PHYSICAL REVIEW VOLUME 158, NUMBER 5 25 JUNE 1967

Self-Consistent Equations and Self-Coupling of Vector Mesons*

O. FLEISCHMAN,† R. MUSTO, L. O'RAIFEARTAIGH,‡ AND P. S. RAO
Physics Department, Syracuse University, Syracuse, New York
(Received 30 January 1967)

A system of n vector mesons in self-interaction is considered, and the problem of deducing the existence of a symmetry group for the interaction by the use of self-consistency equations is investigated. Two results are established. The first is that a technical condition of complete antisymmetry for the coupling constants, which was used as an assumption in an earlier "bootstrap" demonstration of the existence of a symmetry group by Cutkosky, can be derived from other assumptions. The latter have an immediate physical interpretation and for this reason may be more plausible. Secondly, it is shown that for the physically interesting case $n=8$ at any rate, the existence of the symmetry group for the interaction follows also from the so-called Smushkevich principle.

I. INTRODUCTION

RECENTLY a number of papers[1,2] have appeared in which the existence of an internal symmetry group for strong interactions has been deduced from self-consistency, or bootstrap, equations for the coupling constants. Perhaps the most well-known of these derivations is that due to Cutkosky,[1] who for the case of n vector mesons in trilinear interaction with themselves deduced the invariance of the interaction under a compact semisimple Lie group of order n. In spite of the elegance of Cutkosky's approach, however, his derivation is subject to the limitation that in addition to some rather natural physical assumptions (the conservation of one additive quantum number and the assumption that the states with which one starts are the lowest energy bound states) which are used to supplement the bootstrap hypothesis, a very strong algebraic assumption is used. This assumption, which at least on the surface, has no physical motivation, is that the array $C_{\alpha\beta\gamma}$ of all coupling constants is antisymmetric in all three indices. The antisymmetry in *two* indices is all that is required by physical consideration (Lorentz invariance) and the strength of the assumption of antisymmetry in all three indices can be seen by noting that what one is ultimately attempting to show is that the $C_{\alpha\beta\gamma}$ are the necessarily completely antisymmetric structure constants of a compact semisimple Lie group.

The primary purpose of the present paper is to show that the rather unphysical complete antisymmetry assumption just mentioned can be derived from other assumptions which have an immediate physical interpretation and seem to us somewhat more plausible. These assumptions we shall call the labeling assumption and off-mass-shell assumption, respectively, and they are as follows:

(a) Labeling Assumption: We assume that the particles involved are labeled by *additive conserved* quantum numbers (Q, Y etc.) (the antiparticles having labels $-Q$, $-Y$ etc.) and that this labeling is unique for all except neutral particles i.e., those for which Q, Y etc. are all zero. Further, we assume that there exists at least one neutral.

Assumption (a) is clearly satisfied for the physically interesting vector-meson octet (K^*,ρ,ω). Note that mere labeling implies nothing but invariance under the gauge groups generated by Q, Y, etc., and in particular implies nothing about invariance under the full isotopic spin group or under $SU(3)$.

(b) Off-Mass-Shell Assumption: We assume that the interaction Hamiltonian is such that the condition

$$\langle\alpha|\int T(H(x)H(y))d^4(xy)|\beta\rangle=\text{const}\times\delta_{\alpha\beta}, \quad (1.1)$$

where $|\alpha\rangle$, $\alpha=1\cdots n$, are the one-particle vector-meson states, is satisfied, *both on and off the mass shell.* This condition may be looked on as the generalization of Cutkosky's normalization condition (4) for the completely antisymmetric case, or as a type of bootstrap condition for the propagator graph of Fig. 1. Equation (1.1) can also be regarded as a second-order consequence of the so-called Smushkevich principle, which demands that the interaction be such that the n vector mesons be mass-degenerate before and after the interaction is switched on.

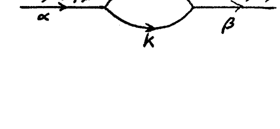

FIG. 1. Propagator graph for Eq. (1.1).

---

* Work supported in part by the U. S. Atomic Energy Commission.

† Present address: Department of Physics, Boston University, Boston, Massachusetts.

‡ On leave of absence from Dublin Institute for Advanced Studies.

[1] R. Cutkosky, Phys. Rev. **131**, 1888 (1963).

