Single-Collision-Time Theory of Sound Propagation in Liquid He^4 below 0.6°K*

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(Received 23 January 1967)

A unified derivation of expressions for the velocity and the attenuation of sound in liquid He^4 at $T<0.6$°K is presented. The derivation, which is valid for both the hydrodynamic and the collisionless regions, is carried out within the framework of the simple conserving collision-time model. It is shown that using this model one can reproduce the results obtained by Khalatnikov and Chernikova under the assumption of the existence of an equilibrium of collinear phonons. Since applying the collision-time approximation does not involve any assumption of this kind, it is argued that the assumption is in fact unnecessary. The theoretical derivation is accompanied by a discussion of the experimental results obtained by Abraham *et al.* for the attenuation of collisionless sound. Special emphasis is placed on the effects of scattering of phonons from boundaries. It is shown on the basis of these results that the parameter γ which determines the dispersion of the phonon spectrum is likely to be smaller than 2×10^{35} g^{-2} cm^{-2} sec^2.

I. INTRODUCTION

THE theory of sound propagation in superfluid He^4 has been thoroughly investigated in a number of studies reported over the last quarter of a century.[1–13] Although these deal with many aspects of the theory, none except that of Khalatnikov and Chernikova[13] gives a unified derivation of expressions for the sound velocity and the attenuation which is valid for both large and small values of $\omega_s\tau$. (ω_s is the angular frequency of the sound wave and τ is some characteristic lifetime.) A new derivation of a similar kind is given in this paper.

The study we present here is based on the use of three coupled linearized equations: the equation of continuity, the equation of motion of the condensate, and the kinetic equation for the phonons. All effects which are due to rotons are neglected. In the course of the derivation we simplify the kinetic equation by assuming a conserving collision-time model for the collision operator. The algebraic manipulations that follow this simplification are basically similar to the ones performed by Khalatnikov and Andreev[8] who studied the sound velocity using a collisionless kinetic equation.

Deriving the sound velocity and the sound attenuation in this way differs in one important respect from the derivation given by Khalantnikov and Chernikova. These authors use a complicated collision integral whose structure is determined from a microscopic theory of the interactions between phonons.[14] The use of this collision integral adds to the complexity of the set of equations under consideration. In order to simplify these equations Khalatnikov and Chernikova assume that the small-angle phonon-phonon scattering establishes equilibrium of collinear phonons within one period of the wave motion. On this assumption the distribution function for phonons moving in a given direction has a Bose-Einstein form, characterized by a temperature which depends on the direction of the phonon motion. The simplified set of equations obtained as a result of using this distribution function is solved for the sound velocity and the attenuation. As will be shown later, these solutions are similar to the ones we obtain using the collision-time approximation.

Because of the nature of the assumption involved, the derivation given by Khalatnikov and Chernikova applies only for values of $\omega_s\tau$ which satisfy the condition $\omega_s\tau\ll\tau/t$ (or $\omega_s t\ll1$) where t is a time characterizing the establishment of equilibrium in a given direction. Since $\tau/t\gg1$,[15] this condition is satisfied in the entire hydrodynamic region ($\omega_s\tau\ll1$) and in a part of the collisionless region ($1\ll\omega_s\tau\ll\tau/t$). The derivation we present in this paper is likely to hold in an even larger range of $\omega_s\tau$ values. Since we make no assumptions about the structure of the phonon distribution, it seems that this derivation may also be applied in the part of the collisionless region where $\omega_s\tau\gtrsim\tau/t$. This conclusion is of interest in view of the fact that experiments

* This research was supported in part by the U. S. Air Force Office of Scientific Research under Grant No. AF-328-67.
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14 I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965), Chap. 7, p. 40.
15 Explicit estimates of τ and t are given in Sec. III.

have already been performed under conditions such that this inequality is satisfied.[16]

The derivations and considerations reviewed here are based on the assumption that the rotons may be neglected. As this assumption is justified only at temperatures lower than 0.6°K, rotons should be taken into account when higher temperatures are considered. This can be done by coupling a kinetic equation for rotons to the three equations mentioned earlier. The resulting set of equations is quite complicated and will not be considered here. We would like to mention, however, that expressions for the sound velocity and the sound attenuation at $T>0.6$°K were deduced by Khalantikov and Chernikova from this set of equations. The method they used here was similar to the one applied by them for temperatures below 0.6°K.

The study presented in this paper contains some quantitative considerations concerning the attenuation of collisionless sound. Special emphasis is placed on the effects of scattering of phonons from boundaries. Taking into account this boundary scattering we discuss the experimental results obtained recently by Abraham, Eckstein, Ketterson and Vignos.[17] It is argued on the basis of these results that the parameter γ which determines the dispersion of the phonon spectrum is likely to be smaller than 2×10^{35} g^{-2} cm^{-2} sec^2.

The discussion that follows is divided into several sections. The formulation of the theory is briefly reviewed in Sec. II. In Sec. III we introduce the conserving-collision-time model. Section IV consists of a derivation of an algebraic expression, from which explicit expressions for the sound velocity and the sound attenuation are derived in Sec. V. In Sec. VI we present some quantitative considerations concerning the attenuation of the collisionless sound.

