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Decays $\pi^0 \to 2\gamma$, $\eta \to 2\gamma$ and Sum Rules for Nucleon Compton Scattering

HEINZ PAGELS
The Rockefeller University, New York, New York
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Motivated by the success of the Goldberger–Treiman calculation of charged-pion decay and the assumption of pion-pole dominance of the divergence of the axial current subsequently introduced, we study the decays of the neutral pion and eta in the same way. Examining Compton scattering of real photons from the nucleons presents a fruitful analogy to the nuclear β-decay process. Identifying the dominant nucleon and meson-pole-term contributions to Compton scattering and utilizing the content of the exact low-energy theorem on Compton scattering, a no-subtraction hypothesis, supported by the Regge-pole phenomenology, then enables us to establish exact sum rules for the lifetimes $\tau_{\pi^0\to 2\gamma}$ and $\tau_{\eta\to 2\gamma}$ and also the Drell–Hearn sum rules for $\kappa_p^2 \pm \kappa_n^2$. We also consider in detail the relation of our sum rule to the Goldberger–Treiman calculation of π^0 decay and forward-Compton-scattering sum rules for systems of spin=1. Neglecting continuum contributions, which are small, we find from our sum rules $\kappa_p^2=\kappa_n^2$ and $\tau_{\pi^0\to 2\gamma}^{-1}=\pi\alpha^2 m_\pi^3\kappa_p^2/4g_{\pi N}^2 M_N^2 \simeq 3.1$ eV or $\tau_{\pi^0\to 2\gamma}\simeq 2.2\times 10^{-16}$ sec in approximate agreement with the experimental value $\tau_{\pi^0\to 2\gamma}=(1.0\pm 0.5)\times 10^{-16}$ sec. Better photopion-production data for the nonresonant multipoles would enable us to accurately estimate the continuum contributions. Including the dominant contributions to the photopion production continuum, we find that the Drell-Hearn sum rule for $\kappa_p^2+\kappa_n^2$ is well satisfied. We can estimate the $\eta \to 2\gamma$ lifetime but our result depends on a knowledge of the eta-nucleon coupling and on nonresonant background contributions to the photopion multipoles. Once better photopion production data for the nonresonant multipoles $E_l\pm$, $M_l\pm$, $l\leqslant 2$ for energies <1 BeV become available, these sum rules can offer reliable theoretical estimates for the decays $\pi^0 \to 2\gamma$, $\eta \to 2\gamma$.

I. INTRODUCTION

ABOUT eight years ago Goldberger and Treiman,[1] using the techniques of dispersion relations, derived a relation between the decay lifetime of the charged pion into leptons $\tau_{\pi^-\to e^-+\bar\nu}$, the pion-nucleon coupling constant $g_{\pi N}^2/4\pi \simeq 14$ and the Fermi constant for Gamow-Teller transitions $G_A \simeq (1.18) \times (1.015\times 10^{-5}) M_p^{-2}$. The result of their calculation may be expressed by

$$\frac{1}{\tau_{\pi^-\to e^-+\bar\nu}}=\frac{M_N^2 G_A^2 m_\pi^3}{4\pi g_{\pi N}^2}\left(\frac{m_e}{m_\pi}\right)^2\left(1-\frac{m_e^2}{m_\pi^2}\right)^2, \quad (1.1)$$

which is found to be within 25% of the observed decay rate for charged pions. The essential physical ingredients of the Goldberger-Treiman calculation, stated in a somewhat different language than given originally,[2] center around the hypothesis that the pole term from charged pions dominates the induced pseudoscalar amplitude for the process of nuclear β decay, $n \to p+e^- +\bar\nu$ [Fig. 1(a)]. This pole-term contribution is given by the product of the pion-nucleon coupling constant $g_{\pi N}$ and the amplitude for pion decay which is to be calculated. Appealing to the fact that this amplitude at threshold is simply proportional to the experimentally determined constant G_A, and assuming a strongly convergent dispersion relation with neglect of all but the pion-pole-term contribution, one then obtains the result Eq. (1.1). To reiterate: The necessary physical input going into the calculation was (1) information about the threshold behavior of the amplitude for $n\to p+e^-+\bar\nu$, in this case taken from experiment, (2) the assumption that single-pion exchange dominates the induced pseudoscalar amplitude, and (3) a no-subtraction hypothesis for the dispersion relation and neglect of the continuum contributions. This technique then enables us to compute the decay amplitude for charged pions.

Motivated by the success of the Goldberger-Treiman calculation of the decay rate for charged pions, we now direct our attention along similar lines in this investigation to the decay of the neutral pion, $\pi^0 \to 2\gamma$ (and $\eta\to 2\gamma$). Assuming, as is usual, that the decay of the π^0 is purely electromagnetic in origin, then Compton scattering of real photons from the nucleons offers a suggestive analogy to the nuclear β-decay process. As we will show in the sequel, one of the six invariant amplitudes describing the nucleon Compton effect receives a contribution from the pole term arising from π^0 exchange, the residue of which is proportional to $g_{\pi N}$ and the pion-decay amplitude $\pi^0\to 2\gamma$ that we want to calculate [Fig. 2(a)]. Moreover, the threshold behavior of this amplitude is known exactly and is regulated by the low-energy theorem for Compton scattering of Gell-Mann, Goldberger, and Low[3] which implies the threshold behavior is exactly given by the low-energy behavior of the Born terms.[4] Assuming the amplitude requires no subtractions, an assumption supported by the Regge-pole model for high-energy scattering, we can

[1] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).
[2] J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento, **17**, 757 (1960).
[3] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954); F. E. Low, *ibid.* **96**, 1428 (1954).
[4] The Born terms and pole terms for Compton scattering are, of course, not the same differing in as much as they treat numerators differently. Consequently, they may differ in their threshold behavior.

in principle calculate the lifetime for neutral pion decay. The π^0 pole, of course, does not contribute to the low-energy theorem; however, knowing the relevant amplitude at *two* points, namely at zero momentum and infinite momentum where we assume it vanishes, we can get a nontrivial condition on the π^0 lifetime. This is also true for Drell-Hearn-type sum rules where one uses the low-energy theorem to which the N^* does not contribute plus the asymptotic behavior to relate the anomalous moments to the γNN^* vertex. Since the continuum contribution, in a first approximation of keeping only the lightest intermediate state, can be shown to be related to the photopion production amplitude in the $T=\frac{1}{2}$ channel it can be expected to be small. Within the compass of these approximations, which do not essentially differ from those of the pion-pole dominance model, we obtain for the lifetime of the π^0

$$\frac{1}{\tau_{\pi^0\to2\gamma}}=\frac{\pi\alpha^2m_\pi\kappa_p{}^2}{4g_{\pi N}{}^2}\left(\frac{m_\pi}{M_N}\right)^2\simeq3.1\text{ eV} \tag{1.2}$$

or $\tau_{\pi^0\to2\gamma}\simeq2.2\times10^{-16}$ sec, where $\alpha=1/137$ and $\kappa_p=1.79$ is the anomalous magnetic moment of the proton in units of $e/2M_N$. This result is not in disagreement with the observed lifetime $\tau_{\pi^0}\simeq(1.0\pm0.5)\times10^{-16}$ sec. Similar considerations also apply to the decay $\eta\to2\gamma$ however here the continuum contribution is much larger related as it is to photopion production in the $T=\frac{3}{2}$ channel which includes a large resonant contribution from the $M_{1+}{}^{(3/2)}$ transition (N^*) which we will estimate. The $\eta\to2\gamma$ decay-rate calculation also requires knowledge of the coupling of the η to nucleons, which is not known experimentally but can be related to the known pion-nucleon coupling using $SU(3)$ symmetry of the meson-baryon couplings.

In the next section we will make use of the work of Hearn and Leader[5] to examine the pole-term contributions to the isoscalar and isovector parts of the invariant amplitudes for the nucleon-Compton effect and the content of the low-energy theorem. Adopting a no-subtraction philosophy for these amplitudes, we can then establish nontrivial sum rules for the residues of the pole terms the isoscalar and isovector parts of which yield the sum rules for the decays $\eta\to2\gamma$ and $\pi^0\to2\gamma$ and the Drell-Hearn sum rules[6] for $\kappa_p{}^2+\kappa_n{}^2$ and $\kappa_p{}^2-\kappa_n{}^2$, respectively. Nowhere do we make use of current algebra or the PCAC (partially conserved axial-vector current) approximation. We also examine the issue of subtractions in the light of the Regge-pole model for the high-energy asymptotic behavior of the scattering amplitude.

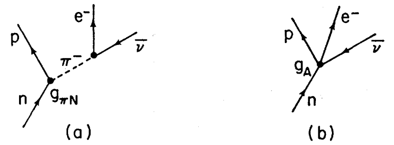

FIG. 1. Decay of the charged pion.