[2] R. Cutkosky, Ann. Phys. (N. Y.) **23**, 415 (1963); R. H. Capps, Phys. Rev. Letters **10**, 312 (1963); E. C. G. Sudarshan, *ibid.* **9**, 286 (1964); J. J. Sakurai, *ibid.* **10**, 446 (1963); E. C. G. Sudarshan, L. S. O'Raifeartaigh, and T. S. Santhanam, Phys. Rev. **136**, B1092 (1964); E. C. G. Sudarshan, Curr. Sci. **34**, 202 (1965); P. Narayanaswamy and T. S. Santhanam, Nuovo Cimento (to be published); E. Abers, F. Zachariasen, and C. Zemach, Phys. Rev. **132**, 1831 (1963); J. C. Polkinghorne, Ann. Phys. (N. Y.) **34**, 153 (1965); Hong-Mo Chan, P. DeCelles, and J. E. Paton, Phys. Rev. Letters **11**, 521 (1963); Nuovo Cimento **33**, 70 (1964); J. S. Dowker and J. E. Paton, *ibid.* **30**, 450 (1963); J. S. Dowker, *ibid.* **34**, 773 (1964); J. S. Dowker and P. A. Cook, *ibid.* **37**, 335 (1965); H. Leutwyler and E. C. G. Sudarshan, Phys. Rev. (to be published).

FIG. 2. Graphical representation of Eq. (1.3).

The explicit form of the (interaction-picture) Hamiltonian used in (1) is assumed to be

$$H(x)=C_{\alpha\beta\gamma}\phi_\alpha{}^\mu(x)\phi_\beta{}^\nu(x)\phi_\gamma{}^\nu(x)_{,\mu}\,, \tag{1.2}$$

where $\phi_\alpha{}^\mu(x)$, $\mu=0,1,2,3$, $\alpha=1\cdots n$ are the vector-meson fields, and the summation convention is used. From the divergence condition $\phi_\alpha{}^\nu(x)_{,\nu}=0$ for the (free) vector-meson fields,[3] it follows that no generality is lost by assuming $C_{\alpha\beta\gamma}$ to be antisymmetric in β and γ. In fact, this is the reason for using *vector*-meson fields.

The main result of this paper is a demonstration that for a Hamiltonian of the form (1.2), conditions (a) and (b) above are already sufficient to imply that the $C_{\alpha\beta\gamma}$ are antisymmetric in all three indices. Cutkosky's original argument based on his bootstrap condition

$$\mathrm{tr}C_\alpha C_\beta C_\gamma=\mu C_{\alpha\beta\gamma}\,, \tag{1.3}$$

where μ is a number, and C_α the matrix with entries $C_{\alpha\beta\gamma}$, corresponding to the graph of Fig. 2, then shows that the $C_{\alpha\beta\gamma}$ must be the structure constants of a compact semisimple Lie group.

A secondary purpose of the paper is to investigate the following question: If the vertex bootstrap Eq. (1.3) above is dropped, but the consequences of the Smushkevich condition are assumed to be true in all orders of perturbation, does it still follow that the $C_{\alpha\beta\gamma}$ must be the structure constants of a compact semisimple Lie group? An alternative way of asking the same question is to ask whether the bootstrap conditions (1.3), which are for 3-point functions, can be replaced by conditions on 2-point functions. The interest in this question stems from the observation that in many cases in which the existence of a symmetry has been derived from a bootstrap hypothesis, it has been derivable from the Smushkevich principle also. The vector-meson case provides a more stringent test of this than has hitherto been proposed. Unfortunately, it also provides more complicated algebra than has hitherto been encountered. For that reason, we have been able to investigate only the cases $n=3,4,5,6$, and the physically most interesting case, $n=8$. For these cases, we have been able to show that the Smushkevich principle does indeed reduce the $C_{\alpha\beta\gamma}$ to structure constants.

II. SELF-CONSISTENCY EQUATIONS IN SECOND ORDERS

Following the discussion of the previous section, we assume the off-mass-shell condition (b) of Eq. (1.1) and note that since this condition holds both on and off the mass shell, and since we are assuming mass degeneracy, it must hold in particular for the *absorptive* part of the expression on the left-hand side of Eq. (1.1). Thus, if p^2 denotes the four-momentum squared of the one-particle states $|\alpha\mu\rangle$, the relation

$$\langle\alpha\mu|\int d^4(xy)H(x)H(y)|\beta\mu\rangle=\mathrm{const}\times\delta_{\alpha\beta} \tag{2.1}$$

must hold for all values of p^2. In contrast to Eq. (1.1) which is divergent, Eq. (2.1) is not only convergent but is easily calculated. A straightforward calculation, yields (see Appendix),

$$2C_{\alpha bc}C_{\beta bc}I_1(p^2)+C_{b\alpha c}C_{b\beta c}I_2(p^2)+C_{b\alpha c}C_{c\beta b}I_3(p^2)+4C_{\alpha bc}C_{b\beta c}I_4(p^2)=\mathrm{const}\times\delta_{\alpha\beta}\left(\frac{p^2}{p^2-4m^2}\right)^{1/2}, \tag{2.2}$$

