma_T=\gamma_T(M,k,\cos\theta)\quad(T=0,1,2)\,, \quad (2.3)$$

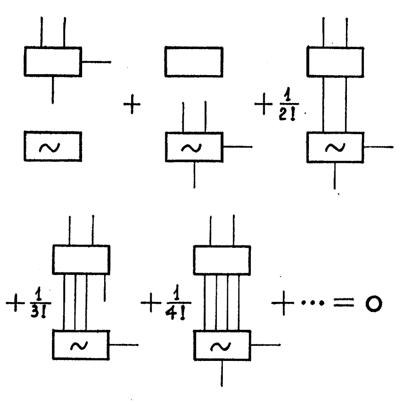

FIG. 4. The Fourier transform of Eq. (1.4), in the peripheral production case. The three dots stand for the diagrams with five or more intermediate lines. Many diagrams are altogether absent for reasons described in the text. The diagram with three internal lines is worth noting as a curiosity, because it does not contribute to the on-shell (i.e., usual) unitarity condition. A subdiagram without lines stands for $\lambda_0=\lambda_0^{\sim}=1$.

[11] See, for example, P. Singer in Lectures at the Liperi Summer School, Liperi, Finland, 1966 (unpublished); C. Goebel, Phys. Rev. Letters 1, 337 (1958); G. F. Chew and F. F. Low, Phys. Rev. **113**, 1640 (1959).

where

$$(p_4)^2=M^2, \quad |\mathbf{p}_1|=|\mathbf{p}_2|=k, \qquad (2.4)$$
$$\theta=\text{angle between } -\mathbf{p}_3 \text{ and } +\mathbf{p}_1.$$

The form (2.3) stems from three-dimensional rotation invariance.

We note that, *if* $M=m$, then each γ_T $(T=0,1,2)$ is invariant under the transformation $p_1\to -p_1, \cdots, p_4\to -p_4$. This is because the switch to energies of the wrong sign can be made good by the further exchange $P_1\leftrightarrow P_3$, $P_2\leftrightarrow P_4$ (which includes isospin) in each Λ_T of Eq. (2.2). On the other hand, if $m\neq M$, the argument breaks down because the above-mentioned exchange would have to map a real pion into a virtual one. In this more general case, we have to invoke time-reversal invariance (which is presumably valid here), or some weaker invariance involving time reversal, say under CPT. This means, for a given isospin component,

$$\lambda_n(p_1,\cdots,p_n)=\lambda_n(-p_1,\cdots,-p_n), \qquad (2.5)$$

which again implies the invariance of γ_T. Owing to this, the second diagram of Fig. 4 is the complex conjugate of the first.

Below the inelastic threshold, and with the help of Eq. (2.1), the equation of Fig. 4 gives us, for each separate T and in the center-of-mass system,

$$2\,\mathrm{Im}\gamma_T(M,k,\cos\theta)$$
$$=-\frac{1}{2(2\pi)^6}\int\frac{d^3q_1}{2q_1^0}\int\frac{d^3q_2}{2q_2^0}\gamma_T^*(M,|\mathbf{q}|,\cos\sphericalangle(\mathbf{q}_1,-\mathbf{p}_3))$$
$$\times\gamma_T(m,k,\cos\sphericalangle(\mathbf{q}_1,\mathbf{p}_1))(2\pi)^4\delta(p_1+p_2-q_1-q_2) \qquad (2.6)$$
$$=\frac{-k}{64\pi^2(k^2+m^2)^{1/2}}\int d\Omega_q\gamma_T^*(M,k,\cos\sphericalangle(\mathbf{q},-\mathbf{p}_3))$$
$$\times\gamma_T(m,k,\cos\sphericalangle(\mathbf{q},-\mathbf{p}_1)), \qquad (2.7)$$

where $\int d\Omega_q$ stands for an integration over all orientations of $\mathbf{q}$.

We can now expand in Legendre polynomials:

$$\gamma_T(M,k,z)=\sum_l \mathcal{A}_{Tl}(M,k)P_l(z). \qquad (2.8)$$

Even when $M\neq m$, we may conclude from the Bose-Einstein symmetry of the two outgoing pions that $\sum_l$ ranges only over even (odd) l when T is even (odd).

Inserting Eq. (2.8) into Eq. (2.7), choosing $\mathbf{p}_1$ as the polar axis in $\mathbf{q}$ space, and using the addition theorem for the P_l:

$$P_l(\cos\sphericalangle(\mathbf{q},-\mathbf{p}_3))$$
$$=(4\pi/(2l+1))\sum_{m=-l}^{l}Y_{lm}^*(\theta,0)Y_{lm}(\theta_q,\varphi_q), \qquad (2.9)$$

we find, on comparing coefficients of $P_l(\cos\theta)$ (suppressing the index T),

$$\mathrm{Im}\mathcal{A}_l(M,k)=\frac{-k\mathcal{A}_l^*(M,k)\mathcal{A}_l(m,k)}{32\pi(k^2+m^2)^{1/2}(2l+1)}. \qquad (2.10)$$

For $M=m$ we obtain the usual unitarity condition

$$\mathcal{A}_l(m,k)=-32\pi(2l+1)(k^2+m^2)^{1/2}$$
$$\times k^{-1}e^{i\delta_l(k)}\sin\delta_l(k) \qquad (2.11)$$

(δ_l real); for $M\neq m$, and using Eq. (2.11), we have

$$\mathrm{Im}\mathcal{A}_l(M,k)=\mathcal{A}_l^*(M,k)e^{i\delta_l(k)}\sin\delta_l(k)=0, \qquad (2.12)$$

which implies

$$\mathcal{A}_l(M,k)=\kappa_l(M,k)e^{i\delta_l(k)} \qquad (2.13)$$

(κ_l real). This is the content of generalized unitarity *plus* time reversal invariance, below the inelastic threshold: The phase of the partial-wave amplitude does not change as one pion goes off the mass shell. This reproduces a known result of potential scattering theory.[9]

In addition, we may note that this statement also applies to some cases where both incoming pions are virtual, for example when they both have positive energy. In such cases, no other diagrams occur than those of the type shown in Fig. 4. In conclusion, it should be stressed that generalized unitarity deserves to be better known as a practical method which can be applied to any scattering process, as well as to decay and production processes. In the particular case investigated here, the real function κ_l in Eq. (2.13) appears as a natural one to calculate in various approximation schemes, in particular if δ_l is already known from on-shell considerations.

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