[5] A. C. Hearn and E. Leader, Phys. Rev. **126**, 789 (1962).
[6] S. D. Drell and A. C. Hearn, Phys. Rev. Letters **16**, 908 (1966).

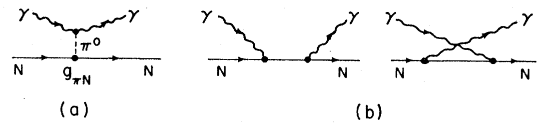

FIG. 2. Pole-term contributions to nucleon Compton scattering.

In Sec. III we estimate the continuum contributions arising from photopion production to the decay lifetimes and the Drell-Hearn sum rule. The final results are compared with the available experimental data.

In Appendix A we consider in detail the relation of our sum rule to the Goldberger-Treiman calculation of $\pi^0\to2\gamma$ decay. In Appendix B we investigate Drell-Hearn-type sum rules for charged systems of spin$=1$ and in particular consider application to bound-state nuclear systems.

II. SUM RULES FOR NUCLEON-COMPTON SCATTERING

In this section we will analyze nucleon Compton scattering making use of the study of Hearn and Leader.[5] For convenience we present some of their results. This analysis enables us to understand in a simple way how the hypothesis of convergent dispersion relations—when incorporated with the identification of the dominant pole-term contributions and the content of the low-energy theorem—provide us with exact sum rules for the residues of the poles. Besides obtaining purely forward-direction Compton scattering sum rules corresponding to the Drell-Hearn sum rules, we also extract information from the low-energy theorem in nonforward directions which then provide the basis for our understanding of the radiative decays of the π^0 and η mesons. Having established the sum rules, we then examine the assumed convergence properties of the amplitudes utilizing Regge-pole phenomenology in the asymptotic region.

A. Derivation of the Sum Rules

From the investigation of Hearn and Leader we learn that the process of physical Compton scattering of real photons from nucleons of mass m (see Fig. 3) can be analyzed in terms of the six Prange invariants $A_i(s,t,\bar{s})$ with

$$\begin{aligned} s&=-(p_1+k_1)^2,\\ \bar{s}&=-(p_1+k_2)^2,\\ t&=-(p_1+p_2)^2, \end{aligned} \tag{2.1}$$

which satisfy $s+\bar{s}+t=2m^2$ (m being the nucleonic

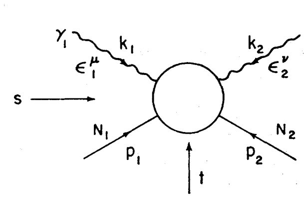

FIG. 3. Nucleon Compton scattering.

mass). In the barycentric system of the scattering processes these variables are related to the total energy W, photon energy p, and scattering angle θ according to

$$\begin{aligned} s&=W^2=[(p^2+m^2)^{1/2}+p]^2\,,\\ \bar{s}&=-2p^2(1+\cos\theta)+[(p^2+m^2)^{1/2}-p]^2\,,\\ t&=-2p^2(1-\cos\theta)\,,\end{aligned} \tag{2.2}$$

or

$$\begin{aligned} p&=(s-m^2)/2\sqrt{s}\,,\\ \cos\theta&=[(s-m^2)^2+2st]/(s-m^2)^2\,.\end{aligned} \tag{2.3}$$

The Feynman scattering amplitude in the s channel is given by

$$\langle\gamma_2N_2|F|\gamma_1N_1\rangle=\epsilon_{2\gamma}{}^{\dagger}\bar{u}_2(-p_2)F_{\mu\nu}u_1(p_1)\epsilon_{1\mu}\,, \tag{2.4}$$

where ϵ_1 and ϵ_2 are the polarization vectors of γ_1 and γ_2, the u_1 and u_2 are initial and final Dirac spinors, and in terms of the set of invariant amplitudes free from kinematical singularities,[7]

$$\begin{aligned} F_{\mu\nu}={}&A_1(s,t,\bar{s})\frac{P_\mu'P_\nu'}{P'^2}+A_2(s,t,\bar{s})\frac{N_\mu N_\nu}{N^2}+A_3(s,t,\bar{s})\\ &\times\frac{(P_\mu'N_\nu-P_\nu'N_\mu)i\gamma_5}{(P'^2N^2)^{1/2}}+A_4(s,t,\bar{s})\frac{P_\mu'P_\nu'i\gamma\cdot K}{P'^2}\\ &+A_5(s,t,\bar{s})\frac{N_\mu N_\nu i\gamma\cdot K}{N^2}+A_6(s,t,\bar{s})\\ &\times\frac{(P_\mu'N_\nu+P_\nu'N_\mu)i\gamma_5 i\gamma\cdot K}{(P'^2N^2)^{1/2}}\,.\end{aligned} \tag{2.5}$$

Here $P_\mu'=P_\mu-(P\cdot K)/K^2K_\mu$, $P_\mu=\frac{1}{2}(p_{1\mu}-p_{2\mu})$, $K_\mu=\frac{1}{2}(k_{1\mu}-k_{2\mu})$, $N_\mu=\epsilon_{\mu\nu\rho\sigma}P_\nu'K_\rho Q_\sigma$, $Q_\mu=k_{1\mu}+k_{2\mu}$, and $(P'^2N^2)^{1/2}=+\frac{1}{2}(m^4-s\bar{s})$. The isotopic-spin decomposition of the invariant amplitudes in Eq. (2.5) is given by

$$A_i=A_i{}^sI+A_i{}^v\tau_3\,, \tag{2.6}$$

so that for Compton scattering from photons and neutrons, one has $A_i{}^{p,n}=A_i{}^s\pm A_i{}^v$. The isoscalar ($s$) and isovector ($v$) parts of the Compton amplitude in this analysis refer, respectively, to the sum and difference of the scattering on photons and neutrons and have nothing directly to do with the isotopic character of the photons. Crossing symmetry, exchanging initial and final photons, implies

$$A_i(s,t,\bar{s})=\eta_iA_i(\bar{s},t,s)\,, \tag{2.7}$$

with $\eta_i=+1$, $i=1$, 2, 3, 6 and $\eta_i=-1$, $i=4$, 5.

With Hearn and Leader, we will first assume that the amplitudes $A_i(s,t,\bar{s})$ satisfy a Mandelstam representation of the form

$$\begin{aligned} A_i(s,t,\bar{s})={}&R_i\left(\frac{1}{s-m^2}+\frac{\eta_i}{\bar{s}-m^2}\right)+\frac{r_i}{t-m_\pi{}^2}+\frac{v_i}{t-m_\eta{}^2}+\frac{1}{\pi^2}\\ &\times\int_{s_0}^{\infty}ds'\int_{s_0}^{\infty}d\bar{s}'\frac{\chi_i(\bar{s}',s')}{(s'-s)(\bar{s}'-\bar{s})}+\frac{1}{\pi^2}\int_{s_0}^{\infty}ds'\\ &\times\int_{4m_\pi{}^2}^{\infty}dt'\frac{\rho_i(s',t')}{t'-t}\left(\frac{1}{s'-s}+\frac{\eta_i}{s'-\bar{s}}\right),\end{aligned} \tag{2.8}$$

where R_i, r_i, v_i are residues of the pole terms, $s_0=(m+m_\pi)^2$ is the threshold for photopion production and $\chi_i(\bar{s}',s')=\eta_i\chi_i(s',\bar{s}')$. A similar representation can be written for the amplitude $A_i(s,\cos\theta)=A_i(s,t,\bar{s})$ considered as a function of s and $\cos\theta$.[5] We will also be considering the function $A_i(s)=A_i(s,0,\bar{s})=A_i(s,1)$ which has cuts for $-\infty<s<-s_0+2m^2$ and $s_0<s<\infty$ and $A_i(s)=\eta_iA_i(\bar{s})$. This $A_i(s)$ for $i=1$, 2, 3, 6 has a representation of the form

$$\begin{aligned} A_i(s)={}&-\frac{r_i}{m_\pi{}^2}-\frac{v_i}{m_\eta{}^2}+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'\,\mathrm{Im}A_i(s')}{s'-s}\\ &+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'\,\mathrm{Im}A_i(s')}{s'-\bar{s}}\end{aligned} \tag{2.9}$$

so that as $s\to m^2$ at threshold

$$A_i(m^2)=-\frac{r_i}{m_\pi{}^2}-\frac{v_i}{m_\eta{}^2}+\frac{2}{\pi}\int_{s_0}^{\infty}\frac{ds\,\mathrm{Im}A_i(s)}{s-m^2}\,.$$

This equation is then the basis of our sum rules. As we will see the threshold values $A_i(m^2)$ are exactly specified by the low-energy theorem.