where

$$\begin{aligned}
I_1(p^2)&=[4m^2-p^2]\left[3-\frac{p^2}{m^2}+\frac{1}{4}\frac{p^4}{m^4}\right],\\
I_2(p^2)&=-p^2\left[8-\frac{p^2}{m^2}-\frac{1}{4}\frac{p^4}{m^4}\right],\\
I_3(p^2)&=\frac{p^4}{m^2}\left[1-\frac{p^2}{4m^2}\right],\\
I_4(p^2)&=p^2\left[2-\frac{3}{2}\frac{p^2}{m^2}+\frac{1}{4}\frac{p^4}{m^4}\right].
\end{aligned} \tag{2.3}$$

Since the 4 functions in Eqs. (2.3) are clearly functionally independent, and Eq. (2.2) is assumed to hold for all p^2, we obtain from (2.2) the four separate equations

$$\begin{aligned}
C_{\alpha bc}C_{\beta bc}&=\lambda\delta_{\alpha\beta}\,,\\
C_{b\alpha c}C_{b\beta c}&=\lambda\delta_{\alpha\beta}\,,\\
C_{b\alpha c}C_{c\beta b}&=\mu\delta_{\alpha\beta}\,,\\
C_{\alpha bc}C_{b\beta c}&=\mu\delta_{\alpha\beta}\,,
\end{aligned} \tag{2.4}$$

where λ and μ are constants. That the *same* constant appears in the first two and last two equations, respectively can be seen by contracting each of Eqs. (2.4) with respect to α, β and using the two-index antisymmetry condition $C_{\alpha\beta\gamma}+C_{\alpha\gamma\beta}=0$, which can be assumed according to the discussion following Eq. (1.2).

We now show that a necessary and sufficient condition for the $C_{\alpha\beta\gamma}$ to be antisymmetric in all indices is that

$$\lambda+\mu=0. \tag{2.5}$$

For this purpose we write the two center equations of (2.4) as

$$\begin{aligned}
\mathrm{tr}C_\alpha C_\beta&=\lambda\delta_{\alpha\beta}\,,\\
\mathrm{tr}C_\alpha\tilde{C}_\beta&=\mu\delta_{\alpha\beta}\,,
\end{aligned} \tag{2.6}$$

[3] We are not assuming of course that the divergence condition holds for the interacting, or Heisenberg fields. If it does, then it follows immediately that the interaction is $SU(n)$-invariant. See V. I. Ogieretski and I. V. Polubarinov, Ann. Phys. (N. Y.) **25**, 358 (1963).

where C_α is the matrix with elements $C_{b\alpha c}$, and $\tilde{C}_\alpha$ its transpose. From Eq. (2.6) we obtain immediately

$$\mathrm{tr}(C_\alpha+\tilde{C}_\alpha)^2=(\lambda+\mu)\,. \tag{2.7}$$

Since $C_\alpha+\tilde{C}_\alpha$ is real and Hermitian, it follows that it is zero if and only if $\lambda+\mu=0$. On the other hand, $C_{b\alpha c}$ which is antisymmetric in α and c, is antisymmetric in all indices if and only if it is antisymmetric in b and c, i.e., if and only if $C_\alpha+\tilde{C}_\alpha=0$. This establishes the result.

We add the following observation. Since the right-hand side of (2.8) is independent of α, $(\lambda+\mu)=0$, if for any *single* α, $C_\alpha+\tilde{C}_\alpha=0$. In other words, $C_\alpha+\tilde{C}_\alpha=0$ for any single α, implies $C_\alpha+\tilde{C}_\alpha=0$ for all α.

We do not know the *necessary* conditions that $C_\alpha+\tilde{C}_\alpha$ be zero for a single α. However, in the next section we shall show that a sufficient condition is that the set of n mesons considered satisfy the labeling assumption (a) of the introduction.

