Ward Identities, Unsubtracted Dispersion Relations, and Feynman Graphs*

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We study the interplay of commutators involving a conserved current, Feynman graphs, and dispersion relations in an attempt to formulate general rules for when the invariants involved in the decomposition of matrix elements of retarded products of currents obey unsubtracted dispersion relations. Our main conclusions are two: (1) The assumption of unsubtracted dispersion relations is incorrect when one of the currents is conserved and the other is the source of a strongly interacting particle on its mass shell. (2) When one current is conserved and the other is the source of a system which interacts only weakly or electromagnetically, the assumption may be correct to lowest order in the nonstrong interactions. These relations are, however, on a much different and less firm footing than the sum rules using the hypothesis of partially conserved axial-vector current (like that of Adler and Weisberger). In particular, assuming the unsubtracted dispersion relations, we find that the result (e.g., the sum rules of Cabibbo and Radicati) has a structure such that, *to all orders in any field theory*, the $N^*(\omega)$ intermediate-state graphs project to zero, that is, they fail to contribute to the charge radius of the nucleon (pion), whereas, calculated dispersively, they do contribute. We also show that the real part of amplitudes of type (2) has a fixed power behavior in energy.

I. INTRODUCTION

It is evident by now that the sum rules following from current commutation relations[1] are very close in spirit to the early Ward[2] and Ward-Takahashi[3] identities.[4–6] That is to say, one studies the divergences of retarded or time-ordered products of currents whose divergences, and whose equal-time commutators with other structures in the matrix element, are known. In the case of partially conserved currents, of course, one has extra (known) terms beyond those in the original Ward identities.

It is well known that the Ward identities themselves are in general powerless to give more than low-energy theorems.[7] The reason for this is that, in general, a Ward identity relates an $(n+1)$-point function involving a *timelike vector meson* to an (n)-point function without that meson. Because timelike vector-meson amplitudes are experimentally inaccessible, it is essentially impossible to feed experimental data in at the $(n+1)$-point function level to calculate the (n)-point function. On the other hand, near zero vector-meson four-momentum, there are sufficient analytic relations between the physical (transverse) and the timelike vector-meson scattering to allow the derivation of low-energy theorems. Only at zero vector-meson four-momentum can the (n)-point function be determined from the transverse $(n+1)$-point function. In this sense then, one can say that the implications of equal-time current commutation relations, and the knowledge of the divergences of these currents, are the low-energy theorems. In order to give the Ward identities teeth, that is, to go further and derive sum rules, one needs assume dispersion relations; in particular, unsubtracted dispersion relations (USDR) are usually essential to derive any nontrivial sum rule at all.[8] The USDR, if correct, establishes definite relations between the timelike and the transverse vector-meson scattering, enough so that scattering data at the $(n+1)$-point level can be fed through to calculate the (n)-point function with relative ease. Evidently, it is crucial to know in which cases an assumption of USDR is correct.

There is a feeling in the literature that these questions are in general too difficult to answer, and that one may as well assume USDR whenever needed. One of our main points in this discussion is that, without any detailed dynamics, one can already say a few things about when it is correct to assume USDR, and when it is wrong or suspicious. This is bound up with a careful

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1 M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

2 J. C. Ward, Phys. Rev. **77**, 293 (1950).

3 Y. Takahashi, Nuovo Cimento **6**, 370 (1957).

4 S. Fubini, Nuovo Cimento **43A**, 475 (1966).

5 S. Fubini and G. Segrè, Nuovo Cimento **65**, 641 (1966).

6 K. Raman and E. C. G. Sudarshan, Phys. Letters **21**, 4 (1966); M. Nauenberg, *ibid.* **22**, 201 (1966).

7 F. E. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. Goldberger, *ibid.* **96**, 1433 (1954); F. E. Low, *ibid.* **110**, 974 (1958); N. Kroll and M. A. Ruderman, *ibid.* **93**, 233 (1954); A. Klein, *ibid.* **99**, 998 (1955); S. Adler and Y. Dothan, *ibid.* **151**, 1267 (1966).

8 Many of these sum rules have been obtained by evaluating a commutator in the $|\mathbf{p}|\to\infty$ frame [S. Fubini and G. Furlan, Physics (N.Y.) **1**, 223 (1965)]. This corresponds to USDR for a particular set of invariant amplitudes; while the $|\mathbf{p}|\to\infty$ technique sometimes makes derivations simpler, the dispersive approach allows us to see the relevance of the Ward identities and will consequently be used throughout this paper.

discussion of which Feynman graphs contribute to which pieces in the Ward identities.