II. THE GENERAL THEORY

In this section we briefly describe Khalatnikov's formulation of the theory of sound propagation in superfluid helium.[18] Let us consider a situation where a sound signal is sent into the liquid by some external source. It is assumed that the signal is weak and hence that it induces only small deviations from equilibrium in the liquid. With this assumption it is sufficient to consider the linear response of the liquid to a monochromatic signal. We denote the angular frequency of the sound wave and its wave number by ω_s and k, respectively. It is customary to represent the attenuation of the wave by taking k to be complex:

$$k\equiv k_1+ik_2. \tag{1}$$

The sound velocity is defined by

$$S\equiv\omega_s/k_1. \tag{2}$$

The ratio k_2/k_1 gives a dimensionless measure of the attenuation. It is meaningful to talk about "sound propagation" only when

$$k_2/k_1\ll1. \tag{3}$$

The nature of the restoring forces responsible for the sound-wave motion depends on the quantity $\omega_s\tau$. (τ is a time characterizing the collisions between the quasiparticles.) In the collisionless region, where $\omega_s\tau\gg1$, a given quasiparticle is acted upon by a restoring force which is due to the averaged field of the other quasiparticles. The resulting "collisionless sound" mode is thus similar in its origin to the "zero-sound" mode found in Fermi liquids. In the hydrodynamic region ($\omega_s\tau\ll1$), on the other hand, the restoring forces are mainly due to frequent collisions between the thermally excited quasiparticles. There are two modes of hydrodynamic sound, the well-known "first" and "second" sound. These two modes correspond respectively to "in-phase" and "out-of-phase" motions of the normal and superfluid components.

For each of the sound modes mentioned above there is a different sound velocity S. The various sound velocities and the corresponding attenuations depend exclusively on the equilibrium properties of the liquid. It is the object of the theory we describe here to find the explicit form of this dependence.

The equilibrium state of superfluid helium at rest may be specified by means of a mass density (ρ_0), a temperature T, and a quasiparticle distribution function (n_p^0). At low temperatures ($T<0.6$°K) where rotons can be neglected n_p^0 is the equilibrium phonon distribution:

$$n_p^0=[\exp(\epsilon_p/k_BT)-1]^{-1}. \tag{4}$$

The phonon energy is given by

$$\epsilon_p=S_0p(1-\gamma p^2), \tag{5}$$

where S_0 is the isothermal sound velocity at $T=0$.[19] S_0 and hence ϵ_p depend on the liquid density. The dispersion of the phonon spectrum is determined by the positive quantity γ. An estimate made by Landau and Khalatnikov[20] gives: $\gamma\approx2.8\times10^{37}$ g^{-2} cm^{-2} sec^2. As noted by Khalatnikov and Chernikova, this estimate is very crude; the actual value of γ may be much

[16] We shall say more about this in Sec. III.

[17] B. M. Abraham, Y. Eckstein, J. B. Ketterson, and J. H. Vignos, Phys. Rev. Letters **23**, 1039 (1966).

[18] I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965), Chap. 22, p. 135.

[19] Equation (5) has been used extensively in the literature as a standard description of the phonon spectrum at finite temperatures as well as at $T=0$. It should however be noted that since it involves the $T=0$ isothermal sound velocity, this equation provides a correct description of the spectrum only at $T=0$. The problem of giving a correct expression for ϵ_p at finite temperatures has not yet been settled. Thus, although Eq. (5) is in most cases a reasonable approximation for $T=0$, there may be situations where it is inapplicable.

[20] I. M. Khalatnikov and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **19**, 637 (1949) [English translation in *Collected Papers of L. D. Landau* (Pergamon Press, Ltd., Oxford, England, 1965), p. 494].

smaller.[7,13] It is usually assumed that the dispersion in ϵ_p is very small:

$$\gamma p^2 \ll 1. \tag{6}$$

Even with the high estimate for γ, (6) is satisfied by phonons whose wave number is smaller than 0.5 Å^{-1}. In particular it holds for the thermally excited phonons which play a dominant role in determining the sound velocity and the attenuation.

The sound signal perturbs the equilibrium state of the liquid by inducing space- and time-dependent density fluctuations, which are accompanied by fluctuations in the phonon distribution and by the appearance of a fluctuating longitudinal condensate velocity. Since we have assumed that the signal is weak, it is sufficient to consider the linear response of the liquid to the perturbation. Hence, we may write

$$\begin{aligned} \rho(\mathbf{r},t) &= \rho_0 + \rho' e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_s t)}, \\ n_\mathbf{p}(\mathbf{r},t) &= n_p^0 + n_\mathbf{p}' e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_s t)}, \\ \mathbf{V}_s(\mathbf{r},t) &= \mathbf{V}_s' e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_s t)}, \end{aligned} \tag{7}$$

where ρ, $n_\mathbf{p}$, and $\mathbf{V}_s$ denote the space- and time-dependent density, phonon distribution, and condensate velocity. The deviations of ρ, $n_\mathbf{p}$, and $\mathbf{V}_s$ from their equilibrium values are determined by the small quantities ρ', $n_\mathbf{p}'$, and $\mathbf{V}_s'$. These quantities are not independent. A complete description of their dependence was given by Khalatnikov[18] who used for this purpose the linearized forms of the kinetic equation, the equation of continuity and the equation of motion of the condensate. Following Khalatnikov we may write these equations as follows:

(a) The linearized kinetic equation:

$$(\omega_s - kV_p\cos\theta)n_\mathbf{p}' + kV_p\cos\theta\frac{\partial n_p^0}{\partial\epsilon_p} \times\left(\frac{\partial\epsilon_p}{\partial\rho}\rho' + pV_s'\cos\theta\right) = iI[n_\mathbf{p}'], \tag{8}$$

$V_s' \equiv |\mathbf{V}_s'|$, $V_p \equiv |\mathbf{V}_\mathbf{p}|$, where $\mathbf{V}_\mathbf{p}$ is defined as follows:

$$\mathbf{V}_\mathbf{p} \equiv \partial\epsilon_p/\partial\mathbf{p} = S_0(1-3\gamma p^2)(\mathbf{p}/p); \tag{9}$$

θ is the angle between $\mathbf{k}$ and $\mathbf{p}$. $I[n_\mathbf{p}']$ is the linearized collision integral. By definition it is a linear functional of $n_\mathbf{p}'$.