III. DERIVATION OF COMPLETE ANTISYMMETRY FROM THE LABELING ASSUMPTION

Let ω denote one of the neutral mesons in the set n (by hypothesis there exists at least one neutral). Since the additive quantum numbers are conserved, the ω can couple to the non-neutral mesons only through a coupling of the form

$$\omega\phi_a\phi_{-a}\,, \tag{3.1}$$

where a is the (unique) labeling operator for the non-neutrals. If x denotes the labeling index for the neutrals and C the charge-conjugation operator we have (since the mesons are spin 1)

$$C\phi_x=-\phi_x\,,$$
$$C\phi_a=-\phi_{-a}\,, \tag{3.2}$$

the second equation of which essentially defines ϕ_{-a}. In (3.1) the space-time indices have of course been suppressed. It follows from (3.1) that the only contribution of ω to (1.2) is of the form

$$C_{0a-a}\omega^\mu\phi_a{}^\nu\phi_{-a}{}^\nu{}_{,\mu}+2C_{a0-a}\phi_a{}^\mu\omega^\nu\phi_{-a}{}^\nu{}_{,\mu}+C_{xyz}\phi_x{}^\mu\phi_y{}^\nu\phi_z{}^\nu{}_{,\mu}\,, \tag{3.3}$$

where for definiteness, ω has been identified with ϕ_0. The charge conjugate of this expression is, from Eq. (3.2),

$$-C_{0a-a}\omega^\mu\phi_{-a}{}^\nu\phi_a{}^\nu{}_{,\mu}-2C_{a0-a}\phi_{-a}{}^\mu\omega^\nu\phi_a{}^\nu{}_{,\mu}-C_{xyz}\phi_x{}^\mu\phi_y{}^\nu\phi_z{}^\nu{}_{,\mu}\,. \tag{3.4}$$

Charge-conjugation invariance implies the equality of Eqs. (3.3) and (3.4). Comparing coefficients we obtain

$$C_{xyz}=0$$

and

$$C_{a0-a}=-C_{-a0a}\,. \tag{3.5}$$

FIG. 3. Fourth-order self-mass diagram.

Combining Eq. (3.5) with the relation $C_{a0b}=\delta_{ab}C_{a0-a}$, implicit in Eqs. (3.1) and (3.3) we obtain finally

$$C_{\alpha0\beta}=-C_{\beta0\alpha}=\delta_{\beta-\alpha}C_{\alpha0-\alpha}\,, \tag{3.6}$$

for all α and β.

But Eq. (3.6) is just the relation $C_0+\tilde{C}_0=0$ which, according to the discussion of the previous section, is sufficient to imply that the $C_{\alpha\beta\gamma}$ are antisymmetric in all indices. Thus we have now shown that the two conditions (a) and (b) of the introduction are sufficient to imply the complete antisymmetry of the $C_{\alpha\beta\gamma}$.

As mentioned in the introduction, if one now assumes the vertex bootstrap condition (1.3), Cutkosky's original arguments can now be applied to deduce that the $C_{\alpha\beta\gamma}$ are the structure constants of a compact semisimple Lie group.

IV. SMUSHKEVICH PRINCIPLE

As mentioned in the introduction, in many cases in which the invariance of an interaction under a compact simple Lie group can be derived from a bootstrap hypothesis it can be derived from the Smushkevich principle which is defined in that section. It is interesting to ask whether the same is true of the interaction (1.2). The second-order Smushkevich equations are of course Eq. (1.1), and so if these are taken off the mass shell and allied to the labeling assumption (b) they already yield

(i) complete antisymmetry of the $C_{\alpha\beta\gamma}$, and
(ii) the equation

$$\mathrm{tr}C_\alpha C_\beta=\mathrm{const}\times\delta_{\alpha\beta}\,, \tag{4.1}$$

into which Eqs. (2.4) collapse in the completely antisymmetric case. The Smushkevich principle which involves only self-masses does not yield any conditions in odd orders of perturbation. In fourth order we get

$$\mathrm{tr}C_\alpha C_\gamma C_\beta C_\gamma=\mathrm{const}\times\delta_{\alpha\beta}\,, \tag{4.2}$$

corresponding to the self-mass diagram of Fig. 3. Similarly in sixth order we get

$$\mathrm{tr}C_\alpha C_\gamma C_\delta C_\beta C_\gamma C_\delta=\mathrm{const}\times\delta_{\alpha\beta}\,, \tag{4.3}$$

and so on. It is easy to check that the 4th, 6th, . . . order Smushkevich relations follow from the second-order Smushkevich relations (4.1) and the vertex bootstrap condition (1.3). The question we shall be interested in is whether the converse is true, i.e., whether the vertex bootstrap condition, or equivalently its structure-constant consequence, can be recovered from Eqs. (4.1), (4.2), (4.3), etc.

On account of the complexity of the algebra we have not been able to obtain a general answer to this question. Instead we have investigated the somewhat trivial cases $n=3,4,5,6$, and the nontrivial and physically interesting case $n=8$.