We should say at the outset that we consider the USDR assumption in those sum rules using the hypothesis of partially conserved axial-vector current (PCAC) (e.g., the Adler-Weisberger relation[9]) to rest on fairly reasonable grounds: (a) In these, one assumes that USDR for the mass-shell (say) π-N scattering plus a smooth off-mass-shell continuation (PCAC) implies USDR for zero-mass π-N scattering. (b) Moreover, the Feynman graphical structure of, e.g., the Adler-Weisberger relation, is obvious. (c) Finally, the essential part of the sum rule (making g_A/g_V calculable from π-N scattering data) can be obtained from the associated low-energy π-N scattering theorem[10]—which does not use USDR. We shall not have much to add to the existing discussion of these points. Rather, we shall concern ourselves almost exclusively with the sharply contrasting situation of the more Ward-like sum rules, namely those which involve a commutator of a *conserved current* with some operator. In general, we shall use the term "Ward identity" in the discussion below in referring to any identity between an $(n+1)$-point function and an (n)-point function, *before* any assumption about dispersion relations is made. Our conclusion is that these "conserved-current-commutator" sum rules are on a different, and, we feel, weaker, footing than those which, like Adler and Weisberger, employ PCAC.

For conventience in the discussion of the sum rules involving commutators of at least one conserved current, we distinguish between two classes of identities:

(1) Those identities in which USDR is assumed for the matrix element of a retarded product of a conserved current with the *source of a strongly interacting particle on its mass shell*. For example, in Sec. II, we shall discuss USDR (and the resulting sum rule) for the structure

$$\int e^{i(q_2\cdot x'-q_1\cdot x)}d^4xd^4x'\langle p_2|T\{j_\mu(x)\cdot J_\pi{}^i(x')\}|p_1\rangle, \quad (1.1)$$

where $j_\mu(x)$ is the electromagnetic current and $J_\pi{}^i$ is the source of the pion. The four-momentum of the pion q_2 is taken on the mass shell. Taking $q_{1\mu}$ times an assumed USDR for Eq. (1.1), one can derive a sum rule relating pion photoproduction to the πNN form factor, with the pion off its mass shell. A Feynman graphical analysis of the different pieces in the Ward identity shows that USDR for (1.1) is almost certainly wrong. Moreover, it is wrong even if the theory Reggeizes. Observe that, *had* the sum rule been correct, it would have provided the means for determining an off-mass-shell strong vertex in terms of a mass-shell scattering process. Other sum rules like this could have been cooked up to determine the off-mass-shell behavior of every strongly interacting particle. This is the unphysical sort of thing to which incorrect assumption of USDR may lead. We conjecture in general that, whenever a conserved-current commutation relation and USDR (with no other assumption, such as, e.g., $\partial_\mu A_\mu{}^\alpha \equiv \pi^\alpha$) allow the "measurement" of an off-shell strong process, then the USDR assumption is incorrect.

(2) The second class of identities is that in which USDR is assumed for the retarded product of a conserved current with a current which is the *source of a system which interacts only weakly or electromagnetically*. This is the subject of Sec. III. Analysis of such structures have led to a series of relations between Compton scattering and electromagnetic form factors.[11] For example, the relevant matrix element in the derivation of Cabibbo-Radicati–like relations is

$$\int e^{i(q_2\cdot x'-q_1\cdot x)}d^4xd^4x'\langle p_2|T\{j_\mu{}^i(x),j_\nu{}^j(x')\}|p_1\rangle, \quad (1.2)$$

where $j_\mu{}^i$ is the ith component of the isotopic-spin current. A Feynman-graphical analysis of the various pieces in these Ward identities shows that, *because neither current is the source of a strongly interacting system*, the arguments used to destroy sum rules of type (1) above do not apply *to first order in the nonstrong interactions*. On the other hand, the graphical analysis shows that the USDR assumption is causing some very strange things to happen in the sum rule. For example, Cabibbo and Radicati find that the N^* intermediate state (or the ω intermediate state when the external states are taken as pions) contributes sizeably to the Compton-scattering side of the relation, and hence to the charge radius. The graphical analysis shows that *no Feynman graph with an N^* intermediate state (to all orders in any field theory) can contribute to the Compton-scattering side of the Ward identity*; i.e., the entire set of graphs is projected to zero by the $q_{1\mu}$ operation which relates (1.2) to the form factors. Another way of saying this is that the Ward identity accepts information in general only about timelike photon Compton scattering, and the N^* graphs do not contribute to this. The argument is valid also for the resonant part of a composite N^*. The same surprising analysis can be carried through for the set of all one ω-meson intermediate-state graphs in the case of external pions. In fact, it is shown that *the only single-particle states whose graphs can survive the $q_{1\mu}$ are those*

[9] W. Weisberger, Phys. Rev. Letters **14**, 1047 (1965); S. Adler, *ibid.* **14**, 1051 (1965).

