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Bootstrap Algebras for Baryons and Mesons*

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A dynamical model based on linearized D functions leads to a series of relations called the bootstrap algebras. Under specific conditions these algebras imply that the $\frac{1}{2}^+$ baryons must belong to a representation of the Lie algebra whose structure constants are given by the vector-meson coupling constants. If the u-channel forces admit of a static approximation, the algebras lead to consequences which are the same as those of the spin–internal-symmetry groups, while avoiding the usual difficulties encountered for the couplings in these groups. Several such systems leading to consequences of groups like $SU(6)$, $Sp(16)$, and the strong-coupling group are discussed. The results are shown to be applicable to meson-meson scattering even though the light pseudoscalars are exchanged in the u channel. It is found that the pseudoscalar-meson–vector-meson couplings in the $SU(6)$ model and the strong-coupling model are in remarkable agreement with each other.

I. INTRODUCTION

THE quest for a dynamical basis for internal symmetries has led to several discussions[1–9] of the bootstrap philosophy[10] as the origin of symmetries in the strong-interaction phenomena. The first real progress in this direction was made by Cutkosky[1] who showed that for an idealized world of vector mesons, the bootstrap requirements yield the result that the vector-meson coupling constants are the structure constants of a compact semisimple Lie algebra. It has since been shown[5] that this result is retained in a somewhat more realistic model which allows the existence of both vector and pseudoscalar mesons. It was also found that the pseudoscalar mesons themselves must belong to a representation of the Lie algebra described by the structure constants of the vector mesons.

There has been another class of attempts in which spin–unitary-spin combination arises out of bootstrap requirements. Here impressive results come out with the use of the static model. It has been shown by Cook, Goebel, and Sakita[8] that in the strong-coupling limit, the pseudoscalar mesons and baryon isobars have an underlying noncompact Lie group. Thus, for example, the group for an octet of pseudoscalar mesons scattered by static baryons is $[SU(2)\otimes SU(3)]\times T_{24}$, a direct product of the invariant spin group $SU(2)$ and the invariant internal spin group $SU(3)$ and a semidirect product with T_{24}, the translation group in 24 dimensions. There have been some other attempts[3,6,7] which, starting from some approximate or unrealistic model, give results which are similar to those obtained from $SU(6)$.

In the present study we develop a dynamical model[11] which with the bootstrap requirements yields certain algebraic relations between various coupling constants. These relations are quite general and apply to any multichannel system. It is then shown that under some specific assumptions the baryons must belong to a representation of the Lie group whose structure constants are defined by the vector-meson coupling constants.

If one introduces the static approximation for the exchange of heavy particles in the u channel, one obtains groups which combine spin and unitary spin. In this context, the familiar $SU(6)$, the strong-coupling noncompact group,[8] $R(11)$ the rotation group in 11 dimensions, and $Sp(16)$ the symplectic group in 16 dimensions, are discussed with reference to the baryon isobars. It is found that the results of only the first two groups, the $SU(6)$ and the strong-coupling group, are in reasonable agreement with the experimental results. The extensions of the model to describe the spin–unitary-spin combination for mesons are discussed and it is shown that the results are almost identical for $SU(6)$ and the strong-coupling group.

II. THE DYNAMICS

Consider a multichannel scattering for a system with a given angular momentum, baryon number, etc., the channels being characterized by an index i and the scattering amplitude from channel i to channel j by T_{ij}. We assume Lorentz invariance, Bose statistics, charge conjugation, parity, and time-reversal invariance, and that the fields describing the various mesons are Hermitian. We further assume that all the particles in a state characterized by a given angular-momentum, parity, baryon number, etc., have the same mass; e.g., the 8 pseudoscalar mesons have the same mass; and so

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do the baryons in the $\frac{1}{2}^+$ octet. The unitarity condition for the amplitude T_{ij} is of the form

$$T_{ij}-T_{ij}^*=2iT_{ki}^*T_{kj}\rho_k\,, \tag{2.1}$$

where ρ_k is the phase-space factor in channel k. We use the usual $\mathbf{ND}^{-1}$ method to describe the analyticity and unitarity properties of the T_{ij} amplitudes. In the matrix notation we write

$$\mathbf{T}(s)=\mathbf{N}(s)\mathbf{D}^{-1}(s)\,, \tag{2.2}$$

where

$$\mathbf{N}(s)=\frac{1}{\pi}\int_L\frac{[\mathrm{Im}\mathbf{T}(s')]\mathbf{D}(s')}{s'-s}ds'\,, \tag{2.3}$$

$$\mathbf{D}(s)=\mathbf{C}-\frac{(s-s_r)}{\pi}\int_R\frac{\varrho(s')\mathbf{N}(s')}{(s'-s)(s'-s_r)}ds'. \tag{2.4}$$

The $\mathbf{D}$ functions have only the right-hand singularities and $\mathbf{N}$ functions only the left-hand singularities. The expressions for $\mathbf{D}$ are once subtracted. We further assume that the integrals in (2.4) are slowly varying functions of s and replace them by constants A_{ij};

$$\mathbf{D}(s)\simeq\mathbf{C}+(s-s_r)\mathbf{A}. \tag{2.5}$$

If s_r is taken to be the position of the resonance of the stable particle in the state characterized by the quantum numbers of the system, then the assumption that the T matrix has no other poles in the complex s plane essentially implies[12] that

$$C_{ij}=0. \tag{2.6}$$

The T matrix can then be written as

$$T_{ij}(s)=\frac{1}{\pi(s-s_r)}\int_L\frac{[\mathrm{Im}T(s')]_{ij}(s'-s_r)}{(s'-s)}ds'.$$