(b) The linearized equation of continuity:

$$\omega_s\rho' - \rho_0 kV_s' - k\int p\cos\theta n_\mathbf{p}' d\tau = 0. \tag{10}$$

This form of the linearized equation is due to the longitudinal character of $\mathbf{V}_s'$. The integration in (10) is taken over the whole of momentum space.

(c) The linearized equation of motion:

$$\mathbf{k}c_0^2\frac{\rho'}{\rho_0} - \omega_s\mathbf{V}_s' + \mathbf{k}\int\frac{\partial\epsilon_p}{\partial\rho}n_\mathbf{p}'d\tau = 0, \tag{11}$$

where

$$c_0^2 \equiv S_{is}^2 - \rho_0\int\left(\frac{\partial\epsilon_p}{\partial\rho}\right)^2\frac{\partial n_p^0}{\partial\epsilon_p}d\tau \equiv S_0^2 + \rho_0\int\frac{\partial^2\epsilon_p}{\partial\rho^2}n_p^0 d\tau. \tag{12}$$

S_{is} is the isothermal sound velocity. Since $\mathbf{V}_s'$ is longitudinal we can replace (11) by

$$kc_0^2\frac{\rho'}{\rho_0} - \omega_s V_s' + k\int\frac{\partial\epsilon_p}{\partial p}n_\mathbf{p}'d\tau = 0. \tag{13}$$

Equations (8), (10), and (13) form a closed set of linear relations between ρ', V_s', and $n_\mathbf{p}'$. They are consistent with one another only when a certain condition relating the parameters which appear in them is satisfied. The sound velocity and the sound attenuation can be determined from this consistency condition.

III. THE COLLISION-TIME MODEL

It is not known how to derive the consistency condition for the three linearized equations (8), (10), (13) written in their most general form. However, a derivation is possible under some simplifying assumptions. Such a derivation was given by Khalatnikov and Andreev[8] who assumed that the phonon collisions could be neglected. These authors therefore omitted the collision integral from the kinetic equation. Having done this they obtained a set of equations for which the consistency condition was easily derived. Recently Khalatnikov and Chernikova[13] have taken into account the effects of collisions. As we have mentioned in Sec. I the basic assumption in their work is that phonons moving in a given direction are in equilibrium with one another.

The assumption made by Khalatnikov and Chernikova can be justified as long as

$$\omega_s t \ll 1, \tag{14}$$

where t is a characteristic time for small-angle phonon-phonon scattering. The time characterizing small-angle scattering of a phonon of energy ϵ_p was estimated by Landau and Khalatnikov,[21] who obtained the expression:

$$\frac{1}{t_p} \approx \frac{(u+1)^4}{3456\pi}\frac{1}{S_0\gamma\rho_0}\left(\frac{k_B}{S_0\hbar}\right)^7 T^7 X_p(6+X_p)^3 \text{ sec}^{-1}, \quad X_p \equiv \frac{\epsilon_p}{k_B T}. \tag{15}$$

The parameters entering (15) are

$$S_0 \approx 238m \text{ sec}^{-1}, \quad u \equiv \frac{\rho_0}{S_0}\frac{\partial S_0}{\partial\rho} \approx 2.7, \quad \rho_0 \approx 0.145 \text{ g cm}^{-3}, \quad \gamma \approx 2.8\times10^{37} \text{ g}^{-2}\text{ cm}^{-2}\text{ sec}^2. \tag{16}$$

Therefore,

$$1/t_p \approx 2\times10^5 T^7 X_p(6+X_p)^3 \text{ sec}^{-1}. \tag{17}$$

For thermal phonons $(X_p)_{\text{av}}\approx 3$. Therefore,

$$1/t\approx 4\times 10^{8}T^{7}\ \text{sec}^{-1}. \tag{18}$$

Substituting (18) into (14) gives

$$\omega_s t\approx 2.5\times 10^{-9}\omega_s/T^{7}\ll 1. \tag{19}$$

This inequality is violated at very low temperatures and at very high frequencies. The temperatures and frequencies needed for this to happen can be reached by using current experimental techniques. In fact the sound attenuation has already been measured in a temperature and frequency range (0.12 deg$\leq T\leq$0.45 deg, 30 Mc/sec$\leq\omega_s/2\pi\leq$150 Mc/sec) where (19) does not hold.[21] Noting this, one may ask whether the results obtained by Khalatnikov and Chernikova are valid when an equilibrium of collinear phonons cannot be established within a period of the wave motion. In the following we shall try to answer this question by calculating the sound velocity and the sound attenuation without making any assumption concerning the existence of this equilibrium.

We start by assuming a simple model for the collision integral:

$$I[n_p']=-\frac{1}{\tau}\left[n_p'-\frac{\int n_p'\epsilon_p d\tau}{\int n_p^0(1+n_p^0)\epsilon_p^2 d\tau}n_p^0(1+n_p^0)\epsilon_p -3\frac{\int n_p' p\cos\theta d\tau}{\int n_p^0(1+n_p^0)p^2 d\tau}n_p^0(1+n_p^0)p\cos\theta\right]. \tag{20}$$

This expression includes a single collision time τ. Models of this kind are sometimes referred to as "collision-time" models. Conservation of linear momentum and energy are built into our model, since the collision integral satisfies the conditions[22]:

$$\int I[n_p']\epsilon_p d\tau=0 \quad \int I[n_p']p\cos\theta d\tau=0. \tag{21}$$

We may therefore expect to obtain correct results for the hydrodynamic sound.