The pole terms in Eq. (2.8) correspond to single-nucleon exchange in the direct and crossed channel and π^0 and η exchange in the t channel (Fig. 2). The residues R_i depend only on the total charge and total-nucleon magnetic moments

$$\begin{aligned} &R_1=2mF_1{}^2\,,\quad R_2=0\,,\quad R_3=mF_1(F_1+2mF_2)\,,\\ &R_4=-F_1{}^2\,,\quad R_5=(F_1+2mF_2)^2\,,\\ &R_6=-F_1(F_1+2mF_2)\,,\end{aligned} \tag{2.10}$$

with

$$\begin{aligned} F_1&=\tfrac{1}{2}e(I+\tau_3)\,,\\ F_2&=[\tfrac{1}{2}(\kappa_p+\kappa_n)I+\tfrac{1}{2}(\kappa_p-\kappa_n)\tau_3]e/2m\,,\end{aligned} \tag{2.11}$$

where $\kappa_p=+1.79$, $\kappa_n=-1.91$, $e^2/4\pi=\alpha=1/137$.

The residues of the π^0 and η poles are just the product of the meson-nucleon coupling constant and the meson decay amplitude into two photons $F_\pi(-m_\pi{}^2)$ and $F_\eta(-m_\eta{}^2)$. We find that these are given by

$$\begin{aligned} r_3&=-\tfrac{1}{2}g_{\pi N}m_\pi{}^2F_\pi(-m_\pi{}^2)\tau_3\,,\\ v_3&=-\tfrac{1}{2}g_{\eta N}m_\eta{}^2F_\eta(-m_\eta{}^2)I\,,\end{aligned} \tag{2.12}$$

[7] There is a purely typographical error in Ref. 5 where the spin factors of A_6 are given by $(P_\mu'N_\nu-P_\nu'N_\mu)$.

$r_i=v_i=0$, $i\neq 3$. The function F we have introduced is identical to that used by Goldberger and Treiman in their examination of π^0 decay[8] and is expressed in terms of the lifetime of $\pi^0\to 2\gamma$ and $\eta\to 2\gamma$;

$$\begin{aligned} F_\pi^2(-m_\pi^2)&=64\pi/m_\pi^3\tau_\pi\,,\\ F_\eta^2(-m_\eta^2)&=64\pi/m_\eta^3\tau_\eta\,. \end{aligned}\tag{2.13}$$

We note that the π^0 and η pole contribute only to the isovector and isoscalar parts of one amplitude, $A_3(s,t,\bar{s})$.

Next we turn to the content of the low-energy theorem (let) for Compton scattering.[3] This imposes *exact* conditions on the Compton scattering as the photon frequency $p\to 0$ for fixed $\cos\theta$. Denoting

$$A_i^{\text{let}}=\lim_{p\to 0}A_i(s,t,\bar{s})\,,$$

the low-energy theorem expressed in terms of the invariant amplitudes is

$$\begin{aligned} &A_1^{\text{let}}=\frac{F_1^2}{m}(\cos\theta-1)\,,\quad A_2^{\text{let}}=4F_2(F_1+mF_2)\,,\\ &A_3^{\text{let}}=F_1\left(\frac{F_1}{2m}+F_2\right)(\cos\theta-1)-2F_2(F_1+mF_2)\,,\\ &A_4^{\text{let}}=-\frac{F_1^2}{pm}\,,\quad A_5^{\text{let}}=\frac{4m}{p}\left(\frac{F_1}{2m}+F_2\right)^2,\\ &A_6^{\text{let}}=2F_2^2-\frac{F_1}{m}\left(\frac{F_1}{2m}+F_2\right)(\cos\theta-1)\,. \end{aligned}\tag{2.14}$$

This is necessarily the exact behavior of the A_i as $p\to 0$. On the other hand, the pole terms appearing in Eq. (2.8) in the limit $p\to 0$ yield

$$\begin{aligned} &A_1^{\text{pole}}=\frac{F_1^2}{m}(\cos\theta-1)\,,\quad A_2^{\text{pole}}=0\,,\\ &A_3^{\text{pole}}=F_1\left(\frac{F_1}{2m}+F_2\right)(\cos\theta-1)\\ &\qquad\qquad+\tfrac{1}{2}g_{\pi N}F_\pi(-m_\pi^2)\tau_3+\tfrac{1}{2}g_{\eta N}F_\eta(-m_\eta^2)I\,, \qquad (2.15)\\ &A_4^{\text{pole}}=-\frac{F_1^2}{pm}\,,\quad A_5^{\text{pole}}=\frac{4m}{p}\left(\frac{F_1}{2m}+F_2\right)^2,\\ &A_6^{\text{pole}}=-\frac{F_1}{m}\left(\frac{F_1}{2m}+F_2\right)(\cos\theta-1)\,, \end{aligned}$$

so that for $i=2$, 3, 6 A_i^{pole} does not yield the correct threshold behavior. We now extract the physical content of this observation. Either one assumes that the mismatch between A_i^{let} and A_i^{pole} at threshold implies the need for a subtraction, as did Hearn and Leader, or one assumes no subtraction is required in which case there are consistency conditions or sum rules for Compton scattering and one can in principle calculate the discrepancy $A_i^{\text{let}}-A_i^{\text{pole}}$ from a knowledge of the Compton-scattering continuum.

We will adopt this latter alternative and assume no subtractions are required for the amplitudes $A_3(s,t,\bar{s})$, $A_6(s,t,\bar{s})$, $A_2(s,t,\bar{s})-A_1(s,t,\bar{s})$,[9] an assumption we examine in the next part of this article. Specializing to $\cos\theta=1$ and using Eqs. (2.9), (2.12)–(2.14), we obtain the following sum rules expressed in terms of $A_i(s)$:

$$-\tfrac{1}{2}g_{\pi N}F_\pi(-m_\pi^2)=\frac{e^2}{4m}(2\kappa_p+\kappa_p^2-\kappa_n^2)+\frac{2}{\pi}\int_{s_0}^\infty\frac{\text{Im}A_3^v(s)ds}{s-m^2}\,,\tag{2.16a}$$

$$-\tfrac{1}{2}g_{\eta N}F_\eta(-m_\eta^2)=\frac{e^2}{4m}(2\kappa_p+\kappa_p^2+\kappa_n^2)+\frac{2}{\pi}\int_{s_0}^\infty\frac{\text{Im}A_3^s(s)ds}{s-m^2}\,,\tag{2.16b}$$

$$\frac{e^2}{4m^2}(\kappa_p^2-\kappa_n^2)=\frac{2}{\pi}\int_{s_0}^\infty\frac{\text{Im}A_6^v(s)ds}{s-m^2}\,,\tag{2.17a}$$

$$\frac{e^2}{4m^2}(\kappa_p^2+\kappa_n^2)=\frac{2}{\pi}\int_{s_0}^\infty\frac{\text{Im}A_6^s(s)ds}{s-m^2}\,,\tag{2.17b}$$

$$\frac{e^2}{2m}(2\kappa_p+\kappa_p^2-\kappa_n^2)=\frac{2}{\pi}\int_{s_0}^\infty\frac{\text{Im}(A_2^v(s)-A_1^v(s))ds}{s-m^2}\,,\tag{2.18a}$$

$$\frac{e^2}{2m}(2\kappa_p+\kappa_p^2+\kappa_n^2)=\frac{2}{\pi}\int_{s_0}^\infty\frac{\text{Im}(A_2^s(s)-A_1^s(s))ds}{s-m^2}\,.\tag{2.18b}$$

Our exact result, Eq. (2.16a), for the $\pi^0\to 2\gamma$ lifetime, it turns out, is the essential content of the Goldberger-Treiman calculation,[8] although they begin from a different point of view and make many assumptions which are difficult to justify. The similarity of our calculation and the essential content of the Goldberger-Treiman calculation along with numerical disagreements between our results is elaborated in Appendix A. We also point out that if there are other particles with the

[8] M. L. Goldberger and S. B. Treiman, Nuovo Cimento **9**, 451 (1958).

[9] We consider the contribution A_2-A_1 instead of just A_2 because a possible 0^+ pole in the t channel contributes according to $A_2^p(s,t,\bar{s})=A_1^p(s,t,\bar{s})=c/(t-m^2)$ and not to the other amplitudes.

same quantum numbers as the π and η they will also contribute pole terms to A_3 and must be included in the calculation. The 2^+ mesons do not contribute to A_3 since only odd-parity mesons can contribute poles to this amplitude.

If we had used the Born terms instead of the pole terms in the Mandelstam representation, then since the Born terms taken as $p\to 0$ reproduce the low-energy theorem we would be obliged to use a subtracted form for the continuum contribution to $A_i(s, \cos\theta)$.[10] The no-subtraction hypothesis implies that $A_i\to 0$ as $p\to\infty$. In this limit the Born terms reproduce just the difference between the pole terms at $p\to 0$ and the A_i^{let},[11] and we again obtain the above sum rules. These sum rules do not depend on the technical differences between Born and pole terms but only on the no subtraction assumption.