Therefore

$$\mathrm{Res}T_{ij}(s_r)=\frac{1}{\pi}\int_L\mathrm{Im}T_{ij}^B(s')ds'. \tag{2.7}$$

One may need a cutoff in (2.7), but for our purpose this does not change the basic results. Since the integration is on the left, we have assumed that the integrand in (2.7) is given by the sum of the Born terms obtained from the various exchanges in the t and u channels. It is clear that (2.7) yields various algebraic relations. As a simple example we obtain Cutkosky's results[1] for vector-vector scattering. If $F_{\alpha\beta}{}^{\gamma}$ is the totally antisymmetric coupling constant for 3-vectors, (2.7) yields

$$F_{\alpha\beta}{}^{\epsilon}F_{\gamma\delta}{}^{\epsilon}=a(F_{\alpha\gamma}{}^{\epsilon}F_{\beta\delta}{}^{\epsilon}-F_{\alpha\delta}{}^{\epsilon}F_{\beta\gamma}{}^{\epsilon}) \tag{2.8}$$

(summation over the repeated index). The total antisymmetry of F then implies that (unless $a=-\frac{1}{2}$ which case we ignore)

$$F_{\alpha\beta}{}^{\epsilon}F_{\gamma\delta}{}^{\epsilon}+F_{\alpha\gamma}{}^{\epsilon}F_{\delta\beta}{}^{\epsilon}+F_{\alpha\delta}{}^{\epsilon}F_{\beta\gamma}{}^{\epsilon}=0\,, \tag{2.9}$$

which can be written in the form of the Jacobi identity and the F's can be regarded as the structure constants defining a Lie algebra. We note that (2.9) can also be written in a different form[9] making use of the definition of a crossing matrix;

$$f=C_{st}f+C_{su}f\,, \tag{2.10}$$

where C_{st} is the crossing matrix from t to s channel and C_{su} from u to s channel, and f is the square of the reduced matrix element obtained from $F_{\alpha\beta}{}^{\gamma}$. For the vector mesons under consideration, the relevant element of the crossing matrix is the diagonal element corresponding to the regular representation and this being the same for C_{st} and C_{su}, must be equal to $\frac{1}{2}$. This is true, for example, for $SU(2)$ or $SU(3)$.

We will be making extensive use of the relations of the type (2.10) and hence would like to note two straightforward but useful results applicable to equations of the form

$$f=C_{su}f+C_{st}\gamma. \tag{2.11}$$

Proposition 1.

Consider the scattering process

$$P_k+m_i\rightarrow P_l'+m_j\,, \tag{2.12}$$

where m_i belong to a self-conjugate multiplet, i.e.,

$$Cm=\bar{m}=m\,, \tag{2.13}$$

where C denotes the charge-conjugation operator. Then it has been shown[13] that the column vectors of C_{st} corresponding to states symmetric in the interchanges of m_i and m_j are eigenvectors of C_{su} with eigenvalue $+1$ and those corresponding to states antisymmetric in the interchanges of m_i and m_j are eigenvectors of C_{su} with eigenvalue -1.

Proposition 2.

Let us write (2.11) in the form

$$f=C_{su}f+\gamma'. \tag{2.14}$$

Then since $C_{su}{}^2=1$ (for $\bar{m}=m$), solutions to (2.14) exist if and only if γ' is an eigenstate of C_{su} with eigenvalue -1.

With the above results in mind we consider the problems of meson-baryon and meson-meson scattering.

III. BARYON INTERNAL SYMMETRIES

In this section we discuss the implications of (2.7) to the case in which the only baryons that exist are the $\frac{1}{2}^+$ baryons.

[12] The result (2.6) does not quite follow from the assumption. There are exceptions. However, C_{ij} are arbitrary to some extent, and (2.6) does not obviously violate any of the requirements and it automatically gives a pole at S_r in the T matrix which is what we need. The author is grateful to C. Fronsdal for discussion on this point.

[13] H. S. Mani *et al.*, Ann. Phys. (N. Y.) **36**, 285 (1966); see Biswas *et al.*, in Ref. 9.

Equation (2.7) decouples the multichannel problem into single-channel problems. Therefore for establishing baryon symmetries one can consider the scattering of vector mesons by $\frac{1}{2}^+$ baryons. Let the allowed exchanges in the t channel be those of pseudoscalar, vector, and tensor mesons, and let the baryons be represented by indices i, j, k, the vector mesons by the indices α, β, γ, and the pseudoscalars and tensors by a, b, c. Let the 3-vector vertex be denoted by $F_{\alpha\beta}{}^{\gamma}$, the vector-vector-tensor or pseudoscalar vertex by $G_{\alpha\beta}{}^{a}$, and the baryon-baryon-meson vertex by $g_{ij}{}^{\alpha}$ (for vectors) and $f_{ij}{}^{a}$ (for tensors or pseudoscalars). We assume that the relevant couplings are such that $F_{\alpha\beta}{}^{\gamma}$ is totally antisymmetric and $G_{\alpha\beta}{}^{a}$ is symmetric in α and β. Then the use of (2.7) for the scattering of vector meson α by baryon i into vector meson β and baryon j yields

$$g_{ik}{}^{\alpha}g_{kj}{}^{\beta}=xg_{ik}{}^{\beta}g_{kj}{}^{\alpha}+yF_{\alpha\beta}{}^{\gamma}g_{ij}{}^{\gamma}+zG_{\alpha\beta}{}^{a}f_{ij}{}^{a}. \quad (3.1)$$

Interchange of α, β gives

$$g_{ik}{}^{\beta}g_{kj}{}^{\alpha}=xg_{ik}{}^{\alpha}g_{kj}{}^{\beta}+yF_{\beta\alpha}{}^{\gamma}g_{ij}{}^{\gamma}+zG_{\beta\alpha}{}^{a}f_{ij}{}^{a}. \quad (3.2)$$