[21] In estimating the lifetime t we assumed that $\gamma=2.8\times10^{37}$ g^{-2} cm^{-2} sec^{2}. As mentioned in Sec. II the actual value of γ may be much smaller. We shall see in Sec. VI that this value is likely to be smaller than 2×10^{35}. Decreasing the value of γ shortens the lifetime t and hence increases the range of validity of the inequality $\omega_s t\ll 1$. However, even with $\gamma\approx 1\times10^{34}$ g^{-2} cm^{-2} sec^{-2} the inequality does not hold everywhere in the range of temperatures and frequencies mentioned above.

[22] The origin of the conserving collision-time model is best presented within the framework of a study of the eigenfunction expansion of the collision operator [see for example: G. E. Uhelenbeck and G. W. Ford, *Lectures in Statistical Mechanics* (American Mathematical Society, Providence, Rhode Island, 1963); Chap. 3, pp. 82–3]. The model is obtained when the eigenvalue (relaxation) spectrum is assumed to consist of only two eigenvalues; a zero eigenvalue corresponding to eigenfunctions which are conserved in the collisions (energy and momentum in our case) and a nonzero eigenvalue $(-1/\tau)$ associated with nonconserved eigenfunctions. A similar model was used by A. A. Abrikosov and I. M. Khalatnikov in their study of liquid He^3 [Rept. Progr. Phys. **22**, 329 (1959)].

Within the framework of the collision-time model, transport properties depend on only a single collision time. Thus, in order to get realistic results for these properties one should identify τ with the lifetime for collisions which contribute to transport phenomena. The small-angle scattering mentioned earlier do not contribute to these phenomena. However, contributions do come from large-angle scattering of phonons.[23] The most frequent of these is the large-angle phonon-phonon scattering. We shall therefore identify τ with the lifetime for this process. An estimate of this lifetime was given by Andronikashvili[24]:

$$\frac{1}{\tau}=\frac{9\cdot 13!}{2^{20}}\frac{(u+1)^4}{(\pi\hbar)^7}\frac{1}{\rho_0^2 S_0}\left(\frac{k_B T}{S_0}\right)^9\approx 3\cdot 10^{7}T^{9}\text{sec}^{-1}. \tag{22}$$

Substituting (20) into (8) we get our modified kinetic equation:

$$(\omega-kV_p\cos\theta)n_p'+kV_p\cos\theta\frac{\partial n_p^0}{\partial\epsilon_p}\left(\frac{\partial\epsilon_p}{\partial\rho}\rho'+pV_s'\cos\theta\right) =\frac{i}{\tau}\left[\frac{\int n_p'\epsilon_p d\tau}{\int n_p^0(1+n_p^0)\epsilon_p^2 d\tau}n_p^0(1+n_p^0)\epsilon_p +3\frac{\int n_p' p\cos\theta d\tau}{\int n_p^0(1+n_p^0)p^2 d\tau}n_p^0(1+n_p^0)p\cos\theta\right], \tag{23}$$

where

$$\omega\equiv\omega_s+i/\tau. \tag{24}$$

In the next section we shall derive the consistency condition for the three linearized equations (10), (13), and (23). The sound velocity and the attenuation will be deduced from this condition in Sec. V.

IV. THE CONSISTENCY CONDITION

Let us rewrite Eqs. (10) and (13) in a dimensionless form:

$$\frac{\omega_s}{kc_0}\frac{\rho'}{\rho_0}-\frac{V_s'}{c_0}=\frac{1}{c_0\rho_0}\int n_p' p\cos\theta d\tau, \tag{25}$$

$$\frac{\rho'}{\rho_0}-\frac{\omega_s}{kc_0}\frac{V_s'}{c_0}=-\frac{u}{c_0^2\rho_0}\int n_p'\epsilon_p d\tau. \tag{26}$$

The dimensionless Grüneisen parameter u is a measure of the dependence of the sound velocity on the density; its value is given in (16).

Equations (25)–(26) are linear and homogeneous in the variables ρ'/ρ_0, V_s/c_0, $\int n_p'\epsilon_p d\tau$, and $\int n_p' p\cos\theta d\tau$. We shall now derive from the kinetic equation two

[23] I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York 1965), Chap. 19, p. 122; Chap. 20, p. 127.

[24] E. L. Andronikashvili, Zh. Eksperim. i Teor. Fiz. **18**, 429 (1948).

additional equations of the same kind. To get the first of these we multiply (23) by $\epsilon_p/[c_0^2\rho_0(\omega_s-kV_p\cos\theta)]$ and integrate over momentum space. The angular integration gives

$$\frac{1}{c_0^2\rho_0}\int n_p'\epsilon_p d\tau+\left(\frac{S_0}{c_0}\right)^2\frac{\rho'}{\rho_0}\frac{u}{\rho_0}\int\frac{\partial n_p^0}{\partial\epsilon_p}p^2\omega(\xi_p)d\tau+\left(\frac{S_0}{c_0}\right)\frac{V_s'}{c_0}\frac{1}{\rho_0}\int\frac{\partial n_p^0}{\partial\epsilon_p}p^2\xi\omega(\xi_p)d\tau$$
$$=-\frac{1}{\sigma}\left(\int n_p'\epsilon_p d\tau\Big/\int\frac{\partial n_p^0}{\partial\epsilon_p}\epsilon_p^2 d\tau\right)\frac{S_0^2}{c_0^2\rho_0}\int\frac{\partial n_p^0}{\partial\epsilon_p}p^2\frac{1+\omega(\xi_p)}{\xi}d\tau$$
$$-\frac{3}{\sigma}\left(\int n_p'p\cos\theta d\tau\Big/\int\frac{\partial n_p^0}{\partial\epsilon_p}p^2 d\tau\right)\frac{S_0}{c_0^2\rho_0}\int\frac{\partial n_p^0}{\partial\epsilon_p}p^2\omega(\xi_p)d\tau, \quad (27)$$