In obtaining these conditions on Compton scattering we have not deviated essentially from the original philosophy of pion-pole dominance. We have simply observed that the pole terms do not necessarily reproduce the known low-energy behavior of the amplitude and hence if one assumes the amplitude does not require a subtraction there emerge conditions or sum rules which must be exactly satisfied by the Compton-scattering process. Then one can calculate the residues of the poles from these sum rules.

B. Subtractions

Now we turn to examining the crucial assumption of no subtractions on which our sum rules were derived. This issue is decided on the basis of the high-energy behavior of the invariant amplitudes. We consequently will utilize the Regge-pole model which it is hoped offers at least a phenomenological basis for understanding the behavior of the amplitudes for large s.[12]

First consider the sum rule Eq. (2.17) for κ_p^2 and κ_n^2. Examining the helicity decomposition in the t channel given in Hearn and Leader one can establish a Regge representation for $A_6(s,t,\bar{s})$. Denoting the leading Regge trajectory by $\alpha(t)$, one finds from this representation that as $s\to\infty$, $A_6(s,t,\bar{s})\to s^{\alpha(t)-1}$. Exercising special care in going to the forward direction $t=0$ because of the pecularities of the kinematics with mass-zero photons, one finds $A_6(s)\to s^{\alpha(0)-1}$. Since the Pomeranchuk pole does not contribute to A_6, we have $\alpha(0)<1$ and conclude the sum rule Eq. (2.17) is valid.

If we examine the sum rule, Eq. (2.18), for $2\kappa_p+\kappa_p^2 \pm\kappa_n^2$ in the same way we find that as $s\to\infty$, $A_2(s) -A_1(s)\to s^{\alpha(0)}$ so that the sum rule as written may not be valid.[13] Whether or not some other combination of invariants will exhibit convergent behavior using a Regge model is not known. We do point out as independent, but by no means conclusive, evidence that the amplitude $A_2(s)-A_1(s)$ requires a subtraction, the fact that if one keeps only the resonant $N^*(1238)$ contribution to the continuum which is expected to dominate, then $\text{Im}A_i^v(s)=0$; this in particular implies, from Eq. (2.18v), that $2\kappa_p+\kappa_p^2=\kappa_n^2$, in considerable disagreement with experiment. Of course, a large contribution from the high-energy region or pole-term contribution in the t channel would invalidate this observation. We will not subsequently consider the sum rule Eq. (2.18).

Finally we examine our sum rules, Eq. (2.16), for the meson lifetimes. To find the large-s behavior of $A_3(s,t,\bar{s})$, we assume, of course, that it is the exchange of the π^0 and η Regge poles in the t channel that dominate the amplitude for large s. From the helicity analysis in the t channel, one finds as $s\to\infty$, $A_3^v(s)\to s^{\alpha_\pi(0)}$, $A_3^s(s)\to s^{\alpha_\eta(0)}$, and since $\alpha_\pi(0)<0$, $\alpha_\eta(0)<0$, we conclude that $A_3(s)\to 0$. This behavior apparently contradicts the representation Eq. (2.9) which implies that $A_3(s)\to$ const as $s\to\infty$. However, since we have assumed in establishing the high-energy behavior that the π and η are Regge poles and not fixed poles, the poles appearing in the Mandelstam representation should be replaced by their Regge expressions and the appropriate crossing symmetric Regge representation of Khuri[14] should be used for the amplitude. Since the Regge-pole residue with its characteristic s dependence and the background integral vanish as $s\to\infty$, this implies for $t\to 0$ that $A_3(s)\to 0$. The Regge-pole expression and spectral function do not differ from the fixed pole and $\text{Im}A_3(s)$ in the sub-BeV region for which the sum rules are written, so we will use the representation Eq. (2.9) and not consider possible small high-energy corrections induced by Regge behavior.

To be more explicit, if we assume $A_3^v(\infty)=0$, then we have the representation

$$A_3^v(s)=\frac{1}{\pi}\int\frac{\text{Im}A_3^v(s')ds'}{s'-s}$$

and can directly exhibit the contribution of the Reggized pion to the absorptive part by writing $\text{Im}A_3^v(s) =\text{Im}R_\pi(s)+\text{Im}A_c(s)$, where $\text{Im}A_c(s)$ is the continuum contribution. Then we have

$$A_3^v(s)=R_\pi(s)+\frac{1}{\pi}\int\frac{\text{Im}A_c(s')ds'}{s'-s}$$

and note that for $s=m^2$ the Regge-pole expression $R_\pi(m^2)$ and the fixed-pole expression at $t=0$ do not markedly differ. Furthermore restricting our attention to the sub-BeV continuum contributions which are not

[10] See Ref. 5, Eq. (5.5).

[11] The t-channel Born terms behave like $t/(t-m_\pi^2)$ and, of course, do not contribute as $p\to 0$.

[12] The fermion Regge-pole analysis in the $\bar{s}$ channel has been carried out by V. G. Gorshkov, M. P. Rekalo, and G. V. Frolov, Zh. Eksperim. i Teor. Fiz **45**, 285 (1963) [English transl.: Soviet Phys.—JETP **18**, 199 (1964)]. For the t-channel Regge analysis see V. D. Mur, Zh. Eksperim. i Teor. Fiz. **44**, 2173 (1963) [English transl.: Soviet Phys.—JETP **17**, 1458 (1963)].

[13] The leading trajectories could be the $f_0(1250)$ or $f_1(1525)$ for which $1>\alpha(0)>0$.

[14] N. N. Khuri, Phys. Rev. **132**, 914 (1963).

modified by Regge behavior, we may approximate $\mathrm{Im}A_c \simeq \mathrm{Im}A_3$ which then gives us our sum rule. What we have essentially done is to split the contributions to $A_3^v(m^2)$ into a high-energy contribution from the pion-Regge pole and the low-energy continuum contributions. Alternatively one may reject the Regge model and simply assume the unsubtracted Mandelstam representation for $A_3^v(s,t)$ which would then directly supply the sum rule from Eq. (2.9).

In summary we conclude that the sum rules for τ_π, τ_η [Eq. (2.16)] and $\kappa_p^2 \pm \kappa_n^2$ [Eq. (2.17)] are consistent with Regge behavior, while the $2\kappa_p + \kappa_p^2 \pm \kappa_n^2$ sum rule Eq. (2.18) may not be. This analysis of the asymptotic behavior of the invariant amplitudes is consistent with the work of Mur,[12] who classified the t-channel Regge poles as P trajectories with vacuum quantum numbers $(-1)^J P = +1$, $(-1)^J C = +1$, Q trajectories with negative signature $(-1)^J P = -1$, $(-1)^J C = -1$, and S trajectories with negative parity $(-1)^J P = -1$, $(-1)^J C = +1$. Only Q trajectories contribute to $A_6(s)$ so this amplitude is asymptotically small and only S trajectories contribute to $A_3(s)$. It is clear that an even-parity meson like the $A_2(1230)$ will never show up as a Breit-Wigner resonance in the amplitude A_3, since, for physical $J = \alpha(m_{A_2}^2) = 2$, the residue function of A_3 must vanish because of parity conservation. If one assumed that the A_2 Regge trajectory with $\alpha_{A_2}(0) = +0.4$ could contribute to the asymptotic behavior of $A_3 \to \beta s^{\alpha_{A2}(0)}$ in the s channel (thus destroying the no-subtraction assumption for this amplitude), even though it never showed up as a t-channel resonance, the usual connection between direct-channel asymptotic behavior and crossed-channel poles would be lost removing much of the motivation for the Regge model.[15] If such a connection is lost, one could always imagine a trajectory $\alpha(t)$ with, let us say, $\alpha(0) = 0.3$ but which turned back before crossing $J = \frac{1}{2}$. Such a trajectory could dominate asymptotic behavior, but there is no way of identifying its parameters with a resonance. Admitting such trajectories would remove the usefulness of Regge phenomenology. We will thus assume that the A_2 trajectory will not contribute to the asymptotic behavior of $A_3(s)$. Lacking experimental evidence or any rigorous theory of Regge behavior, we can neither prove nor disprove this assumption. We do point out that if one assumes that the A_2 trajectory does not turn back but goes on, as apparently some of the nucleon trajectories appear to do, the vanishing of the residue function of A_3 at all odd-J values because of a signature factor, and at even-J values because of parity, is sufficient to rule out any contribution of the A_2 trajectory to our amplitude.