Subtracing (3.1) from (3.2) and using the symmetry properties of F and G, we get

$$(g_{ik}{}^{\alpha}g_{kj}{}^{\beta}-g_{ik}{}^{\beta}g_{kj}{}^{\alpha})=rF_{\alpha\beta}{}^{\gamma}g_{ij}{}^{\gamma}, \quad (3.3)$$

where $r=y/(1+x)$. Thus, unless $x=-1$, or $y=0$, relation (3.3) implies[14] that the baryons belong to a representation of the Lie algebra defined by the structure constants $F_{\alpha\beta}{}^{\gamma}$. Many other results may be obtained by considering other scattering processes such as pseudoscalar meson–baryon scattering but these are not of great interest since our present model is unrealistic in the sense that baryon resonances have been ignored. Our purpose was only to point out that for sufficiently simple models the bootstrap requirements induce internal symmetries for baryons.

For all the following considerations we will implicitly assume given internal symmetries and discuss the implications of the bootstrap principle with reference to (2.7) for the couplings of particles with different spin.

IV. COMBINING SPIN AND INTERNAL SYMMETRIES FOR BARYONS

In the following considerations we find it very convenient to describe the various relations in terms of crossing matrices and reduced matrix elements analogous to (2.10). Equation (2.7) being decoupled, we can discuss each scattering amplitude separately. We assume that the Born term in (2.7) is saturated by the exchanges, in the u channel, of various baryons of different spins denoted by the indices x, y and by the exchanges of various mesons of spins r in the t channel. We will confine ourselves to meson-baryon isobar scattering in the p wave. Then for the scattering of some state i into i, the relation (2.7) yields (after the use of crossing matrices)

$$\Gamma^{x}=D_{y}{}^{x}C_{su}\Gamma^{y}+A_{r}{}^{x}C_{st}\gamma^{r}, \quad (4.1)$$

where C_{su} and C_{st} are the crossing matrices from u to s and t to s channel, respectively, Γ^{x} is the square of the coupling of the state i to the baryon isobar of spin x, γ^{r} is the product of the couplings of the meson of spin r to the two external mesons and the external baryon and antibaryon. The numbers $D_{y}{}^{x}$ and $A_{r}{}^{x}$ are constants which are obtained by the integrations on the right-hand side of (2.7), of the imaginary parts of the various Born terms. Of course (4.1) can easily be generalized to the scattering in any partial wave but for our present purpose we need only the p-wave scattering amplitudes. The only approximation that has gone into the derivation of (4.1) is linearization (2.5) of the D elements. However, (4.1) as it stands is still too complicated to be transparent. In order to see the wealth of information hidden in (4.1) we make the assumption that the forces which arise from the exchange of baryon isobars in the u channel, which give the first term on the right-hand side of (4.1), can be approximated by their static limit. This then means that the $D_{y}{}^{x}$ is actually the spin crossing matrix from the u to s channel. Thus, symbolically,

$$\Gamma=(C_J\times C_U)_{su}\Gamma+(A\times C_U)_{st}\gamma, \quad (4.2)$$

where subscripts J and U stand for spin and internal symmetry, respectively, and the multiplication stands for the direct product.

Now (4.2) is an inhomogeneous equation of the type (2.14) considered in proposition 2 of Sec. II. Then by proposition 2 the solution to (4.2) exists if and only if $(A\times C_U)_{st}\gamma$ is an eigenvector of $(C_J\times C_U)_{su}$ with eigenvalue -1. Furthermore by proposition 1 of Sec. II, we know that the column vectors of $(C_U)_{st}$ are eigenstates of $(C_U)_{su}$ with eigenvalues ±1; $+1$ if the column corresponds to the exchange of a meson whose coupling to the external mesons is symmetric in the internal-symmetry indices of the external mesons and -1 for antisymmetry in the same indices. Therefore in view of these observations the column vectors of A must be eigenstates of $(C_J)_{su}$ with eigenvalues ±1. More specifically the column vector of A, corresponding to the exchange of a particle which couples symmetrically in the internal-symmetry indices to the external mesons, is an eigenstate of $(C_J)_{su}$ with eigenvalue -1; it is an eigenstate of $(C_J)_{su}$ with eigenvalue $+1$ if the coupling of the exchanged meson to the external mesons is antisymmetric in internal-symmetry indices of the external particles.

It is plausible from the above observations that by the proper choice of the exchanged mesons the second term on the right-hand side of (4.2) can be formally regarded as coming from a coupling which is totally antisymmetric (including spin and internal symmetry),

[14] Polkinghorne has obtained this result starting from different assumptions. J. C. Polkinghorne, Ann. Phys. (N. Y.) 34, 153 (1965).

and hence meson couplings can be looked upon (once again formally only) as structure constants of a spin–internal-symmetry group.

We will now elucidate these considerations by applying them to models leading to various groups of physical interest and discuss the various consequences. The models we discuss will lead to $SU(6)$, $SO(11)$, $Sp(16)$ which are all compact[15] and $[SU(2)\otimes SU(3)]\times T_{24}$ which is the strong-coupling noncompact group. Of these, only $SU(6)$ and the strong-coupling results give physically reasonable results.

A. "$SU(6)$" for Baryons

As far as baryon classification and couplings are concerned, we find that $SU(6)$ is probably the most physically appealing group. Some bootstrap aspects of $SU(6)$ have been discussed in Refs. 3, 4, and 6.