In all these cases it will be assumed from the start that $C_{\alpha\beta\gamma}$ is completely antisymmetric. This assumption we can regard either as a direct assumption (following Cutkosky) or as a consequence of our conditions (a) and (b) of Sec. I. For the physically interesting case $n=8$, we assume in addition (again following Cutkosky) that there is one conserved additive quantum number. We make this assumption a little stronger than Cutkosky's by assuming also that there are four neutrals. This latter assumption is motivated by the observation that if the conserved quantum number is identified with the charge, then for the physical 8 (K^*, ρ, ω, with ω-ϕ mixing neglected) four of the particles *are* neutral. On the other hand, we do not make Cutkosky's "lowest-bound-state" hypothesis.

Under these conditions we find that for the rather trivial cases $n=3,4,5,6$ the second-order Smushkevich Eqs. (4.1) are already enough to ensure that the $C_{\alpha\beta\gamma}$ are the structure constants of a compact semisimple Lie group. In particular the $C_{\alpha\beta\gamma}$ are zero for $n=4,5$ and are the structure constants of $O(3)$ and $O(4)$ for $n=3$ and 6, respectively. In the more interesting case $n=8$, the second- and fourth-order Smushkevich relations (4.1) and (4.2) are together sufficient to imply that the $C_{\alpha\beta\gamma}$ are the structure constants of a compact simple Lie group, $SU(3)$ in this case. Thus for $n=3,4,5,6$, and 8 at any rate, the Smushkevich principle seems to be "as good as" the bootstrap condition in forcing the invariance of the interaction.

The calculation for the case $n=8$ is sketched in the next section. For the full details for $n=8$ and for the case $n=6$ the reader is referred to Ref. 4.

V. THE SMUSHKEVICH PRINCIPLE FOR THE CASE $n=8$

In this section we consider the case of 8 vector mesons in self-interaction by means of the Hamiltonian (1.2) where, as mentioned in the last section, we assume that

(a) $C_{\alpha\beta\gamma}$ is antisymmetric in all indices, and

(b) one additive quantum number (to be identified with charge and with respect to which 4 of the 8 particles are neutral) is conserved.

Charge conjugation then demands that of the remaining four particles two be positively charged and two negatively. For definiteness, we identify the neutrals with linear combinations of the fields ϕ_i, $i=1\cdots4$, and identify the charged particles with $(\phi_5\pm i\phi_6)/\sqrt{2}$ and $(\phi_7\pm i\phi_8)/\sqrt{2}$, respectively. The most general set of $C_{\alpha\beta\gamma}$ satisfying (a) and (b) above can then be seen to be

$$
\begin{bmatrix} \begin{matrix} 0&0&0&0\\0&0&0&0\\0&0&0&x\\0&0&-x&0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0&y&0&0\\-y&0&0&0\\0&0&0&z\\0&0&-z&0 \end{matrix} \end{bmatrix}
\begin{bmatrix} \begin{matrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0&a_2&b_2&c_2\\-a_2&0&-c_2&b_2\\-b_2&c_2&0&d_2\\-c_2&-b_2&-d_2&0 \end{matrix} \end{bmatrix}
\begin{bmatrix} \begin{matrix} 0&0&0&-x\\0&0&0&0\\0&0&0&0\\x&0&0&0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0&a_3&b_3&c_3\\-a_3&0&-c_3&b_3\\-b_3&c_3&0&d_3\\-c_3&-b_3&-d_3&0 \end{matrix} \end{bmatrix}
$$

$$
\begin{bmatrix} \begin{matrix} 0&0&x&0\\0&0&0&0\\-x&0&0&0\\0&0&0&0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0&a_4&b_4&c_4\\-a_4&0&-c_4&b_4\\-b_4&c_4&0&d_4\\-c_4&-b_4&-d_4&0 \end{matrix} \end{bmatrix}
\begin{bmatrix} 0 & \begin{matrix} 0&-y&0&0\\0&-a_2&-b_2&-c_2\\0&-a_3&-b_3&-c_3\\0&-a_4&-b_4&-c_4 \end{matrix} \\ \begin{matrix} 0&0&0&0\\y&a_2&a_3&a_4\\0&b_2&b_3&b_4\\0&c_2&c_3&c_4 \end{matrix} & 0 \end{bmatrix}
\begin{bmatrix} 0 & \begin{matrix} y&0&0&0\\a_2&0&c_2&-b_2\\a_3&0&c_3&-b_3\\a_4&0&c_4&-b_4 \end{matrix} \\ \begin{matrix} -y&-a_2&-a_3&-a_4\\0&0&0&0\\0&-c_2&-c_3&-c_4\\0&b_2&b_3&b_4 \end{matrix} & 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 0 & \begin{matrix} 0&0&0&-z\\b_2&-c_2&0&-d_2\\b_3&-c_3&0&-d_3\\b_4&-c_4&0&-d_4 \end{matrix} \\ \begin{matrix} 0&-b_2&-b_3&-b_4\\0&c_2&c_3&c_4\\0&0&0&0\\z&d_2&d_3&d_4 \end{matrix} & 0 \end{bmatrix}
\begin{bmatrix} 0 & \begin{matrix} 0&0&z&0\\c_2&b_2&d_2&0\\c_3&b_3&d_3&0\\c_4&b_4&d_4&0 \end{matrix} \\ \begin{matrix} 0&-c_2&-c_3&-c_4\\0&-b_2&-b_3&-b_4\\-z&-d_2&-d_3&-d_4\\0&0&0&0 \end{matrix} & 0 \end{bmatrix}, \tag{5.1}
$$