C. Forward Compton Scattering

It is instructive to examine the Compton-scattering amplitude in the forward direction which has the general form[16]

$$f = \chi_f^* [h_1(p)\boldsymbol{\varepsilon}_1^* \cdot \boldsymbol{\varepsilon}_2 + h_2(p) i\boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}_1^* \times \boldsymbol{\varepsilon}_2)] \chi_i. \quad (2.19)$$

Here p, as before, is the frequency of the photon. The functions $h_i(p)$ are analytic in the cut p plane and may be expressed in terms of the invariant amplitudes according to

$$4\pi h_1(p) = (2E)^{-1}\{m[A_2(s) - A_1(s)] \\ -\tfrac{1}{2}(s-m^2)[A_5(s) - A_4(s)]\}, \quad (2.20)$$

$$4\pi h_2(p) = -\frac{(s-m^2)}{2E} A_6(s), \quad E = (p^2+m^2)^{1/2}. \quad (2.21)$$

The low-energy theorem informs us that at threshold the $h_i(p)$ are expressed directly in terms of the static electromagnetic properties of the target particle, its charge, and anomalous magnetic moment:

$$\begin{aligned} 4\pi h_1(0) &= -F_1^2/m, \\ 4\pi h_2'(0) &= -2F_2^2. \end{aligned} \quad (2.22)$$

Furthermore the absorptive parts of $h_i(p)$ are simply related to physical cross sections, since we are dealing with elastic-forward scattering, and from the work of Drell and Hearn,[6] or from the helicity analysis in Hearn and Leader,[5] one has for $p > 0$

$$\begin{aligned} \mathrm{Im}h_1(p) &= \frac{p}{4\pi}\frac{\sigma_+(p)+\sigma_-(p)}{2} = \frac{p}{4\pi}\sigma_{\mathrm{tot}}(p), \\ \mathrm{Im}h_2(p) &= \frac{p}{4\pi}\frac{\sigma_-(p)-\sigma_+(p)}{2}. \end{aligned} \quad (2.23)$$

Here $\sigma_\pm(p)$ is the total cross section for circularly polarized photons with spin parallel and antiparallel to the target spin and in principle is directly open to experimental measurement.

From this we see immediately using Eq. (2.21) and (2.23) that the sum rule Eq. (2.17) is completely identical to that of Drell and Hearn,

$$\frac{e^2}{4m^2}(\kappa_p^2 \pm \kappa_n^2) = \frac{1}{\pi}\int_{m_\pi}^{\infty} \frac{dp}{p}[\sigma_+^{s,v}(p) - \sigma_-^{s,v}(p)], \quad (2.24)$$

when written in the more conventional form. Here $\sigma_\pm^p = \sigma_\pm^s + \sigma_\pm$, $\sigma_\pm^n = \sigma_\pm^s - \sigma_\pm^v$. This is the only forward-direction sum rule one can write for physical Compton scattering.[17] One cannot assume an unsubtracted dispersion relation for $h_1(p)$, otherwise there emerges the contradiction $-2\pi^2\alpha/m = \int dp\, \sigma_{\mathrm{tot}}(p) > 0$ which also

[15] The author would like to thank Professor S. Treiman for discussions on this point.

[16] M. Gell-Mann, M. L. Goldberger, and W. Thirring, Phys. Rev. **95**, 1612 (1954).

[17] For the non-Abelian Compton effect there are additional sum rules, M. A. B. Bég, Phys. Rev. Letters **17**, 333 (1966). Bég also obtains the sum rule for $\kappa_p^2 + \kappa_n^2$ as one of his sum rules.

serves to inform us that at least one dispersion relation for the $A_i(s)$ must have a subtraction; the combination given by Eq. (2.20). One can, however, incorporate information from the low-energy theorem for nonforward directions, and this is the origin of the sum rules for the π^0 and η radiative decays. Here the continuum contributions are no longer simply related to directly measureable total cross sections, but this in no way prevents our estimating the continuum contribution, or relating it to experimental quantities, as we do in the next section.

We also remark, as is well known, that the low-energy theorem provides one with an experimental definition of the total physical charge of a particle. For forward scattering it also provides an unambiguous definition of the anomalous magnetic moment of a particle and a sum rule for this quantity. Not only is this the case for spin $\frac{1}{2}$, but also evidently for higher-spin systems as well. For example, for a spin-1 system one can show that

$$\frac{\pi^2\alpha(1-\kappa)^2}{m^2}=\int_0^\infty\frac{dp}{p}[\sigma_+(p)-\sigma_-(p)],\qquad(2.25)$$

where the total magnetic moment of the spin-1 particle is $\mu=(1+\kappa)e/2m$ and $\sigma_\pm(p)$ are the total cross sections for photon-helicity parallel and antiparallel to the target spin. The sum rules for the magnetic moments of spin-$\frac{1}{2}$ and spin-1 particles have the immediate physical significance that in the absence of strong interactions one recovers from the Drell-Hearn sum rule the fact that the normal Dirac moment of a spin-$\frac{1}{2}$ particle is $\mu=e/2m$, while from Eq. (2.25) we have that the normal moment of a spin-1 particle is $\mu=e/m$. In Appendix B we derive and discuss the spin-1 sum rule, Eq. (2.25), further.

III. CONTINUUM CONTRIBUTIONS TO THE SUM RULES

In writing our sum rules, we have dispersed in the s channel rather than the t channel. This has the advantage that to a first approximation retaining only the intermediate pion-nucleon state, the absorptive parts $\mathrm{Im}A_i(s)$ are directly related to photopion production amplitudes. Keeping only the $J=\frac{1}{2}$ and $J=\frac{3}{2}$, partial waves should suffice for the low-energy continuum contribution. We will consider separately the continuum contributions to the isovector and isoscalar parts of the sum rules.

A. Isovector Continuum Contributions

The isovector sum rules for the π^0 lifetime and $\kappa_p{}^2-\kappa_n{}^2$ are

$$\frac{4g_{\pi N}}{m_\pi{}^2}\left(\frac{\pi m_\pi}{\tau_\pi}\right)^{1/2}=\frac{e^2}{4m}(2\kappa_p+\kappa_p{}^2-\kappa_n{}^2)+\frac{2}{\pi}\int_{(m+m_\pi)^2}^\infty\frac{\mathrm{Im}A_3{}^v(s)ds}{s-m^2},\qquad(3.1)$$

$$\frac{e^2}{4m^2}(\kappa_p{}^2-\kappa_n{}^2)=\frac{2}{\pi}\int_{(m+m_\pi)^2}^\infty\frac{\mathrm{Im}A_6{}^v(s)ds}{s-m^2}.\qquad(3.2)$$

Retaining just $J=\frac{1}{2}$, $\frac{3}{2}$ partial waves the absorptive parts in the low-energy region are expressed in terms of the CGLN multipoles[18] $M_{l\pm}{}^{(i)}$, $E_{l\pm}{}^{(i)}$ according to[19]

$$\begin{aligned}\mathrm{Im}A_3{}^v(s)=4\pi W\left(\frac{q}{p}\right)\mathrm{Re}[&2(M_{1-}{}^{(1/2)}M_{1-}{}^{(0)*}-M_{1+}{}^{(1/2)}M_{1+}{}^{(0)*}-E_{0+}{}^{(1/2)}E_{0+}{}^{(0)*}\\&+E_{2-}{}^{(1/2)}E_{2-}{}^{(0)*})-6(M_{1+}{}^{(1/2)}E_{1+}{}^{(0)*}+E_{1+}{}^{(1/2)}M_{1+}{}^{(0)*}+E_{2-}{}^{(1/2)}M_{2-}{}^{(0)*}\\&+M_{2-}{}^{(1/2)}E_{2-}{}^{(0)*}+3E_{1+}{}^{(1/2)}E_{1+}{}^{(0)*}-3M_{2-}{}^{(1/2)}M_{2-}{}^{(0)*})],\qquad(3.3)\end{aligned}$$

$$\begin{aligned}\mathrm{Im}A_6{}^v(s)=4\pi\left(\frac{q}{p}\right)\mathrm{Re}[&2(-M_{1-}{}^{(1/2)}M_{1-}{}^{(0)*}+M_{1+}{}^{(1/2)}M_{1+}{}^{(0)*}-E_{0+}{}^{(1/2)}E_{0+}{}^{(0)*}\\&+E_{2-}{}^{(1/2)}E_{2-}{}^{(0)*})-6(M_{1+}{}^{(1/2)}E_{1+}{}^{(0)*}+E_{1+}{}^{(1/2)}M_{1+}{}^{(0)*}-E_{2-}{}^{(1/2)}M_{2-}{}^{(0)*}\\&-M_{2-}{}^{(1/2)}E_{2-}{}^{(0)*}+E_{1+}{}^{(1/2)}E_{1+}{}^{(0)*}+M_{2-}{}^{(1/2)}M_{2-}{}^{(0)*})].\qquad(3.4)\end{aligned}$$

Here q is the momentum of the pion in the barycentric system for photopion production.