[10] Y. Tomozawa, Nuovo Cimento **46**, 707 (1966); S. Weinberg, Phys. Rev. Letters **17**, 616 (1966); A. P. Balachandran, M. Gundzik, and F. Nicodemi, Nuovo Cimento **44**, 1257 (1966).

[11] N. Cabibbo and L. Radicati, Phys. Letters **19**, 697 (1966); R. Dashen and M. Gell-Mann, in Proceedings of the Conference on Symmetries of Strong Interactions, Coral Gables, 1966 (unpublished); J. D. Bjorken, Phys. Rev. **148**, 1467 (1966); I. Muzinich, *ibid.* **151**, 1206 (1966); C. Bouchiat and P. Meyer (unpublished); M. Gourdin, Nuovo Cimento **47**, 145 (1967); S. Okubo (unpublished); F. Buccella, G. Veneziano, and R. Gatto, Nuovo Cimento **42A**, 1019 (1966).

with the isospin of the external state (nucleon or pion in the two cases).

What is happening is that the USDR assumption (if correct) is requiring a relation between the resonant and nonresonant parts of (say) the 33 wave—such that, although the Ward identity projects the resonant part to zero, using the imaginary part of the resonance effectively introduces the nonresonant part (which can contribute). In a situation like this, one's customary graphical intuition is useless. We feel that one of two conclusions should be drawn from these observations: (1) Either such strange goings-on should be regarded as throwing doubt in general on sum rules derived using conserved-current commutation relations and USDR, or (2) if such extraordinary antigraphical tricks turn out to give agreement with experiment,[12] then other problems, in which graphical and pological intuition has evidently ground to a halt, should be tackled with these methods—with the hope that a bit of this "magic" will succeed where straightforward methods have not.

Toward deciding between these two views, we also point out in Sec. III that USDR in these Cabibbo–Radicati–like cases certainly receives no support from Adler-Weisberger–type arguments: USDR on the ρ-meson mass shell plus any simple continuation to zero meson-mass fails to yield the sum rules. We shall also note that, contrary to the Adler-Weisberger case, *there is*, in this case, *no way to calculate the charge radii from Compton scattering without having to make the USDR assumption*—that is, through the corresponding low-energy theorem. The relevant theorem, recently derived by Bég,[13] applies to the structure (1.2), which is not measurable in Compton scattering. In fact it is necessary to perform a series of first- and second-order weak-interaction experiments to determine its value.[14]

Finally, in an Appendix, we include a derivation of the original Takahashi identity which corrects compensating errors of two factors of Z_3 in the original treatment.[3] The derivation is an interesting exercise in the kind of intuition emphasized in this paper, namely, the relations between processes involving timelike and transverse vector mesons.

II. ONE CURRENT IS SOURCE OF STRONGLY INTERACTING PARTICLE

A. Derivation of a Sum Rule

In this section, we shall first proceed formally, by deriving a sum rule in the "usual manner," and then criticize the derivation. Consider the time-ordered product

$$T_\mu{}^i=-i\int d^4z\, e^{-ik\cdot z}\langle p_2|\,T\{J_\pi{}^i(0),j_\mu(z)\}\,|p_1\rangle\,,\quad (2.1)$$

where the initial and final states are nucleons of momentum p_1 and p_2, $J_\pi{}^i(0)$ is the source of the pion field with isotopic spin i, and j_μ the source of the electromagnetic field. The latter's isovector component is related to the isotopic-spin current $j_\mu{}^i$ by

$$j_\mu{}^V=ej_\mu{}^3.\quad (2.2)$$

We assume four-momentum conservation, $p_1+k=p_2+q$, where q is the pion's four-momentum, and also that all particles are on the mass shell, namely $p_1{}^2=p_2{}^2=M^2$, $q^2=\mu^2$, and $k^2=0$.