where

$$\omega(\xi_p)\equiv-1+\frac{\xi}{2}\ln\frac{\xi+1}{\xi_p-1}, \quad (28)$$

$$\xi_p\equiv\frac{\omega}{kV_p}\approx\frac{\omega_s}{kS_0}\left(1+\frac{i}{\omega_s\tau}\right)(1+3\gamma p^2), \quad (29a)$$

$$\xi\equiv\frac{\omega}{kS_0}\equiv\frac{\omega_s}{kS_0}\left(1+\frac{i}{\omega_s\tau}\right), \quad (29b)$$

$$\frac{1}{\sigma}\equiv-\frac{i}{kS_0\tau}. \quad (29c)$$

In deriving (27) we have neglected contributions coming from terms containing the small factor γp^2. The only place where we retained this factor is the logarithm in (28), whose behavior depends strongly on γp^2 when $\omega_s\tau\gg1$. To evaluate integrals of the type $\int(\partial n_p^0/\partial\epsilon_p)\log(\xi_p-1)p^4dp$ which occur in (27) we employ the method used by Khalatnikov and Andreev.[8] According to this method the quantity ξ_p appearing in the integral is replaced by ξ_{pT} where p_T is the mean thermal momentum:

$$p_T\equiv3(k_BT/S_0)\approx1.7\cdot10^{-20}\text{ g cm sec}^{-1}. \quad (30)$$

Evaluating the integrals in this manner, we find

$$\left(1+\frac{1+\omega(\xi_T)}{\xi\sigma}\right)\frac{1}{c_0^2\rho_0}\int n_p'\epsilon_p d\tau+\frac{3\omega(\xi_T)}{\sigma}\frac{S_0}{c_0}\frac{1}{c_0\rho_0}\int n_p'p\cos\theta d\tau-3\frac{\rho_n}{\rho_0}\omega(\xi_T)\frac{S_0}{c_0}\left(u\frac{S_0}{c_0}\frac{\rho'}{\rho_0}+\xi\frac{V_s'}{c_0}\right)=0, \quad (31)$$

where

$$\omega(\xi_T)\equiv\omega(\xi_p)|_{p=p_T}, \quad (32)$$

$$\frac{\rho_n}{\rho_0}\equiv-\frac{1}{3\rho_0}\int\frac{\partial n_p^0}{\partial\epsilon_p}p^2d\tau. \quad (33)$$

ρ_n is the mass density of the normal fluid.[1] At this point one can estimate the errors introduced by the neglect of terms proportional to γp^2 in the derivation of (27). These terms are of order $\int(\partial n_p^0/\partial\epsilon_p)p^2\gamma p^2d\tau$. The final results we obtain for the sound velocity and attenuation will therefore be correct only to zero order in $\gamma p_T^2(\rho_n/\rho_0)$. For this reason only first-order expansions of these results in ρ_n/ρ_0 and $\rho_n/\rho_0\ln\rho_n/\rho_0$ will be considered.

Equation (31) is our third equation relating ρ'/ρ_0, V_s'/c_0, $\int n_p'\epsilon_p d\tau$, and $\int n_p'p\cos\theta d\tau$. The fourth equation is similar to (31);

$$\frac{\omega(\xi_T)}{\sigma}\frac{c_0}{S_0}\frac{1}{c_0^2\rho_0}\int n_p'\epsilon_p d\tau+\left(1+\frac{3\xi\omega(\xi_T)}{\sigma}\right)\frac{1}{c_0\rho_0}\int n_p'p\cos\theta d\tau-3\frac{\rho_n}{\rho_0}\left(u\frac{S_0}{c_0}\xi\omega(\xi_T)\frac{\rho'}{\rho_0}+(\xi^2\omega(\xi_T)-\tfrac{1}{3})\frac{V_s'}{c_0}\right)=0. \quad (34)$$

To get (34) we multiplied the kinetic equation (23) by $[p\cos\theta/c_0\rho_0(\omega_s-kV_p\cos\theta)]$ and then integrated over momentum space. Eliminating the integrals $\int n_p'\epsilon_p d\tau$ and $\int n_p'p\cos\theta d\tau$ from Eqs. (25), (26), (31), and (34) gives two linear homogeneous equations in ρ'/ρ_0 and V_s'/c_0:

$$\left[\left(1+\frac{1}{\xi\sigma}\right)+\frac{\omega(\xi_T)}{\xi\sigma}\left(1-3u\frac{S_0}{c_0}\xi\frac{\omega_s}{kc_0}\right)+3\frac{\rho_n}{\rho_0}\left(u\frac{S_0}{c_0}\right)^2\omega(\xi_T)\right]\frac{\rho'}{\rho_0}$$
$$-\left[\left(1+\frac{1}{\xi\sigma}\right)\frac{\omega_s}{kc_0}+\frac{\omega(\xi_T)}{\xi\sigma}\left(\frac{\omega_s}{kc_0}-3u\frac{S_0}{c_0}\xi\right)-3\frac{\rho_n}{\rho_0}u\frac{S_0}{c_0}\xi\omega(\xi_T)\right]\frac{V_s'}{c_0}=0, \quad (35)$$

$$\left[u\frac{S_0}{c_0}\frac{\omega_s}{kc_0}+\frac{\omega(\xi_T)}{\xi\sigma}\left(3u\frac{S_0}{c_0}\xi^2\frac{\omega_s}{kc_0}-\xi\right)-3\frac{\rho_n}{\rho_0}\left(u\frac{S_0}{c_0}\right)^2\xi\omega(\xi_T)\right]\frac{\rho'}{\rho_0}$$

$$-\left[u\frac{S_0}{c_0}+\frac{\omega(\xi_T)}{\xi\sigma}\left(3u\frac{S_0}{c_0}\xi^2-\frac{\omega_s}{kc_0}\xi\right)+3\frac{\rho_n}{\rho_0}u\frac{S_0}{c_0}(\xi^2\omega(\xi_T)-\tfrac{1}{3})\right]\frac{V_s'}{c_0}=0. \quad (36)$$