First we note that large resonant $M_{1+}{}^{(3/2)}$, $E_{1+}{}^{(3/2)}$ transitions corresponding to the $N^*(1238)$ do not contribute at all to the isovector sum rules. Thus if we retain *only* the N^* contribution we have from Eq. (3.2),

$$\kappa_p{}^2=\kappa_n{}^2.\qquad(3.5)$$

Incorporating this result with Eq. (3.1), the rate for $\pi^0\to2\gamma$ is

$$\frac{1}{\tau_{\pi^0}}=\left(\frac{m_\pi}{m}\right)^2\left(\frac{\alpha^2}{g_{\pi N}{}^2/4\pi}\right)\frac{\kappa_p{}^2m_\pi}{16}\simeq3.1\text{ eV}\qquad(3.6)$$

[18] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[19] There are some typographical errors in Ref. 5. In Eq. (4.17) the first two terms should read $+2\psi_{\lambda\lambda N}{}^{J(-)}\psi_{\lambda'\lambda N}{}^{J(0)*}+2\psi_{\lambda\lambda N}{}^{J(0)}\psi_{\lambda'\lambda N}{}^{J(-)*}$. In Eq. (4.18) in the last equation one should replace $\psi_{-3/2,-1/2}{}^J$ with $-\psi_{-3/2,-1/2}{}^J$. We also note that parity conservation implies that the photopion partial helicity amplitudes satisfy $\psi_{\lambda\lambda'}{}^J=-\psi_{-\lambda-\lambda'}{}^J$. We would like to thank Professor A. C. Hearn for pointing these facts out.

or

$$\tau_{\pi^0} \simeq 2.2\times 10^{-16}\ \text{sec},$$

the result quoted in the Introduction. Had we not used Eq. (3.5) and neglected the continuum, then

$$\tau_{\pi^0}{}^{-1} = \left(\frac{m_\pi}{m}\right)^2 \frac{m_\pi}{64} \frac{\alpha^2}{(g_{\pi N}{}^2/4\pi)} (2\kappa_p + \kappa_p{}^2 - \kappa_n{}^2)^2 \simeq 2.3\ \text{eV}$$

or $\tau_{\pi^0} \simeq 2.9\times 10^{-16}$ sec. Experimentally the result for the anomalous moments Eq. (3.5) is valid to 14% which is a measure of the error in the *amplitudes* in this approximation and does not differ markedly from the error in the amplitude in the Goldberger-Treiman result for charged-pion decay.

There is considerable scatter in the experimental values for the $\pi^0 \to 2\gamma$ mean lifetime[20] with a typical quoted value of 2.2×10^{-16} sec obtained using emulsion techniques to the minimum value of 0.74×10^{-16} sec[21] obtained using the Primakoff effect. An approximate experimental value is given by $(1.0\pm 0.5)\times 10^{-16}$ sec.[22] A 20% contribution from the continuum in the right direction could bring the calculated value Eq. (3.6) into better agreement with experiment. However, we emphasize that the experimental value must await future experiments for more precise resolution.

It is difficult to present reliable estimates of the small continuum contribution to the isovector sum rule. We can expect it to be small because in the low-energy region the multipoles $M_{l\pm}{}^{(0)}$ and $E_{l\pm}{}^{(0)}$ are proportional to the small isoscalar anomalous moment $\kappa_p + \kappa_n$. The $N^{**}(1520)$ dominates the contribution to the $M_{2-}{}^{(1/2)}$ and $E_{2-}{}^{(1/2)}$ multipoles which enter the continuum contribution multiplied by the small and difficult to estimate $M_{2-}{}^{(0)}$ and $E_{2-}{}^{(0)}$ transition multipoles. The dominant nonresonant s-wave threshold contribution from $E_{0+}{}^{(i)}$ transitions enters $\mathrm{Im}A_6{}^v$ and $\mathrm{Im}A_3{}^v$ with the same sign, and if taken alone, improves the prediction Eq. (3.5) while increasing our estimate of the π^0 lifetime. Because the isovector continuum is so small, it is difficult at this time to give a reliable estimate, and we must await more refined experimental and theoretical determinations of the small photopion multipoles before even the sign of these contributions can be given with some certainty.

B. Isoscalar Continuum Contributions

Next we consider the isoscalar sum rules for $\eta \to 2\gamma$ and $\kappa_p{}^2 + \kappa_n{}^2$,

$$\frac{4g_{\eta N}}{m_\eta{}^2}\left(\frac{\pi m_\eta}{\tau_\eta}\right)^{1/2} = \frac{e^2}{4m}(2\kappa_p + \kappa_p{}^2 + \kappa_n{}^2) + \frac{2}{\pi}\int_{(m+m_\pi)^2}^{\infty} \frac{\mathrm{Im}A_3{}^s(s)ds}{s-m^2}, \quad (3.7)$$

$$\frac{e^2}{4m^2}(\kappa_p{}^2 + \kappa_n{}^2) = \frac{2}{\pi}\int_{(m+m_\pi)^2}^{\infty} \frac{\mathrm{Im}A_6{}^s(s)ds}{s-m^2}. \quad (3.8)$$

The absorptive parts in terms of the CGLN photopion multipoles are given by

$$\mathrm{Im}A_3{}^s(s) = 4\pi W\left(\frac{q}{p}\right)\{\tfrac{2}{3}[-|E_{0+}{}^{(3/2)}|^2 + |M_{1-}{}^{(3/2)}|^2 - |M_{1+}{}^{(3/2)}|^2 + |E_{2-}{}^{(3/2)}|^2 - 9|E_{1+}{}^{(3/2)}|^2 + 9|M_{2-}{}^{(3/2)}|^2$$
$$-6\,\mathrm{Re}(M_{1+}{}^{(3/2)}E_{1+}{}^{(3/2)*} + E_{2-}{}^{(3/2)}M_{2-}{}^{(3/2)*})] + \tfrac{1}{3}[(\tfrac{3}{2})\to(\tfrac{1}{2})] + 3[(\tfrac{3}{2})\to(0)]\}, \quad (3.9)$$

$$\mathrm{Im}A_6{}^s(s) = 4\pi\left(\frac{q}{p}\right)\{\tfrac{2}{3}[-|E_{0+}{}^{(3/2)}|^2 - |M_{1-}{}^{(3/2)}|^2 + |M_{1+}{}^{(3/2)}|^2 + |E_{2-}{}^{(3/2)}|^2 - 3|E_{1+}{}^{(3/2)}|^2 - 3|M_{2-}{}^{(3/2)}|^2$$
$$-6\,\mathrm{Re}(M_{1+}{}^{(3/2)}E_{1+}{}^{(3/2)*} - E_{2-}{}^{(3/2)}M_{2-}{}^{(3/2)*})] + \tfrac{1}{3}[(\tfrac{3}{2})\to(\tfrac{1}{2})] + 3[(\tfrac{3}{2})\to(0)]\}. \quad (3.10)$$

We see that the large magnetic-dipole transition $M_{1+}{}^{(3/2)}$ enters the sum rules for the η lifetime and the Drell-Hearn sum rule with opposite sign. Denoting the continuum contribution by

$$C_{3,6}{}^s = \frac{2}{\pi}\int_{(m+m_\pi)^2}^{\infty} \frac{\mathrm{Im}A_{3,6}{}^s(s)ds}{s-m^2}, \quad (3.11)$$

we estimate its contribution to the sum rules retaining just the large N^* and N^{**} contributions from the multipoles $M_{1+}{}^{(3/2)}E_{1+}{}^{(3/2)}$ and $M_{2-}{}^{(1/2)}E_{2-}{}^{(1/2)}$ and the nonresonant S-wave contribution $E_{0+}{}^{(i)}$ up to 47 MeV above threshold. Our results are given in Table I. We have used Walker's parameterized form for the N^* and N^{**} photopion multipoles[23] and the CGLN expressions

TABLE I. Continuum contribution to isoscalar sum rules.

Multipoles	$m^2C_6{}^s$	$mC_3{}^s$
$N^*\ M_{1+}{}^{(3/2)}$	0.146	−0.190
$N^*\ M_{1+}{}^{(3/2)}E_{1+}{}^{(3/2)}(a)$	0.206	−0.117
$N^{**}\ M_{2-}{}^{(1/2)}E_{2-}{}^{(1/2)}(b)$	0.002	0.000
Threshold < 47 MeV. $E_{0+}{}^{(3/2)}E_{0+}{}^{(3/2)}(c)$	−0.051	−0.067
Total = $(a)+(b)+(c)$	+0.157	−0.184

[20] R. G. Glasser, N. Seeman, and B. Stiller, Phys. Rev. **123**, 1014 (1961); J. Tietge and W. Püschel, *ibid.* **127**, 1324 (1962); H. Shwe, F. M. Smith, and W. H. Barkas, *ibid.* **136**, B1839 (1961); **125**, 1024 (1962); D. A. Evans, *ibid.* **139**, B982 (1965).

[21] G. Belletini, C. Bemponad, P. L. Braccini, and L. Foa, Phys. Letters **22**, 333 (1966); Nuovo Cimento **40**, 1139 (1965).