Let us consider a world in which the only mesons that exist are the eight pseudoscalar mesons, nine vector mesons, and nine tensor mesons belong to **8** and **1** representations of $SU(3)$; the relevant baryons are the octet of $\frac{1}{2}^+$ and decuplet of $\frac{3}{2}^+$ baryons. Since the various channels in our model are decoupled, we can consider them separately. We first discuss (4.2) for pseudoscalar-meson and $\frac{1}{2}^+$ baryon scattering in p wave. There are two angular-momentum states $\frac{1}{2}^+$ and $\frac{3}{2}^+$ and

$$(C_J)_{su}=\begin{pmatrix}-\frac{1}{3} & \frac{4}{3}\\ \frac{2}{3} & \frac{1}{3}\end{pmatrix}, \tag{4.3}$$

whose eigenstates are

$$\begin{pmatrix}1\\1\end{pmatrix}$$

with eigenvalue $+1$, and

$$\begin{pmatrix}1\\-\frac{1}{2}\end{pmatrix}$$

with eigenvalue -1. Now since the tensor mesons couple symmetrically to two pseudoscalars while vectors couple antisymmetrically, Eq. (4.2) reduces to

$$\Gamma=\begin{bmatrix}-\frac{1}{3} & \frac{4}{3}\\ \frac{2}{3} & \frac{1}{3}\end{bmatrix}\times(C_V)_{su}\Gamma+\begin{bmatrix}a_1\\a_2\end{bmatrix}\times\left[\alpha\begin{bmatrix}1\\-\frac{1}{3}\\0\\0\\\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix}+\beta\begin{bmatrix}0\\0\\2/\sqrt{5}\\-2/\sqrt{5}\\0\\0\\0\end{bmatrix}\right]+\begin{bmatrix}b_1\\b_2\end{bmatrix}\times\left[\gamma\begin{bmatrix}1\\\frac{1}{5}\\-\frac{2}{5}\\-\frac{2}{5}\\\frac{1}{2}\\-\frac{3}{10}\\0\end{bmatrix}+\delta\begin{bmatrix}0\\0\\-2/\sqrt{5}\\+2/\sqrt{5}\\0\\0\\0\end{bmatrix}+\epsilon\begin{bmatrix}\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\0\end{bmatrix}\right], \tag{4.4}$$

where the multiplication is the direct product, and (4.4) has a solution if and only if

$$\begin{pmatrix}a_1\\a_2\end{pmatrix}$$

is proportional to

$$\begin{pmatrix}1\\1\end{pmatrix},$$

and

$$\begin{pmatrix}b_1\\b_2\end{pmatrix}$$

to

$$\begin{pmatrix}1\\-\frac{1}{2}\end{pmatrix};$$

in (4.4) the components correspond to forces in states **1**, **27**, **10***, **10**, A, S, Q, where A is the amplitude corresponding to antisymmetric-antisymmetric octet coupling, S is the symmetric-symmetric octet coupling, and Q is the symmetric-antisymmetric octet coupling. Without loss of generality we take

$$\begin{pmatrix}a_1\\a_2\end{pmatrix}$$

to be

$$\begin{pmatrix}1\\1\end{pmatrix},$$

and

$$\begin{pmatrix}b_1\\b_2\end{pmatrix}$$

to be

$$\begin{pmatrix}1\\-\frac{1}{2}\end{pmatrix}.$$

However the solution of (2.21) is not unique since the corresponding homogeneous equation has seven linearly independent solutions.[16] From the point of physical interest we will look for a solution in which only the $\frac{1}{2}^+$ octet baryons and $\frac{3}{2}^+$ decuplet baryons exist. Such a solution exists and is unique. The solution is formally the same as the one obtained by Fulco and Wong[6] though the assumptions and the physical ideas going into the two derivations are quite different. We thus have

$$\Gamma_{10}:\Gamma_S:\Gamma_A:\epsilon:\gamma:\delta:\beta:\alpha=1:5/4:1:\tfrac{4}{3}:\tfrac{5}{6}:\tfrac{1}{6}\sqrt{5}:0:\tfrac{3}{4}. \tag{4.5}$$

The consequences of this result are

1. The D/F ratio for both pseudoscalar- and tensor-

[15] D. W. Joseph, Phys. Rev. **139**, B1406 (1965).

[16] R. C. Hwa and S. H. Patil, Phys. Rev. **138**, B933 (1965).

meson coupling to $\frac{1}{2}^+$ baryons is $\frac{3}{2}$. On the other hand, the vector-meson coupling to $\frac{1}{2}^+$ baryons is pure F type.

2. The ratio of the couplings of $\frac{1}{2}^+$ and $\frac{3}{2}^+$ baryons to pseudoscalar mesons and $\frac{1}{2}^+$ baryons is

$$\Gamma_8/\Gamma_{10}=9/4\,, \tag{4.6}$$

which gives

$$g_{N,N\pi}{}^2/g_{N^*,N\pi}{}^2=25/8\,,$$

which is somewhat large compared to the Chew-Low static theory result of 2.

While the model gives essentially the same results as $SU(6)$ there are two important differences. In our model, the exchange of tensor mesons in the t channel is necessary whereas in $SU(6)$ the tensor mesons do not belong to the **35** representation. However, the introduction of tensor nonet avoids the difficulty for couplings, which arises in $SU(6)$ and we get a nontrivial result.