Postulating the equal-time commutation relation

$$\delta(z_0)[j_0(z),J_\pi{}^i(0)]=ie\epsilon^{3ij}\delta^4(z)J_\pi{}^j(0),\quad (2.3)$$

we have the "Ward identity"

$$k_\mu T_\mu{}^i=-ie\epsilon^{3ij}\langle p_2|\,J_\pi{}^j(0)\,|p_1\rangle\,.\quad (2.4)$$

Equation (2.4) by itself is essentially empty so we proceed to assume USDR. It is convenient to find suitable invariants first. Let $F_\mu{}^i$ be defined by

$$T_\mu{}^i=\frac{M}{(E_1E_2)^{1/2}}\bar{u}(p_2)\gamma_5F_\mu{}^iu(p_1)\,.\quad (2.5)$$

We then multiply $T_\mu{}^i$ by $\gamma_5u(p_2)$ on the left, $u(p_1)$ on the right, and take the trace. The averaged $T_\mu{}^i$, which we call $\bar{T}_\mu{}^i$,

$$\bar{T}_\mu{}^i=\frac{M}{(E_1E_2)^{1/2}}\,\mathrm{Tr}\{\gamma_5\Lambda(p_1)\gamma_5F_\mu{}^i\Lambda(p_2)\}\,,\quad (2.6)$$

can then be written as

$$\left(\frac{E_1E_2}{M^2}\right)^{1/2}T_\mu{}^i=P_\mu A_1{}^i(\nu,t)+Q_\mu A_2{}^i(\nu,t)+\Delta_\mu A_3{}^i(\nu,t)\,,\quad (2.7)$$

where

$$P=\tfrac{1}{2}(P_1+P_2),\quad Q=\tfrac{1}{2}(k+q),\quad \Delta=\tfrac{1}{2}(k-q)\,,$$

$\nu=P\cdot Q$, and $t=-\Delta^2$. If we assume the A's obey fixed-t unsubtracted dispersion relations in ν and that the integral over the ν discontinuity in A ($A\nu$) converges, we may write the limit of A as

$$A^i=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A_\nu{}^i(\nu',t)d\nu'}{\nu'-\nu}\underset{\nu\to\infty}{\longrightarrow}-\frac{1}{\pi\nu}\int_{-\infty}^{\infty}A_\nu{}^id\nu'\,,\quad (2.8)$$

for $A^i=A_{1,2,3}{}^i$. However as $k\cdot P=(Q+\Delta)\cdot P=Q\cdot P=\nu$, it follows that[15]

$$\left(\frac{E_1E_2}{M^2}\right)^{1/2}k_\mu T_\mu{}^i=-\frac{1}{\pi}\int_{-\infty}^{\infty}A_{1\nu}{}^i(\nu',t)d\nu'\,.\quad (2.9)$$

[12] F. Gilman and H. Schnitzer, Phys. Rev. **150**, 1362 (1966).
[13] M. A. Bég, Phys. Rev. Letters **17**, 333 (1966).
[14] S. Adler, Phys. Rev. **143**, 1144 (1966).

[15] Equation (2.7) could actually be derived for arbitrary ν (see Ref. 5). This limit merely makes the calculation easier.

If we define the pion-nucleon form factor $F_\pi(t)$,

$$\langle p_2|J_\pi{}^j(0)|p_1\rangle=\frac{M}{(E_1E_2)^{1/2}}\bar{u}(p_2)\gamma_5u(p_1)F_\pi{}^j(t)\,,\quad(2.10)$$

and perform the same averaging over spins, we find from Eqs. (2.4) and (2.9) that, e.g., in the case of π^+ emission

$$ie\frac{t}{2M^2}F_\pi{}^+(t)=-\frac{1}{\pi}\int_{-\infty}^{\infty}A_{1\nu}{}^+(\nu',t)d\nu'.\quad(2.11)$$

Relating the discontinuity of A to the discontinuity of the Chew-Goldberger-Low-Nambu[16] invariants (CGLN) [which we call a^r, b^r, c^r, and d^r, where r denotes a particular isospin invariant (see Ref. 16)], using crossing symmetry, and calculating explicitly the nucleon pole contribution, Eq. (2.11) becomes

$$e\{tF_{\pi^+}(t)-\tfrac{1}{2}(t+\mu^2)g_{NP\pi^+}\}=-\frac{4\sqrt{2}}{\pi}(t-\mu^2)\times\int_{\nu_0}^{\infty}[a^-(\nu',t)+tb^-(\nu',t)]d\nu'.\quad(2.12)$$

The $t-\mu^2$ comes from a $k\cdot q$ factor appearing in the invariants.