[22] The author would like to thank Professor S. Taylor of Stevens Institute for helpful discussions on the experimental status of the π^0 decay. See, also, P. Stamer, S. Taylor, E. L. Koller, T. Huetter, J. Grauman, and D. Pardoulas, Phys. Rev. **151**, 1108 (1966).

[23] Quoted in S. L. Adler and F. J. Gilman, Phys. Rev. **152**, 1460 (1966).

for the threshold multipoles[18] which can be expected to be valid in the low-energy region.

The left-hand side of the isoscalar Drell-Hearn sum rule [Eq. (3.8)] is $(e^2/4m^2)(\kappa_p^2+\kappa_n^2)=0.156/m^2$, and we find from Table I the continuum contribution $C_6^s=0.157/m^2$. Unless there is a conspiracy of contributions from the nonresonant background or a large contribution for the high-energy region >1 BeV, we conclude that the isoscalar Drell-Hearn sum rule is well satisfied, in agreement with earlier calculations.[6]

For the $\eta\rightarrow 2\gamma$ lifetime there is a large cancellation between the continuum contribution which we obtain from Table I and the nucleon terms,

$$\frac{4g_{\eta N}}{m_\eta^2}\left(\frac{\pi m_\eta}{\tau_\eta}\right)^{1/2}=[0.240-0.184]/m$$

or

$$\frac{1}{\tau_\eta}=\left(\frac{g_{\pi N}}{g_{\eta N}}\right)^2\times 70\text{ eV} \tag{3.12}$$

expressed in terms of the ratio of experimentally determinable coupling constants. For the lifetime one has

$$\tau_\eta=(g_{\eta N}/g_{\pi N})^2\times 9.4\times 10^{-18}\text{ sec}. \tag{3.13}$$

Since the coupling of the η to nucleons is not known experimentally we appeal to $SU(3)$ symmetry which predicts $g_{\eta N}=-(1/\sqrt{3})(1-4f)g_{\pi N}$ where f is the $F/(F+D)$ ratio.[24] For $F/D=\frac{2}{3}$, $\tau_\eta=1.1\times 10^{-18}$ sec, and for $F/D=\frac{1}{2}$, $\tau_\eta=0.35\times 10^{-18}$ sec, so the lifetime depends sensitively on the F/D ratio when expressed in this way.

We also comment that since the continuum contribution is large and negative the $\eta\rightarrow 2\gamma$ lifetime Eq. (3.13) depends on the cancellation of the nucleon terms and the continuum, and therefore is sensitive to nonresonant background terms in the continuum which are difficult to estimate at this time. This sensitivity to the isoscalar background continuum does not appear so dramatically in the isoscalar Drell-Hearn sum rule since one is not computing a small difference between large terms and for this reason it can be expected to be reliability tested. There are thus two features of the $\eta\rightarrow 2\gamma$ lifetime sum rule which prevent us from giving a reliable number. First the η-nucleon coupling must be known and secondly the nonresonant as well as the resonant multipoles must be known with some precision because of the cancellation feature discussed above. If such data becomes available in the near future, we can then give a reliable estimate for the $\eta\rightarrow 2\gamma$ lifetime and then from the known branching ratio offer a prediction of the total lifetime which has as yet not been measured experimentally.

[24] It is dangerous to take $SU(3)$ predictions for meson-baryon coupling constants. See for example M. Losiguoli, M. Restignoli, G. Snow, and G. Violini, Phys. Letters 21, 229 (1966). From their work one sees that $g_{p\Lambda K}^2/4\pi=4.8\pm 1.0$ and the bound $g_{p\Sigma^0 K}^2/4\pi<3.2$ are not consistent with $SU(3)$.

IV. CONCLUSIONS

We have obtained sum rules satisfied by Compton scattering which relate the $\pi^0\rightarrow 2\gamma$ and $\eta\rightarrow 2\gamma$ lifetimes to nucleon anomalous moments and the Compton continuum. There also emerges in the same way the Drell-Hearn sum rule for $\kappa_p^2\pm\kappa_n^2$. To a first rough approximation, we can estimate the π^0 and η lifetime; to do better than this requires more refined data for the nonresonant photopion multipoles and knowledge of the coupling $g_{\eta N}$.

From the point of view of exact $SU(3)$ symmetry the $\eta\rightarrow 2\gamma$ and $\pi^0\rightarrow 2\gamma$ amplitudes have the ratio $1/\sqrt{3}$.[25] This result does not emerge in a simple way from our exact sum rules which extract information about the three-point function or vertex by studying the four-point function. Of course, starting from the vertex and calculating in any $SU(3)$ symmetric way, one will obtain the $SU(3)$ symmetric result for the ratio of the matrix elements; however, it is not obvious how this result arises from the sum rules on the scattering amplitude. If one generalizes the Drell-Hearn sum rule to the baryon **8** in exact $SU(3)$ and saturates with the decuplet, then, as is known, there emerges an overdetermined system of equations the only consistent solution to which is that all the anomalous moments must vanish. Our investigation has nothing new to add to this result except that if one neglects the continuum in the π^0 and η sum rules and considers Compton scattering from the baryon **8** in exact $SU(3)$, then the only solution is that all anomalous moments must vanish.

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APPENDIX A: GOLDBERGER-TREIMAN CALCULATION OF $\pi^0\rightarrow 2\gamma$

Goldberger and Treiman have examined the $\pi^0\rightarrow 2\gamma$ decay amplitude[8] described by the analytic function $F((k+q)^2)$ (in their notation). Assuming that the $\pi^0\rightarrow 2\gamma$ vertex is saturated by the $\bar{N}N$ intermediate state enables them to relate $\mathrm{Im}F$ to the pion-nucleon vertex described by $K[(\bar{N}+N)^2]$ and the $\bar{N}+N\rightarrow 2\gamma$ amplitude Q so that symbolically, $\mathrm{Im}F\sim\int \mathrm{Re}K\,\mathrm{Re}Q$. The $\bar{N}+N\rightarrow 2\gamma$ amplitude $\mathrm{Re}Q$, with the neglect of continuum, is found to contain the nucleon-pole terms, the pion pole which contains $g_{\pi N}F$ as its residue and a subtraction term which is estimated from perturbation theory in terms of the nucleon anomalous moments. Assuming F requires no subtractions the dispersion relation then establishes an inhomogenious linear equation for F which is solved [Eq. (28) of Ref. 8] in terms

[25] S. Okubo and B. Sakita, Phys. Rev. Letters 11, 50 (1963).

of a divergent integral I_1 [if $K((\bar{N}+N)^2)=1$]. In the limit $I_1\to\infty$, F remains finite and they obtain their expression for the lifetime.

However this is easily seen to be simply a consequence of the observation that $\mathrm{Im}F\to 0$ in the high-energy limit implies $(\bar{N}+N)^2\times\mathrm{Re}Q\to 0$ in this limit which imposes a condition on F, just their final result. Hence the essential conclusion and content of their calculation is that an appropriate projection of the Compton amplitude, related by crossing to $\bar{N}+N\to 2\gamma$, vanishes in the high-energy limit. In the limit $(\bar{N}+N)^2\to\infty$ the quantity $(\bar{N}+N)^2\,\mathrm{Re}Q$ which must vanish is just the difference between the pion-pole residue and the subtraction constant, the nucleon pole terms having vanished in this limit.

Goldberger and Treiman find for the quantity $(\bar{N}+N)^2\,\mathrm{Re}Q$ which is the appropriate projection of the Compton amplitude, here expressed in terms of the Mandelstam variables,

$$(\bar{N}+N)^2\,\mathrm{Re}Q=+\frac{e^2}{m}[2\kappa_p-(\kappa_p{}^2-\kappa_n{}^2)]-e^2m(1+\kappa_p)$$
$$\times\left(\frac{1}{s-m^2}+\frac{1}{\bar{s}-m^2}\right)+\frac{tg_{\pi N}F}{t-m_\pi{}^2},\quad (\mathrm{A1})$$

whereas the Born term for the amplitude $A_3{}^v$ in our calculation is given by

$$-2A_3{}^{v\,\mathrm{Born}}=+\frac{e^2}{2m}[2\kappa_p+(\kappa_p{}^2-\kappa_n{}^2)]-e^2m(1+\kappa_p)$$
$$\times\left(\frac{1}{s-m^2}+\frac{1}{\bar{s}-m^2}\right)+\frac{tg_{\pi N}F}{t-m_\pi{}^2}\quad (\mathrm{A2})$$

and the subtraction term in Eq. (A2) is regulated by the low-energy theorem. We find however from direct calculation that $\mathrm{Re}Q$ is not given by Eq. (A1) but instead by

$$(\bar{N}+N)^2\,\mathrm{Re}Q=+\frac{e^2}{2m}[2\kappa_p+(\kappa_p{}^2-\kappa_n{}^2)]-e^2m(1+\kappa_p)$$
$$\times\left(\frac{1}{s-m^2}+\frac{1}{\bar{s}-m^2}\right)+\frac{tg_{\pi N}F}{t-m_\pi{}^2},\quad (\mathrm{A3})$$

which differs from Eq. (A1) by the sign of $\kappa_p{}^2-\kappa_n{}^2$ and a crucial factor of 2 in the subtraction term which shows up as a factor of 4 difference in the Goldberger-Treiman result for the π^0 lifetime and our result.