The same calculations can be carried out for the scattering of p-wave vector-meson octet by $\frac{1}{2}^+$ baryons. We will discuss only the $M1$ transitions, i.e., the orbital angular momentum 1 combines with the vector-meson spin to give an angular momentum of 1. The calculations are essentially identical to the calculation for the pseudoscalars except that we have additional exchanges of pseudoscalar mesons in the t channel. However, the coupling of two vectors and a pseudoscalar is symmetric in the internal-symmetry indices of the vector mesons; hence the effect of this additional force is to change the magnitudes of γ and δ by the same amount since D/F ratios for both pseudoscalars and tensors was found to be the same. However, the solution to (4.4) assuming the existence of only the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet, is unique and the various ratios of α, β, etc., are the same as in the case (4.5). This then implies that the pseudoscalar octet does not couple to two vector octets. We note however that if we had allowed for the exchange of an additional pseudoscalar singlet (which is physically reasonable) then the forces due to pseudoscalar-nonet exchange can be proportional to those due to tensor-nonet exchange and a solution exists for which the pseudoscalar-octet coupling to two vector octets is nonzero. In either of the two cases, the external vector-meson coupling to baryons is the same as the pseudoscalar-meson coupling to baryons. Specifically the D/F ratio is $\frac{3}{2}$ and

$$g_{N,N\rho}{}^2/g_{N^*,N\rho}{}^2=25/8\,. \tag{4.7}$$

One notes that the D/F ratio for the t-channel vector mesons is different from the D/F ratio for the external vector mesons. But this is not a contradiction, since for the static baryons the real vector-meson coupling is essentially magnetic while the t-channel vector is virtual and in that case the coupling is probably mostly electric. With the assumption of photon vertices being dominated by vector mesons, and the external vector-meson coupling D/F ratio of $\frac{3}{2}$, the magnetic moment ratio for proton and neutron is the same as the one obtained by Fulco and Wong[6] and the $SU(6)$ result

$$\mu_p/\mu_n=-\tfrac{3}{2}\,. \tag{4.8}$$

We now calculate the amplitudes for scattering of the pseudoscalar octet by the $\frac{3}{2}^+$ baryon decuplet. The allowed $SU(3)$ channels in the s and u channels are

$$\mathbf{8}\times\mathbf{10}=\mathbf{8}+\mathbf{10}+\mathbf{27}+\mathbf{35}\,, \tag{4.9}$$

whereas in the t channel they are

$$\mathbf{1}+\mathbf{8}+\mathbf{8}'+\mathbf{27}\,. \tag{4.10}$$

One **8** couples symmetrically with the external mesons while the other couples antisymmetrically. The allowed angular momenta in the s and u channels are

$$1\times\tfrac{3}{2}=\tfrac{1}{2}+\tfrac{3}{2}+\tfrac{5}{2} \tag{4.11}$$

and the corresponding crossing matrix is

$$(C_J)_{su}=\begin{bmatrix} 1/6 & -2/3 & 3/2 \\ -1/3 & 11/15 & 3/5 \\ 1/2 & 2/5 & 1/10 \end{bmatrix}. \tag{4.12}$$

Then (2.7) leads us to [in analogy to (4.4)]

$$\gamma=(C_J)_{su}\times(C_U)_{su}\gamma+\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\times\begin{bmatrix} 6 \\ 3 \\ 1 \\ -3 \end{bmatrix} +\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\times\begin{bmatrix} 2 \\ 3 \\ -3 \\ 1 \end{bmatrix}+\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\times\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{4.13}$$

where the 4-vector components correspond to representations **8**, **10**, **27**, **35**, respectively. Now

$$\begin{bmatrix} 6 \\ 3 \\ 1 \\ -3 \end{bmatrix}$$

is an eigenstate of $(C_U)_{su}$ with eigenvalue -1 while the remaining two 4-vectors in (4.13) are eigenstates of $(C_U)_{su}$ with eigenvalue $+1$. Hence, by the earlier discussion in this section, solution to (4.13) exists if and only if a_i is of the form

$$\alpha\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}+\beta\begin{bmatrix} 1 \\ 0 \\ 5/9 \end{bmatrix},$$

b_i is of the form

$$\delta\begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix},$$

and c_i is of the form

$$\epsilon\begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix};$$

i.e., a_i is the general eigenstate of $(C_J)_{su}$ with eigenvalue $+1$ and b_i and c_i are eigenstates of $(C_J)_{su}$ with eigenvalue -1. Assumption of the existence of only the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet once again yields a unique solution

$$\gamma_8:\gamma_{10}:\alpha:\beta:\delta:\epsilon=60:60:3:0:1:2\,, \tag{4.14}$$

where γ_8 and γ_{10} are the couplings of the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet, respectively, to $\frac{3}{2}^+$ decuplet and octet of mesons. From this we get

$$g_{N,N^*\pi}{}^2/g_{N^*,N^*\pi}{}^2=32/25\,. \tag{4.15}$$

The calculations are repeated for p-wave vector octet and $\frac{3}{2}^+$ decuplet scattering. We confine ourselves to that amplitude in which the orbital angular momentum combines with the vector-meson spin to give unit angular momentum. The comments we made about the pseudoscalar exchanges in the case of vector $-\frac{1}{2}^+$ baryon scattering are applicable here also. In any case, the ratio of the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet couplings is the same as (4.14). As a consequence

$$g_{N,N^*\rho}{}^2/g_{N^*,N^*\rho}{}^2=32/25\,. \tag{4.16}$$

Assumption of the dominances of photon vertices by vector mesons then yields

$$\mu_p=\mu_{N^{*+}}=-\mu_{\Omega^-}=\tfrac{2}{3}\sqrt{2}\mu_{N^{*+}\to p\gamma}\,. \tag{4.17}$$

All these results agree with those from $SU(6)$.