B. Analysis of Sum Rule

Having derived the sum rule, let us now see why it is wrong, more precisely, why it is incorrect to assume $A_{1,2,3}$ obey unsubtracted dispersion relations. This will be clear from a discussion of which Feynman graphs contribute to which pieces of the "Ward identity" as given in Eq. (2.4). Let us begin with the expression for the pion photoproduction scattering amplitude,

$$\epsilon_\mu S_\mu{}^i=-\int d^4xd^4y\Box_x{}^2(\Box_y{}^2+\mu^2)\times\langle p_2|T\{\pi^i(y),A_\mu(x)\}|p_1\rangle\frac{\epsilon_\mu e^{-ik\cdot x+iq\cdot y}}{(4k_0q_0)^{1/2}},\quad(2.13)$$

and let the d'Alembertians operate on the time ordering. As always, this divides the S matrix into a current-current (really source-source) term plus an equal-time term which is only a function of momentum transfer squared. Explicitly,

$$\epsilon_\mu S_\mu{}^i=-i(2\pi)^4\frac{\delta^{(4)}(p_1+k-p_2-q)}{(4k_0q_0)^{1/2}}\times\epsilon_\mu[T_\mu{}^i(\nu,t)+f_\mu{}^i(t)],\quad(2.14)$$

with $f_\mu{}^i(t)$, the equal-time term, being given by

$$f_\mu{}^i(t)=ie(2q_\mu-k_\mu)\epsilon^{3ij}\langle p_2|\pi^j(0)|p_1\rangle.\quad(2.15)$$

[16] G. F. Chew, M. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

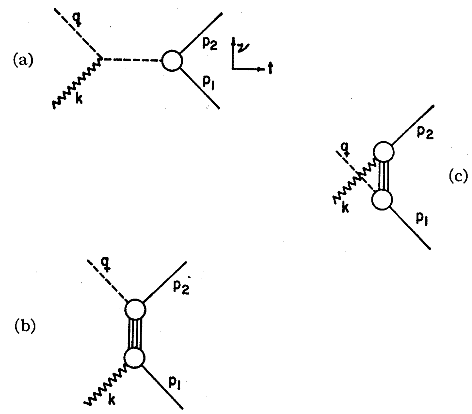


FIG. 1. (a) Graphs that contribute to equal-time term f_μ. (b), (c) Contributions to current-current term T_μ.

Note that

$$\begin{aligned}k_\mu f_\mu{}^i(t)&=ie\epsilon^{3ij}(t-\mu^2)\langle p_2|\pi^j(0)|p_1\rangle\\&=ie\epsilon^{3ij}\langle p_2|J_\pi{}^j(0)|p_1\rangle.\end{aligned}\quad(2.16)$$

Comparing Eqs. (2.16) and (2.4), we see that $k_\mu T_\mu{}^i=-k_\mu f_\mu{}^i$ which, of course, is just the statement that the S matrix is gauge-invariant, $k_\mu S_\mu{}^i=0$. Diagrammatically, one can see in a varity of ways that $\epsilon_\mu f_\mu$ is the sum of all graphs for which the electromagnetic current and the pion source join (into a pion) before going on into the "guts" of the process; the corresponding graphs are displayed in Fig. 1(a). $\epsilon_\mu T_\mu$ is easily seen to consist of all other graphs in S, namely, those for which an interaction occurs before the electromagnetic current joins the pion source. Their representation, in Figs. 1(b) and (c), emphasizes that the current-current term contains all the graphs which have discontinuities in ν (or equivalently in s and u). As stated earlier $f_\mu(t)$ is independent of ν and therefore does not vanish as $\nu\to\infty$.

The reason now why $T_\mu{}^i$ cannot satisfy an unsubtracted dispersion relation in ν for fixed t is that it contains many graphs whose large-ν behavior is at least as singular as that of $\epsilon_\mu f_\mu$. Examples of such graphs are the vertex corrections of Fig. 2(a). Without

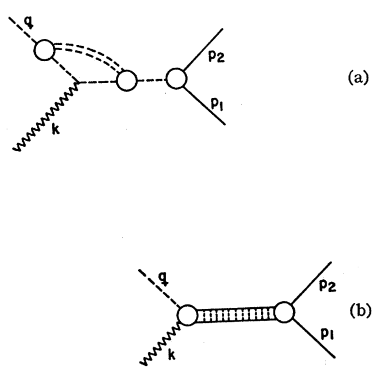


FIG. 2. (a) Graphs in T_μ, as singular as those in f_μ. (b) Graphs in T_μ which, when added to f_μ, would give Reggeization.