For these equations to be consistent with one another their determinant must vanish. Using the explicit expression for the determinant we obtain the following algebraic representation of the consistency condition:

$$\left(\left(\frac{\omega_s}{kc_0}\right)^2-1+\frac{\rho_n}{\rho_0}\right)\phi=3\frac{\rho_n}{\rho_0}\left(\frac{S_0}{c_0}\right)^2\left\{u^2+\left(\frac{\omega_s}{kS_0}\right)^2\left[2u+\left(\frac{c_0}{S_0}\right)^2\right]\right\}-3u^2\left(\frac{S_0}{c_0}\right)^2\left(\frac{\rho_n}{\rho_0}\right)^2, \quad (37)$$

where

$$\phi\equiv\frac{1+i\omega_s\tau[\omega(\xi_T)]^{-1}}{i\omega_s\tau-1}+\frac{3}{i\omega_s\tau}\left(\frac{\omega_s}{kS_0}\right)^2. \quad (38)$$

Expressions for the sound velocity and the sound attenuation will be derived from Eq. (37) in the next section.

V. THE CALCULATION OF THE SOUND VELOCITY AND THE SOUND ATTENUATION

The unknown in Eq. (37) is ω_s/kS_0. Using (2) and (3) we can write this unknown as

$$\frac{\omega_s}{kS_0}\approx\frac{S}{S_0}\left(1-i\frac{k_2}{k_1}\right). \quad (39)$$

Hence solving (37) for ω_s/kS_0 will yield the sound velocity S and the corresponding attenuation. In the following we shall present the solutions for both the hydrodynamic region ($\omega_s\tau\ll1$) and the collisionless region ($\omega_s\tau\gg1$).

A. The Hydrodynamic Region

Assuming that $\omega_s\tau\ll1$ we now replace the function ϕ which appears in (37) by the leading terms in its expansion in powers of $\omega_s\tau$. Keeping only terms of first order in ρ_n/ρ_0 and $\omega_s\tau$, we obtain the following equation:

$$\left[\left(\frac{S_0}{c_0}\right)^2\left(\frac{\omega_s}{kS_0}\right)^2-1+\frac{\rho_n}{\rho_0}\right]\left[3\left(\frac{\omega_s}{kS_0}\right)^2-1+\tfrac{4}{5}i\omega_s\tau\right]$$

$$=3\frac{\rho_n}{\rho_0}\left[u^2+\left(\frac{\omega_s}{kS_0}\right)^2(2u+1)\right]. \quad (40)$$

This is a quadratic equation in $(\omega_s/kS_0)^2$. Its two solutions correspond to the two modes of hydrodynamic sound normally referred to as "first"and "second" sound. Solving Eq. (40) and using (12) we get

$$\frac{S_1}{S_{is}}\approx1+\frac{(3u+1)^2}{4}\frac{\rho_n}{\rho_0}, \quad (41)$$

$$(k_2/k_1)_1\approx\tfrac{3}{10}(u+1)^2\rho_n/\rho_0\omega_s\tau, \quad (42)$$

$$\frac{S_2}{S_{is}}\approx\frac{1}{\sqrt{3}}\left\{1-\tfrac{3}{4}[(u+1)^2+V^2]\frac{\rho_n}{\rho_0}\right\}, \quad (43)$$

$$(k_2/k_1)_2\approx\tfrac{2}{5}\omega_s\tau. \quad (44)$$

S_1 and S_2 denote the first and second sound velocities. $(k_1/k_2)_1$ and $(k_1/k_2)_2$ denote the corresponding attenuations. V^2 is defined as follows:

$$V^2=\frac{1}{2}\frac{\rho_0^2}{S_0}\frac{\partial^2S_0}{\partial\rho_0^2}. \quad (45)$$

B. The Collisionless Region

Assuming that $\omega_s\tau\gg1$ one can expand ϕ (38) in terms of the quantity $1/\omega_s\tau$. The leading term in this expression is the logarithm contributed by the function $\omega(\xi_T)$. Keeping only this term in (37) we get an equation from which the collisionless sound velocity and the corresponding attenuation can be determined:

$$\left[\left(\frac{S_0}{c_0}\right)^2\left(\frac{\omega_s}{kS_0}\right)^2-1+\frac{\rho_n}{\rho_0}\right]=-\frac{3}{2}\frac{\rho_n}{\rho_0}\left[u^2+\left(\frac{\omega_s}{kS_0}\right)^2(2u+1)\right]$$

$$\times\ln\frac{1}{2}\left[\frac{\omega_s}{kS_0}\left(1+\frac{i}{\omega_s\tau}\right)(1+3\gamma p_T^2)-1\right]. \quad (46)$$