We have reproduced almost all the good results of $SU(6)$ by our physically oriented bootstrap algebra. In particular we have been able to express the couplings of all the octet of vector mesons to all the baryons and baryon isobars in terms of a single scale parameter. Assumption of photon vertices being dominated by vector mesons then gives all the magnetic couplings in terms of one parameter. Following Kuo and Yao,[17] we can then make a two-parameter calculation of electromagnetic mass differences;

$$\delta m=a\mathbf{M}\cdot\mathbf{M}+bQ^2\,. \tag{4.18}$$

The only intermediate states allowed in our model are the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet baryons. One is then able to reproduce almost all the results of Kuo and Yao.[17] One interesting point to note is that in most cases the contribution to the mass differences by the Q^2 term in (2.36) is less than 30% and is of opposite sign to that of the $\mathbf{M}\cdot\mathbf{M}$ term.

As examples of "simple" alternative compact groups[15] we will now consider $R(11)$ and $Sp(16)$.

B. $R(11)$ for Baryons

The adjoint representation of the rotation group in 11 dimensions contains an octet and a singlet of vectors and an octet, a **10** and a $\overline{\mathbf{10}}$ of pseudoscalars. The spinor representation contains two spin-$\frac{1}{2}$ octets.

We consider the scattering of an octet of pseudoscalars and $\frac{1}{2}^+$ octet. In the t channel we allow for the exchange of an octet and a **10** and $\overline{\mathbf{10}}$ of vectors and a nonet of tensors. In the s and u channels we allow for the existence of only $\frac{1}{2}^+$ octet. Then the equation to solve is

$$\Gamma=\begin{pmatrix}-1/3 & 4/3\\ 2/3 & 1/3\end{pmatrix}\times(C_U)_{su}\Gamma+\begin{pmatrix}a_1\\a_2\end{pmatrix}$$
$$\times\left[t_s\begin{bmatrix}1\\1/5\\-2/5\\-2/5\\1/2\\-3/10\\0\end{bmatrix}-t_a\begin{bmatrix}0\\0\\-1/\sqrt{5}\\1/\sqrt{5}\\0\\0\\0\end{bmatrix}+e\begin{bmatrix}1\\1\\1\\1\\1\\1\\0\end{bmatrix}\right]+\begin{pmatrix}b_1\\b_2\end{pmatrix}\times\left[v_a\begin{bmatrix}0\\-1/3\\0\\0\\1/2\\1/2\\0\end{bmatrix}-v_s\begin{bmatrix}0\\0\\1/\sqrt{5}\\-1/\sqrt{5}\\0\\0\\0\end{bmatrix}-n\begin{bmatrix}5/4\\-1/12\\1/4\\1/4\\0\\-1/2\\0\end{bmatrix}\right]. \tag{4.19}$$

As before, assumption that a solution exists leads us to

$$\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}1\\-\frac{1}{2}\end{pmatrix}$$

and

$$\begin{pmatrix}b_1\\b_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}.$$

The column indices are the same as in (4.4) The solution to (4.19) even with only the $\frac{1}{2}^+$ octet existing has an additional ambiguity due to reasons pointed out by Joseph[15] that the spinor representation has two octets and these octets cannot be distinguished by any additive quantum number. The result is

$$\begin{gathered}t_s=0\,,\quad t_a:\Gamma_Q:v_s=2:\tfrac{3}{2}:1\,,\\ \Gamma_s:\Gamma_A:e:v_a:n=15/8:9/8:1/2:1:1\,,\end{gathered} \tag{4.20}$$

where the ratio of Γ_s/Γ_Q is not determined. Let us assume, however, that the ratio of Γ_s/Γ_A given above is the corresponding ratio for the physical $\frac{1}{2}^+$ baryon octet coupling to pseudoscalar mesons. Then we can calculate the D/F ratio which turns out to be

$$D/F=\sqrt{3}\,, \tag{4.21}$$

[17] T. Kuo and T. Yao, Phys. Rev. Letters 14, 79 (1965).

which is in quite good agreement with the $SU(6)$ prediction of $\frac{3}{2}$.

The calculations for $M1$ transitions in vector-meson–$\frac{1}{2}^+$ baryon scattering are essentially the same as for pseudoscalar–$\frac{1}{2}^+$ baryon scattering. The D/F ratio for $\frac{1}{2}^+$ baryon coupling to vectors is also $\sqrt{3}$ and consequently the assumption of photon vertices being dominated by vector mesons leads to the result

$$\mu_p/\mu_n = -\tfrac{1}{2}(1+\sqrt{3}) \tag{4.22}$$

which is close to the experimental value of -1.47.

C. $Sp(16)$ for Baryons

The adjoint representation of the symplectic group in 16 dimensions contains, in addition to the multiplets in the adjoint representation of $R(11)$, a 27-dimensional vector multiplet. On the other hand the spinor representation contains only one $\frac{1}{2}^+$ octet. The calculations for the scattering of pseudoscalar octet and $\frac{1}{2}^+$ octet are similar to those in $SU(6)$ and $R(11)$ and we only quote the results

$$\Gamma_s/\Gamma_A = 15/39\,, \quad D/F = 3/\sqrt{13}\,. \tag{4.23}$$

These results are not in good agreement with the experimental results. The corresponding calculations for vector mesons yield

$$\mu_p/\mu_n = -(1+\sqrt{13})/2\,, \tag{4.24}$$

which also is not in good agreement with the experimental result of -1.47.

D. $[SU(2)\otimes SU(3)]\times T_{24}$

This strong-coupling noncompact group[8] whose corresponding model in bootstrap algebra does not have any t-channel exchanges, has already been extensively analyzed.[9] We only note that the results for D/F ratio and the various magnetic-moment ratios in this group are very encouraging.

V. COMBINING SPIN AND INTERNAL SYMMETRIES FOR MESONS

The problem of induction of internal symmetries for mesons has been discussed at some length by Cutkosky and others.[1,5,14] We will therefore confine ourselves to the problem of combining spin and internal symmteries.