where we have utilized the fact that since the neutrals are not completely identified, we are free to make orthogonal transformations in the space spanned by the ϕ_i, in order to set some of the coefficients in the matrices (5.1) equal to zero. This freedom has actually not been utilized to the maximum. The maximum use[4]

[4] R. Musto, Syracuse University thesis, Part I, Sec. 3, 1967 (unpublished).

would correspond to setting also

$$b_2=b_3=0. \tag{5.2}$$

However, for reasons of symmetry we do not use Eq. (5.2) for the moment. Altogether therefore there are 13 independent coupling constants in Eq. (5.1). What has to be shown is that the Smushkevich principle determines all of these in terms of one over-all constant.

We first apply the second-order Smushkevich relation (4.1). These yield[4] the sets of equations

$$a_i^2+2(b_i^2+c_i^2)+d_i^2=\tfrac{1}{2}x^2+\delta_{i2}x^2,\quad i=2,3,4, \tag{5.3}$$

$$y^2+z^2=\tfrac{1}{2}x^2,$$

$$y^2-z^2=\sum_i (d_i^2-a_i^2),$$

$$ya_i+zb_i=0,\qquad i=2,3,4,$$

$$a_ia_j+2(b_ib_j+c_ic_j)+d_id_j=0,\qquad i,j=2,3,4, \tag{5.4}$$

and

$$\sum_i b_i(a_i+d_i)=0,$$

$$\sum_i c_i(a_i+d_i)=0, \tag{5.5}$$

where Eq. (5.3) comes from $\mathrm{tr}C_\alpha C_\alpha$, Eq. (5.4) from $\mathrm{tr}C_iC_j$, $i,j=1\cdots4$, $i\neq j$, and Eq. (5.5) from $\mathrm{tr}C_\alpha C_\beta$, $\alpha,\beta=5\cdots8$, $\alpha\neq\beta$. All other second-order Smushkevich equations are then automatically satisfied.

The Eqs. (5.3), (5.4), and (5.5) can easily be solved by noting that y and z cannot both be zero and that there is no loss of generality in assuming $z\neq0$, by considering a_i, b_i, c_i, and d_i, $i=2,3,4$, as 3-vectors, and by using Eq. (5.2). The solution can be written in the form

$$\begin{pmatrix}d_2\\d_3\\d_4\end{pmatrix}=-\frac{y}{z}\begin{pmatrix}a_2\\a_3\\a_4\end{pmatrix}=-y\epsilon(z/x)\begin{pmatrix}\sqrt{3}\cos\theta\\ \sin\theta\\ 0\end{pmatrix},$$

$$\begin{pmatrix}b_2\\b_3\\b_4\end{pmatrix}=\tfrac{1}{2}x\begin{pmatrix}0\\0\\1\end{pmatrix}\quad \begin{pmatrix}c_2\\c_3\\c_4\end{pmatrix}=\tfrac{1}{2}x\begin{pmatrix}-\sqrt{3}\sin\theta\\ \cos\theta\\ 0\end{pmatrix}, \tag{5.6}$$

where

$$(y^2-z^2)x^2\cos^2\theta=0,$$

$$(y-z)x\cos\theta\sin\theta=0,$$

$$y^2+z^2=\tfrac{1}{2}x^2,$$

and $\epsilon(z/x)=\pm1$ according as $(z/x)\gtrless0$. In deriving Eq. (5.6) some extra freedom allowed by Eqs. (4.3), (4.4), and (4.5) has been used to make orthogonal transformations in the ϕ_α space. Of course, only orthogonal transformations preserving charge conservation, charge conjugation, and Eqs. (4.3), (4.4), and (4.5) are utilized.