When (12) and (39) are taken into account one obtains the following solutions:

$$\frac{S}{S_{is}}\approx1+\frac{3}{4}\frac{\rho_n}{\rho_0}(u+1)^2\ln2\omega_s\tau, \quad (47)$$

and

$$\frac{k_2}{k_1}\approx\frac{3\pi}{8}\frac{\rho_n}{\rho_0}(u+1)^2, \quad (48)$$

when $1\ll\omega_s\tau\ll(3\gamma p_T^2)^{-1}$;

$$\frac{S}{S_{is}} \approx 1 - \frac{3}{4}\frac{\rho_n}{\rho_0}(u+1)^2 \ln\frac{3\gamma p_T^2}{2}, \quad (49)$$

and

$$\frac{k_2}{k_1} \approx -\frac{3}{4}\frac{\rho_n}{\rho_0}(u+1)^2 \frac{1}{3\gamma p_T^2 \omega_s \tau} \quad (50)$$

when $1 \ll (3\gamma p_T^2)^{-1} \ll \omega_s\tau$. It should be noted that in writing the expressions for the sound velocity we have kept only a term proportional to a product of ρ_n/ρ_0 and a logarithm. This is in accordance with the way in which we approximated Eq. (37). Since $\ln 2\omega_s\tau$ and $\ln 3\gamma p_T^2/2$ are large we are able to neglect terms of order ρ_n/ρ_0 which do not contain a logarithm.

C. Discussion

The results we obtained here for the sound velocity and the sound attenuation are identical with those of Khalatnikov and Chernikova[13] to lowest order in ρ_n/ρ_0 and $\omega_s\tau$. As we already mentioned in Secs. I and III these authors calculated the sound velocity and attenuation assuming the existence of an equilibrium of collinear phonons. Hence their results appeared to be valid only in the hydrodynamic region and in a part of the collisionless region where $\omega_s\tau \ll \tau/t$. However, we see now that similar results can be derived from a simple collision-time model whose application is not necessarily equivalent to making this assumption. Therefore, it seems that the assumption is unnecessary and that the results of the calculation are also valid outside the temperature and frequency range in which $\omega_s\tau \ll \tau/t$. It is of interest to compare the expressions obtained here for the attenuation of collisionless sound with the results of calculations done previously by several authors.[4,7,12] The main step in these calculations is the estimation of a characteristic lifetime for the collisions of low-energy (acoustic) phonons with thermal phonons. Knowing this lifetime it is possible to estimate the attenuation using the following relation:

$$k_2/k_1 = 1/\omega_s\theta. \quad (51)$$

θ denotes the characteristic lifetime while ω_s stands for the angular frequency of the acoustic phonon. In order to estimate θ one has to assume a certain mechanism for the collisions between thermal and acoustic phonons. In this context, three- and four-phonon processes were studied. It was shown that the attenuation associated with the three-phonon processes is given by[25]:

$$\frac{k_2}{k_1} = -\frac{3}{4}\frac{\rho_n}{\rho_0}(u+1)^2[\arctan 2\omega_s\tau - \arctan(3\gamma p_T^2 \omega_s \tau)], \quad (52)$$

whereas the attenuation due to the four-phonon processes is[14]:

$$\frac{k_2}{k_1} = \frac{5}{8\pi^3}\frac{(u+1)^4}{\rho_0^2 S_0^2 \gamma}\left(\frac{k_B T}{\hbar S_0}\right)^6. \quad (53)$$

[25] C. J. Pethick (private communication). The original formula derived by Pethick and ter Haar (see Ref. 12) includes an extra factor of $\frac{1}{2}\pi$.

To first order in $1/\omega_s\tau$, Eq. (52) is in a full agreement with the results we obtained earlier for the attenuation of the collisionless sound. This means that our results describe an attenuation which may be interpreted as being due to three-phonon processes. As can be seen from Eqs. (48) and (50) this "three-phonon" attenuation depends on the temperature through ρ_n/ρ_0 in cases where $\omega_s\tau \ll (3\gamma p_T^2)^{-1}$ and through $(\rho_n/\rho_0) \times (3\gamma p_T^2 \omega_s\tau)^{-1}$ in cases where $\omega_s\tau \gg (3\gamma p_T^2)^{-1}$. Since ρ_n/ρ_0 is proportional to T^4, p_T^2 to T^2, and τ to T^{-9}, we find a variation of the attenuation as T^4 when $\omega_s\tau \ll (3\gamma p_T^2)^{-1}$ and as T^{11} When $\omega_s\tau \gg (3\gamma p_T^2)^{-1}$.

It was noted by Kawasaki[4] that energy can be conserved in a three-phonon collision involving an acoustic phonon, only when $\omega_s\tau < (3\gamma p_T^2)^{-1}$. One can therefore argue that for $\omega_s\tau > (3\gamma p_T^2)^{-1}$ processes other than three-phonon collisions may be important. This raises doubts as to the validity of the "three-phonon" result (e.g., the variation of the attenuation as T^{11}) we obtained for $\omega_s\tau \gg (3\gamma p_T^2)^{-1}$. In view of Kawasaki's remark it seems reasonable to assume that in this case the attenuation should be due to four-phonon collisions and hence proportional to T^6 (53). It is hard to determine which one of the two predictions is correct. However, it is obviously clear that both of them differ from the one obtained for $\omega_s\tau \ll (3\gamma p_T^2)^{-1}$ (a variation as T^4). In other words, even without making specific predictions concerning the temperature dependence of the attenuation in the temperature and frequency range where $\omega_s \gg (3\gamma p_T^2)^{-1}$ it is clear that it should be considerably different than the T^4 dependence expected for the range where $\omega_s\tau \ll (3\gamma p_T^2)^{-1}$.[26]