Consider the situation in which the pseudoscalar mesons (which we assume have small mass) are scattered by some heavy mesons such as vector mesons. Of course, Eq. (2.7) is quite general and is applicable to the present problem. One would *a priori* think that Eq. (4.2) which comes out of the assumption that all the particles exchanged in the u channel are heavy, cannot be applied to our case since the same light pseudoscalars as the external ones can, in most cases, also be exchanged in the u channel. However, we will show, by taking specific cases, that (4.2) is indeed valid for most of our calculations.

A. $SU(6)$ for Mesons

Consider the scattering of p-wave pseudoscalar-meson octet by the vector octet. We will assume that the forces from the u-channel exchanges are all due to heavy-particle exchanges (except for the pseudoscalar-octet exchange) and therefore can be adequately described by their static limit. In the t channel we allow for the exchange of a tensor nonet and a vector octet. (The vector singlet does not couple to the two pseudoscalar octets.) Then the equation for this case analogous to (4.4) is

$$\Gamma = \begin{pmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{pmatrix} \times (C_U)_{su}\Gamma + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \left[\alpha \begin{pmatrix} 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1/5 \\ -2/5 \\ -2/5 \\ 1/2 \\ -3/10 \\ 0 \end{pmatrix} \right] + \left[\gamma \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \delta \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right] \times \begin{pmatrix} 1 \\ -1/3 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1/3 \\ 0 \\ 0 \\ 1/2 \\ -1/2 \\ 0 \end{pmatrix}, \tag{5.1}$$

where the last term has been added to compensate for the fact that the u-channel pseudoscalar exchange cannot be approximated by the static limit. The $SU(3)$ column indices are the same as those used for the discussion of baryons. Now quite generally we can write

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = r \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + s \begin{pmatrix} 10 \\ -5 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \tag{5.2}$$

where the first two vectors on the right are eigenstates of $(C_J)_{su}$ the spin crossing matrix, with eigenvalue $+1$, and the last vector is an eigenstate with eigenvalue -1. Similarly one can write the $SU(6)$ column as a linear combination of the seven linearly independent eigenstates v_i of $(C_U)_{su}$,

$$\begin{pmatrix} -1 \\ \frac{1}{3} \\ 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} = \sum x_i v_i , \tag{5.3}$$

where x_i are well-determined nonzero numbers. Three of the v_i are already written in (5.1). The remaining four let us say v_4, v_5, v_6, and v_7 are

$$\begin{pmatrix} 27/8 \\ 7/40 \\ -9/40 \\ -9/40 \\ -9/8 \\ 27/40 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \\ 0 \\ -1/2 \end{pmatrix}, \quad \begin{pmatrix} 5/4 \\ -1/12 \\ 1/4 \\ 1/4 \\ 0 \\ -1/2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \\ 0 \\ 1/2 \end{pmatrix}. \tag{5.4}$$

The first two have eigenvalue $+1$ and the remaining two have eigenvalue -1. Now for the solution to (5.1) to exist, the inhomogeneous term must be an eigenstate of $(C_J)_{su}\times(C_U)_{su}$ with eigenvalue -1. The inhomogeneous terms coming into (5.1) due to direct product of (5.4) with

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix},$$

are linearly independent of the other inhomogeneous terms occurring in (5.1). Therefore these terms by themselves must be eigenstates of $(C_J)_{su}\times(C_U)_{su}$ with eigenvalue -1. Now consider the direct product of the first term in (5.4) with

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

The first term in (5.4) is an eigenstate of $(C_U)_{su}$ with eigenvalue $+1$. Hence solution to (5.1) exists only if

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

is an eigenstate of $(C_J)_{su}$ with eigenvalue -1, i.e., only if $r=s=0$. Now consider the direct product of the third term in (5.4) with

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

The third term in (5.4) being an eigenstate of $(C_U)_{su}$ with eigenvalue -1, the solution to (5.1) exists only if

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

is an eigenstate of $(C_J)_{su}$ with eigenvalue $+1$, i.e., only if $t=0$. Thus the last term in (5.1) which came in because the static approximation for pseudoscalar exchange in the u channel is not valid, is indeed zero if a solution to (5.1) has to exist!

One can now proceed as before. According to the arguments at the beginning of Sec. IV, the solution to (5.1) can be taken to exist if and only if

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ 1 \end{pmatrix}. \tag{5.5}$$

Then the assumption of the existence of only the nonet of vectors and octet of pseudoscalars in the s and u channels gives a unique solution to (5.1) with

$$\delta=0 \quad \text{and} \quad \Gamma_1{}^v:\Gamma_8{}^v:\Gamma_8{}^p:\alpha:\beta:\gamma = 1:5/16:27/32:1:1:5/16:27/32, \tag{5.6}$$

where $\Gamma_1{}^v$, $\Gamma_8{}^v$ are the couplings of the vector singlet and octet, respectively, to the vector octet and pseudoscalar singlet; $\Gamma_8{}^p$ is the coupling of the pseudoscalar octet to the pseudoscalar octet and vector octet. If ω_1 and ω_8 are the isosinglets belonging to the vector singlet and vector octet, respectively,

$$A(\omega_1 \to \pi\rho) = \sqrt{2}A(\omega_8 \to \pi\rho), \tag{5.7}$$

where A stands for the decay amplitude. This means that in the nonet symmetry with the mixing angle $\theta=\tan^{-1}(1/\sqrt{2})$, the ϕ meson decay into $\pi\rho$ is forbidden, a result in agreement with the $SU(6)$ "prediction."