It is clear from Eq. (5.6) that although the second-order Smushkevich equations drastically reduce the number of independent coupling constants in Eq. (5.1), they do not determine them completely in terms of one over-all coupling constant. At this stage, therefore, we use the following members of the fourth-order Smushkevich Eqs. (4.2):

$$\sum_{\alpha=1}^{8}\mathrm{tr}C_1C_\alpha C_1C_\alpha=\sum_{\alpha=1}^{8}\mathrm{tr}C_2C_\alpha C_2C_\alpha=\sum_{\alpha=1}^{8}\mathrm{tr}C_3C_\alpha C_3C_\alpha. \tag{5.7}$$

A straightforward but tedious calculation[4] shows that Eq. (5.6) then reduces to

$$y=\tfrac{1}{2}x\epsilon,$$

$$z=\tfrac{1}{2}x\epsilon,$$

$$\begin{pmatrix}d_2\\d_3\\d_4\end{pmatrix}=-\begin{pmatrix}a_2\\a_3\\a_4\end{pmatrix}=+\tfrac{1}{2}x\epsilon\begin{pmatrix}\sqrt{3}\\0\\0\end{pmatrix},$$

$$\begin{pmatrix}b_2\\b_3\\b_4\end{pmatrix}=\tfrac{1}{2}x\begin{pmatrix}0\\0\\1\end{pmatrix},\quad \begin{pmatrix}c_2\\c_3\\c_4\end{pmatrix}=\tfrac{1}{2}x\epsilon\begin{pmatrix}0\\1\\0\end{pmatrix}, \tag{5.8}$$

where $\epsilon=\pm1$.

Thus the only freedom left is in the sign of ϵ. However, it is easy to show[4] that this freedom corresponds only to a change of sign in some of the fields ϕ_α. Thus up to this change of sign, Eq. (5.8) expresses all of the coupling constants in Eq. (5.1) in terms of one over-all constant as required.

The unique determination of the coupling constants in terms of one over-all constant implies of course that the interaction must be $SU(3)$ invariant, since we know in advance that an $SU(3)$-invariant interaction satisfies the Smushkevich principle to all orders. However, the $SU(3)$ invariance may also be checked directly by showing that the matrices (5.1) with elements (5.8) satisfy the usual $SU(3)$ commutation relations.

ACKNOWLEDGMENTS

The authors wish to thank Dr. H. Leutwyler for correcting an error in the first draft, and to thank him and Dr. E. C. G. Sudarshan for some invaluable discussions.

APPENDIX

Using the Hamiltonian (1.2), the expression for the left-hand side of Eq. (1.1) which corresponds to the Feynman graph of Fig. 1 is easily derived from the Feynman rules. It is, apart from some over-all constants, $\epsilon^{\mu}\epsilon^{\mu'}$ times

$$2C_{\alpha bc}C_{\beta bc}\int k_{\mu}(k_{\mu'}-p_{\mu'})\Delta^{\nu\nu'}(k)\Delta^{\nu\nu'}(k-p)d^4k$$

$$+C_{b\alpha c}C_{c\beta b}p_{\nu}p_{\nu'}\int \Delta^{\nu\nu'}(k)\Delta^{\mu\mu'}(k-p)d^4k$$

$$+C_{a\alpha c}C_{c\beta a}p_{\nu}p_{\nu'}\int \Delta^{\nu\mu'}(k)\Delta^{\mu\nu'}(k-p)d^4k$$

$$+4C_{\alpha bc}C_{b\beta c}p_{\nu'}\int (k_{\mu}-p_{\mu})\Delta^{\nu\nu'}(k)\Delta^{\nu\mu'}(k-p)d^4k\,, \quad \text{(A1)}$$

where $\epsilon(p)\cdot p=\epsilon_{\mu}p_{\mu}=0$ and

$$\Delta_{\mu\nu}(k)=\frac{g_{\mu\nu}-k_{\mu}k_{\nu}/m^2}{k^2-m^2+i\epsilon} \quad \text{(A2)}$$

is the vector-meson propagator. We first note that in general $T_{\mu\mu'}(p^2)=g_{\mu\mu'}I(p^2)+p_{\mu}p_{\mu'}J(p^2)$, and hence $-T_{ii}(p^2)=3I(p^2)-\mathbf{p}^2J(p^2)$, which we then evaluate in the rest frame $\mathbf{p}=0$. The absorptive part of this expression is obtained by putting the intermediate particles on their mass-shells, i.e., by replacing the denominators $(k^2-m^2+i\epsilon)^{-1}$ in the propagator by $\epsilon(k_0)\delta(k^2-m^2)$. This yields finite integrals, which are easily calculable and lead to the results listed in Eq. (2.3). For example, for the coefficient of $C_{b\alpha c}C_{c\beta b}$ in (A1) we obtain

$$T_{ii}(p_0^2)=\int d^4k\,\theta(k_0)\theta(p_0-k_0)\delta(k^2-m^2)\delta[(p-k)^2-m^2]$$
$$\times[p_0^2-(p_0k_0)^2/m^2](-3-\mathbf{k}^2/m^2)$$
$$=-\pi(p_0^2/4-m^2)^{1/2}(p_0-p_0^3/4m^2)(2+p_0^2/4m^2)\,.$$