As remarked previously, the Born terms in the limit $p\to\infty$ reproduce just the difference between A^{let} and A^{pole} at threshold. The assumption of no subtraction for A_3 and neglect of the continuum then implies the Born term vanishes in this limit and as is seen from Eq. (A2) gives us our result for the π^0 lifetime. This is identical to the essential content of the Goldberger-Treiman calculation which implies $(\bar{N}+N)^2\,\mathrm{Re}Q$ vanishes in this same limit i.e., the no-subtraction assumption for $A_3{}^v$. From the point of view advocated in this paper we can see why the Goldberger-Treiman calculation works.

Using the relation between the pion-pole-term residue and the subtraction and substituting this result into the Goldberger-Treiman expression for the absorptive part of the vertex $\mathrm{Im}F$ one then will encounter no divergent integrals and can compute the function $F(-m_\pi{}^2)$ from $\mathrm{Im}F$. Neglecting terms proportional to the small pion mass, one obtains from the dispersion integral $F=(1+\kappa_p)F_{\mathrm{pert}}$, where F_{pert} is the perturbation-theory result. The F obtained in this way is an order of magnitude larger than the F obtained assuming the appropriate projection of the Compton amplitude must vanish at high energy. This simply reflects what a bad approximation saturation of the $\pi^0\to 2\gamma$ vertex with an $\bar{N}N$ state or any baryon-antibaryon state can be.

We conclude from this investigation that the essential content of the Goldberger-Treiman calculation is that an appropriate projection of the Compton amplitude must vanish in the high-energy limit, the same as our no-subtraction hypothesis for $A_3{}^v(s)$. Although the physical assumption of $\bar{N}N$ saturation of the $\pi^0\to 2\gamma$ vertex cannot be justified, their final result is technically the same as ours for reasons given above. Also we believe that there is a crucial numerical error and a not-so-crucial sign error in the Goldberger-Treiman calculation which raises their value of the lifetime by a factor of 4.

APPENDIX B: FORWARD COMPTON SCATTERING FROM SPIN-1 SYSTEMS

Here we examine forward Compton scattering from a spin-1 charged particle of charge e, magnetic moment $\mu=(1+\kappa)e/2m$ and quadrupole moment Q. If $\boldsymbol{\varepsilon}_{i,f}$ is the polarization of the initial and final photons and $\boldsymbol{\lambda}_{i,f}$ is the polarization of the spin$=1$ particle in the initial and final state, then the general form for the forward-scattering amplitude is

$$\begin{aligned}f=&f_1(p)\boldsymbol{\varepsilon}_f{}^*\cdot\boldsymbol{\varepsilon}_i\boldsymbol{\lambda}_f\cdot\boldsymbol{\lambda}_i+f_2(p)(\boldsymbol{\varepsilon}_f{}^*\times\boldsymbol{\varepsilon}_i)\cdot(\boldsymbol{\lambda}_f{}^*\times\boldsymbol{\lambda}_i)\\&+f_3(p)(\boldsymbol{\varepsilon}_f{}^*\cdot\boldsymbol{\lambda}_f{}^*\boldsymbol{\varepsilon}_i\cdot\boldsymbol{\lambda}_i+\boldsymbol{\varepsilon}_i\cdot\boldsymbol{\lambda}_f{}^*\boldsymbol{\varepsilon}_f{}^*\cdot\boldsymbol{\lambda}_i)\\&+f_4(p)(\boldsymbol{\varepsilon}_f{}^*\cdot\boldsymbol{\varepsilon}_i)(\boldsymbol{\lambda}_f{}^*\times\hat{p})\cdot(\boldsymbol{\lambda}_i\times\hat{p}),\end{aligned}\quad (\mathrm{B1})$$

where $\hat{p}$ is a unit vector in the direction of the incident photon and p is the frequency of the photon. One may define the coherent amplitudes for scattering with circularly polarized light with spin parallel f_+ and antiparallel f_- to the spin of the target

$$f_+(p)=f_1(p)-f_2(p)+f_3(p)+f_4(p),$$
$$f_-(p)=f_1(p)+f_2(p)+f_3(p)+f_4(p),$$

so that

$$f_2(p)=\tfrac{1}{2}[f_-(p)-f_+(p)].\quad (\mathrm{B2})$$

Crossing symmetry implies $f_2(p)=-f_2^*(-p)$. Since $f_\pm(p)$ are coherent amplitudes, the optical theorem implies

$$\mathrm{Im} f_2(p)=(p/4\pi)\tfrac{1}{2}[\sigma_-(p)-\sigma_+(p)]. \qquad \text{(B3)}$$

Next we must establish a low-energy theorem for the forward amplitude. Utilizing the methods of Low,[26] Adler and Dothan,[27] and the gauge-invariant electromagnetic vertex for a spin=1 charged particle of Lee and Yang,[28] one can rigorously establish a low-energy theorem for spin-1 particles. Even for a bound-state system no anomalous singularities from "box" diagrams arise with the photons on their mass shell,[29] and one finds

$$4\pi \underset{p\to 0}{f} = -\frac{e^2}{m}\boldsymbol{\varepsilon}_f^*\cdot\boldsymbol{\varepsilon}_i\boldsymbol{\lambda}_f^*\cdot\boldsymbol{\lambda}_i-\frac{e^2p}{4m^2}(1-\kappa)^2 \times(\boldsymbol{\varepsilon}_f^*\times\boldsymbol{\varepsilon}_i)\cdot(\boldsymbol{\lambda}_f^*\times\boldsymbol{\lambda}_i)+O(p^2). \qquad \text{(B4)}$$

Radiation from the quadrupole moment contributes only to $O(p^2)$. Thus we have $4\pi f_1(0)=-e^2/m$, the Thomson limit, and $4\pi f_2'(0)=-(e^2/4m^2)(1-\kappa)^2$.[30]

Assuming $f_2(p)/p\to 0$ as $p\to\infty$ and using the crossing relation and the optical theorem Eq. (B3), one then obtains the exact sum rule

$$\frac{\alpha\pi^2(1-\kappa)^2}{m^2}=\int_0^\infty\frac{dp}{p}[\sigma_+(p)-\sigma_-(p)], \qquad \text{(B5)}$$

which has also been independently obtained by Hosoda and Yamamoto.[31] This would seem to imply that the normal moment for a spin-1 system is $\mu=e/m$.

An obvious application for both the spin-$\frac{1}{2}$ and spin-1 sum rules are nuclear systems with these spins. If one applies Eq. (B5) to the deuteron for which $1-\kappa_d=0.287$, the threshold of the integral corresponding to the deuteron-binding energy, one expects the dominant contribution to be low-energy photodisintegration into np. There is a low-energy theorem for the amplitude for photodisintegration of the deuteron[32] to which one can appeal. Here one finds that in Born approximation the $E1$ transitions do not contribute at all to $\sigma_+-\sigma_-$ and the dominant $M1$ transitions contribute according to $\sigma_+-\sigma_-=-3\sigma_{\mathrm{tot}}|_{M1}$, where $\sigma_{\mathrm{tot}}|_{M1}$ is the total photodisintegration cross section for $M1$ transitions. The sum rule is consequently badly violated for the deuteron from the dominant low-energy contribution. From this point of view moreover it is difficult to see how even the simple additivity of the np magnetic moments produce the deuteron moment as they must exactly with the neglect of the small D-wave probability.

We conjecture that the amplitude $f_2(p)/p$ requires at least one subtraction for weakly bound composite systems, and we must subtract out the free-particle contribution to the amplitude in order to obtain a meaningful sum rule.

Irrespective of the issue of subtractions, the low-energy theorem Eq. (B4) provides an unambiguous definition of the anomalous moment and can be experimentally tested using the techniques of low-energy nuclear physics. Unfortunately the large Rayleigh scattering $\sim p^2$ makes this extremely difficult, obscuring the scattering proportional to $\sim p$.

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$$4\pi^2\alpha\langle s_3\rangle\left(\frac{\mu}{s}-\frac{Z}{m}\right)^2=\int_0^\infty\frac{dp}{p}[\sigma^+(p)-\sigma^-(p)]$$

obtains where μ=total magnetic moment/e, s=total spin of the system, Z=atomic number, $\langle s_3\rangle$=expectation value of the third component of the target spin. The author would like to thank Professor Hosoda and Professor Yamamoto for helpful discussions.

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