VI. ATTENUATION OF COLLISIONLESS SOUND; QUANTITATIVE CONSIDERATIONS

In this section we present some quantitative considerations concerning the attenuation of the collisionless sound. The discussion is centered around the experimental results obtained recently by Abraham, Eckstein, Ketterson, and Vignos.[17] These authors showed that the attenuation varies approximately as T^4 in a range of temperatures and frequencies given by $0.12\text{deg} \leq T \leq 0.45\text{deg}$, $30\text{Mc/sec} \leq \omega_s/2\pi \leq 150\text{Mc/sec}$. In view of this result and in accordance with the considerations presented in the last section one may conclude that the inequality $\omega_s\tau \ll (3\gamma p_T^2)^{-1}$ must hold throughout the range explored in the experiment. A simple calculation in which the estimates (22) and (30) for τ and p_T are used shows that this can be the case only when $\gamma \ll 1 \times 10^{31}$ g^{-2} cm^{-2} sec^2. It should be emphasized that this argument depends strongly on the assumption that the transport phenomena in the superfluid are governed by large-angle phonon-phonon scattering. In the following we shall see that in the

[26] This conclusion was in fact drawn by a number of authors who mentioned that the anticipated T^4 dependence of the attenuation will change in the range where $\omega_s\tau \gg (3\gamma p_T^2)^{-4}$ (see, for example, Refs. 9–12).

range studied by Abraham *et al.* phonon-phonon scattering is likely to be less important than scattering of phonons by the boundaries of the container. Because of this fact, we shall have to modify our estimate of the bound for γ.

The studies presented in this paper did not take into account collisions of phonons with either He^3 impurities or boundaries. Lifetime estimates based on the calculations of Khalatnikov and Zharkov[27] show that for temperatures higher than 0.1 deg and concentrations lower than 10^{-6} the phonon-impurity scattering is much smaller than the large-angle phonon-phonon scattering. The impurity concentrations were much lower than 10^{-6} in the experiments of Abraham *et al.*[28] Thus, the phonon impurity scattering can indeed be neglected in the analysis of their results. The situation is different in the case of collisions with the boundaries. The lifetime for this scattering process is given by

$$\tau_B \approx L/S_0, \tag{54}$$

where L is some relevant linear dimension characterizing the sizes of the container. The length of the container used in the experiment of Abraham *et al.* is of the order of 1 cm. Hence, in this case: $\tau_B \approx 4.2\times10^{-5}$ sec. According to (22), $\tau \approx 3\times10^{-8}\ T^{-9}$ sec. Consequently, for $T<0.4$ deg the collisions with the walls become more frequent than the large-angle phonon-phonon scattering. Thus, we see that boundary scattering cannot be neglected.[29]

It is very hard to give a satisfactory account of the way in which the attenuation is affected by boundary scattering. However, as is done very often in similar cases, one can obtain a crude estimate of this effect by using instead of τ a new characteristic lifetime $\bar{\tau}$ defined as follows:

$$\frac{1}{\bar{\tau}} = \frac{1}{\tau} + \frac{1}{\tau_B} \approx 3\times10^7 T^9 + 2.3\times10^4\ \text{sec}^{-1}. \tag{55}$$

[27] I. M. Khalatnikov and V. N. Zharkov, Zh. Eksperim. i Teor. Fiz. **32**, 1108 (1957) [English transl.: Soviet Phys.—JETP **5**, 905 (1957)].

[28] Y. Eckstein (private communication).

[29] The possible importance of boundary scattering was first mentioned by K. Dransfeld [Z. Physik **179**, 525 (1964)]. It was also discussed by Pethick and ter Haar (see Ref. 12).

Assuming that $\bar{\tau}$ is the relevant lifetime, the attenuation should vary as T^4 when $\omega_s\bar{\tau} \ll (3\gamma p_T^2)^{-1}$. This inequality must therefore apply in the range studied by Abraham *et al.* Using the estimates (30) and (55) for p_T and $\bar{\tau}$ we find that this can happen only for $\gamma \ll 2\times10^{35}$ g^{-2} cm^{-2} sec^2. This bound for γ is higher by four orders of magnitude than the one we obtained when the boundary scattering was neglected. It is, however, lower by two orders of magnitude than the crude estimate, $\gamma \approx 2.8\times10^{37}$ g^{-2} cm^{-2} sec^2 given by Landau and Khalatnikov.[20] As we mentioned earlier, Khalatnikov and Chernikova[13] remarked that the actual value of γ may be much smaller than 2.8×10^{37}. In view of the considerations presented here it seems that the results obtained by Abraham *et al.* provide further support for this.

At this point we note that because of the weak dependence of the attenuation on γ in the region where $\omega_s\bar{\tau} \ll (3\gamma p_T^2)^{-1}$, it is very hard to use the experimental values obtained in this region for a direct calculation of γ. In order to perform a calculation of this kind one would have to know the experimental values of the attenuation in the region where $\omega_s\bar{\tau} \gtrsim (3\gamma p_T^2)^{-1}$. No measurement of the attenuation in this region has yet been reported. Using the equality $\omega_s\bar{\tau} = (3\gamma p_T^2)^{-1}$ and assuming that $\gamma \approx 5\times10^{34}$ g^{-2} cm^{-2} sec^2, one may estimate the frequencies needed for such an experiment to be of the order of 700 Mc/sec at least. For lower value of γ even higher frequencies may be needed. Such frequencies are higher than any used in previous studies of sound attenuation in superfluid helium below 0.6 deg.

ACKNOWLEDGMENTS

The author would like to thank Professor David Pines for his constant encouragement and stimulating advice, Dr. Christofer J. Pethick for useful discussions and for his help in the preparation of the final manuscript, and Dr. Wolfgang Götze, Dr. Pierre C. Hohenberg, and Professor Michael Wortis for illuminating discussions. The author would also like to acknowledge the hospitality of the Physics Division, Aspen Institute for Humanistic Studies, where part of this study was conducted.