PHYSICAL REVIEW VOLUME 158, NUMBER 5 25 JUNE 1967

Partially Conserved Axial-Vector Current, Charge Commutators, Off-Mass-Shell Correction, and the Broken $SU(3)$ Symmetry

S. Matsuda and S. Oneda

Department of Physics and Astronomy, University of Maryland, College Park, Maryland

(Received 15 December 1966; revised manuscript received 6 March 1967)

We point out that the extension of the PCAC (partially conserved axial-vector current) relation $\partial_{\mu}A_{\mu}^{\pi}=C_{\pi}\phi^{\pi}$ to $\partial_{\mu}A_{\mu}^{K}=C_K\phi^K$ and the use of charge commutators typified by $A_K=[V_K,A_{\pi}]$ are useful in the study of broken $SU(3)$ symmetry. The use of the $\partial_{\mu}A_{\mu}^{K}=C_K\phi^K$ condition usually confronts us with a considerable off-mass-shell extrapolation $m_K\to 0$. However, by using the above charge commutators and the approximation we propose, the off-mass-shell extrapolation $m_K\to 0$ may be replaced by a more comfortable one, $m_{\pi}\to 0$, effectively to first order in the symmetry-breaking interaction. This approach is applied to the study of the $SU(3)$ symmetry breaking. Encouraging results have been obtained in the case of $V\to P+P$ (i.e., $K^*\to K+\pi$ and $\rho\to\pi+\pi$) decays and in the direct determination of the $f-f'$ mixing angle from their decay widths. We also make some estimate of the off-shell extrapolation $m_K\to 0$ compared with the case $m_{\pi}\to 0$. Another useful application of the above charge commutators is for the weak leptonic decays. We can derive a set of sum rules for the axial-vector coupling constants of the leptonic decays of hyperons which seem to give new insight into the Cabibbo theory of leptonic interactions.

THERE have been many interesting calculations based on the idea of current algebra.[1] In the actual computations the use of the PCAC (partially conserved axial-vector current) hypothesis is essential. One may take a variety of attitudes toward the use of PCAC.

(I) One point of view is to regard the equation

$$\partial_{\mu}A_{\mu}^{\pi}=C_{\pi}\phi^{\pi} \quad (1)$$

as approximately true.[2] Taking the matrix element of (1) between the proton and the neutron, one obtains a form of Goldberger-Treiman relation

$$C_{\pi}=\sqrt{2}g_A\frac{m_p}{G_{pp\pi}(m_{\pi}^2=0)}m_{\pi}^2\,, \quad (2)$$

where g_A is the ratio of the axial-vector to the vector coupling constant of β decay. The calculation[3] from this standpoint involves the extrapolation of the pion off the mass shell ($m_{\pi}\to 0$).

(II) In a second point of view, $\partial_{\mu}A_{\mu}^{\pi}$ is regarded as a highly convergent operator whose matrix element satisfies an unsubtracted dispersion relation in squared momentum transfer q^2. For small q^2, the dominance of

[1] M. Gell-Mann, Physics **1**, 63 (1964).

[2] Vector and axial-vector currents are denoted by $V_{\mu}^{\pi+}(x)$, $V_{\mu}^{K+}(x),\cdots$, and $A_{\mu}^{\pi+}(x)$, $A_{\mu}^{K+}(x),\cdots$, respectively, normalized so that in a quark model we would have, e.g., $V_{\mu}^{\pi+}(x)=i\bar{q}\gamma_{\mu}\times(\lambda_1+i\lambda_2)q/2$, $A_{\mu}^{K+}(x)=i\bar{q}\gamma_{\mu}(\lambda_4+i\lambda_5)q/2$, etc.; the space integral of, say, $A_0^{\pi+}(\mathbf{x},0)\equiv -iA_4^{\pi+}(\mathbf{x},0)$ is denoted by A_{π^+}. The PCAC relations are used in the form $\partial_{\mu}A_{\mu}^{\pi\pm}=C_{\pi}\phi^{\pi\pm}$ and $\partial_{\mu}A_{\mu}^{K\pm}=C_K\phi^{K\pm}$, where, e.g., $\phi^{\pi+}$ creates π^+ mesons.

[3] For instance, S. Adler, Phys. Rev. **137**, B1022 (1965).