We will consider the bootstrap algebra for the case in which there are no t-channel exchanges, which presumably leads to the results of the strong-coupling group $[SU(2)\otimes SU(3)]\times T_{24}$.

B. The Strong-Coupling Group for Mesons

For the scattering of p-wave pseudoscalar octet by a vector octet, with no t-channel exchanges, the bootstrap equation is

$$\Gamma = \begin{pmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{pmatrix} \times (C_U)_{su}\Gamma + \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1/3 \\ 0 \\ 0 \\ 1/2 \\ -1/2 \\ 0 \end{pmatrix}, \tag{5.8}$$

where the last term is introduced to compensate for the fact that the static approximation for pseudoscalar exchange in the u channel is not valid. However, arguing along the same lines as for $SU(6)$, it is easy to show that solution to (5.8) exists if, and only if,

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = 0. \tag{5.9}$$

The remaining homogeneous solution is not unique; it has **11** linearly independent solutions.[18] If we had a knowledge of the representation of $[SU(2)\otimes SU(3)] \times T_{24}$, to which the mesons belonged, we could have obtained a solution by equating to zero those elements of Γ which do not belong to the representation. However, in general it is not unique as to which representation the mesons belong. Instead we will proceed along our bootstrap way[19] and arrive at essentially a unique answer.

We will first assume that an octet of pseudoscalars, and a nonet of vector mesons, exist in the solution to (5.8). This suggests that the "representation" we desire is the one which has those spin and $SU(3)$ multiplets which contain $Y=0$, $J=I\pm 1$ elements.[20] This already implies that

$$\Gamma_1^2 = 0, \tag{5.10}$$

where the superscript stands for spin and the subscript for the $SU(3)$ representation. We further note that the forces coming from the exchange of the pseudoscalar octet and the vector nonet are significantly attractive in the **27** spin-2 representation (and in vector-nonet and pseudoscalar-octet states), and hence we assume that our "representation" contains a Γ_{27}^2 element. Apart from **27** (2) state we find that the forces in all the other spin-2 states are rather weak [even the forces from **27** (2) exchange]. We therefore look for a solution in which most of the spin-2 states, except the **27** (2) state, are either absent or have small couplings. (We note that the equality of **10** and $\overline{\mathbf{10}}$ couplings, implied by change congregation invariance, itself implies three conditions.) Such a solution does exist and is given by

$$\begin{array}{ccc} J=0 & J=1 & J=2 \\ \begin{pmatrix} 23.2 \\ 0 \\ 0 \\ 0 \\ 23.0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 27.9 \\ 0.5 \\ 3.0 \\ 3.0 \\ 0 \\ 8.52 \end{pmatrix}, & \begin{pmatrix} 0 \\ 6.6 \\ 0 \\ 0 \\ 0.5 \\ 0 \end{pmatrix}, \end{array} \tag{5.11}$$

where the column indices are the same as for baryons except that Γ_Q term is omitted. ($\Gamma_Q=0$ by charge conjugation.) The essential points about the solution are that Γ_1^0, Γ_A^0, Γ_1^1, Γ_S^1, Γ_{27}^2 are the only large terms. ($\Gamma_{10}^1=3$ is relatively small.) We therefore feel that a "representation" which contains these elements and whose other elements are all positive will give essentially the same ratios for the "large" elements. We point out that Γ_1^0 does not correspond to the coupling of X^0 (960 MeV) since X^0 has opposite G parity ($G=+$) to that of our spin-0^- singlet ($G=-$).

From (5.11) we calculate

$$\frac{A(\omega_1 \to \pi\rho)}{A(\omega_8 \to \pi\rho)} = \sqrt{(2.05)}, \tag{5.12}$$

to be compared with the $SU(6)$ value of $\sqrt{2}$. Furthermore, the ratio

$$\Gamma_S^1/\Gamma_A^0 \simeq 0.37 \tag{5.13}$$

is essentially indistinguishable from the $SU(6)$ value of 10/27.

VI. DISCUSSION

The consequences of our bootstrap algebras are the following:

1. In a world with the only baryons being the $\frac{1}{2}^+$ baryons, where the vector-meson coupling constants are the structure constants of a Lie algebra, the $\frac{1}{2}^+$ baryons must belong to a representation of the same algebra.
2. The bootstrap algebra provides a physical way of describing the various spin and internal-symmetry properties of a system. In a model where the forces from the u-channel exchanges can be approximated by their static limits, we obtain an easily solvable system which leads to consequences of various spin–internal-symmetry groups. These consequences are explicitly derived for $SU(6)$, the strong-coupling group, etc. Most of the good features of these groups are retained in our model, e.g., D/F ratios, baryon and baryon isobar coupling ratios, magnetic moments, electromagnetic mass differences, while the usual difficulties encountered for coupling in $SU(6)$, etc., are not present here.
3. The results are easily extended to mesons even though pseudoscalar-meson exchanges in the u channel cannot be approximated by their static limit. The various coupling constants which come out agree with the corresponding group results.
4. The meson-coupling results obtained from a representation for the strong-coupling theory agree remarkably closely with the predictions of $SU(6)$, and hence suggest that the strong-coupling theory deserves serious attention.

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[18] $(C_U)_{su}$ has three eigenstates with eigenvalue -1 and four with eigenvalue $+1$, while $(C_J)_{su}$ has one eigenstate with eigenvalue -1 and two with eigenvalue $+1$. Hence the direct product has eleven eigenstates with eigenvalue $+1$ and ten with eigenvalue -1.

[19] See, e.g., V. Singh and B. Udgaonkar, in Ref. 9.

[20] See C. J. Goebel in Phys. Rev. Letters **16**, 1130 (1966).