a cancellation between these and other graphs in $\epsilon_\mu T_\mu$, there will be no way for $T_\mu \to 0$, the necessary condition for USDR. We are used to talking about subtraction constants being unnecessary if a field theory is Reggeized in a graph summation, but we emphasize that the required cancellation would have nothing to do with Regge behavior of the amplitude; it would in fact contradict a Regge behavior for the S matrix.[17] To see this, assume we have a theory in which the pion Reggeizes; this would come about by having a whole set of graphs in $\epsilon_\mu T_\mu$, such as those of Fig. 2(b), combining with the graphs of $\epsilon_\mu f_\mu$ to give the whole S matrix a $(\nu^{\alpha(t)})$ Regge behavior. Because f_μ goes like a constant at large ν, this would certainly require T_μ to have a subtraction. As we shall see in detail in Sec. III, our inability to extract sum-rule-type information from the "Ward identity," (i.e., to use USDR) is directly traceable to the fact that one of our currents in the retarded product to be dispersed with the source of a strongly interacting particle (the pion) on its mass shell.

Note that if the USDR assumption had been correct, the sum rule (2.12) would have allowed the unambiguous determination of the (pion off-mass shell) pion-nucleon form factor in terms of an on-the-mass-shell pion photoproduction amplitude. Similar combinations of conserved-current commutation relations and USDR could have been arranged to allow the determination of any strongly interacting particle off-mass-shell behavior purely in terms of on-mass-shell quantities. This flies in the face of our experimental sensibilities so badly that we feel it worth conjecturing, that [unless a strong field-theoretical assumption relating weak or electromagnetic currents to strong interacting particles is made (e.g., $\partial_\mu A_\mu{}^\alpha \equiv \pi^\alpha$)[18]] *whenever a sum rule allows the determination of off-mass-shell strong particle behavior from on-mass-shell amplitudes, the USDR assumption is incorrect.*

III. BOTH CURRENTS ARE THE SOURCES OF WEAKLY (OR ELECTROMAGNETICALLY) INTERACTING SYSTEMS

A. Derivation and Preliminary Discussion of a Sample Sum Rule

Let us now turn our attention to the matrix elements of the retarded product of two isotopic-spin currents (for a dispersive approach to other commutators see Ref. 19):

$$\tilde{T}_{\mu\nu}{}^{ij} = i\int d^4z\, e^{-iq_1\cdot z}\langle p_2|T\{j_\mu{}^i(z), j_\nu{}^j(0)\}|p_1\rangle, \quad (3.1)$$

[17] See E. J. Squires, *Complex Angular Momentum and Particle Physics* (W. A. Benjamin, Inc., New York, 1963) for a bibliography of Regge poles.

[18] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 560 (1960).

[19] S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40A**, 1171 (1965); W. Weisberger, Phys. Rev. **143**, 1302 (1966); V. Alessandrini, M. A. Bég, and L. S. Brown, *ibid.* **144**, 1137 (1966).

where, to simplify the kinematics, we shall take the initial and final states to be one-pion states. We define, as in Sec. II,

$$P_\mu = \tfrac{1}{2}(p_1+p_2)_\mu, \quad Q_\mu = \tfrac{1}{2}(q_1+q_2)_\mu, \quad \Delta_\mu = \tfrac{1}{2}(q_1-q_2)_\mu,$$

and $\nu = P\cdot Q$, $t = -\Delta^2$. Expressing $\tilde{T}_{\mu\nu}{}^{ij}$ as

$$(4E_1E_2)^{1/2}\tilde{T}_{\mu\nu}{}^{ij} = P_\mu P_\nu A_1{}^{ij}(\nu,t,q_1{}^2q_2{}^2) + Q_\mu P_\nu A_2{}^{ij} + \Delta_\mu P_\nu A_3{}^{ij} + P_\mu \Delta_\nu A_3{}^{ij} + \cdots, \quad (3.2)$$

we easily see that

$$q_{1\mu}\tilde{T}_{\mu\nu}{}^{ij} = -i\epsilon^{ijk}\langle p_2|j_\nu{}^k(0)|p_1\rangle = -i\epsilon^{ijk}P_\nu F_\pi{}^k(t) \quad (3.3a)$$

$$= -\frac{1}{\pi}P_\nu\int_{-\infty}^{+\infty}\mathrm{Im}A_1{}^{ij}(\nu',t,q_1{}^2,q_2{}^2)d\nu'. \quad (3.3b)$$

In obtaining Eq. (3.3), the equal-time commutation relation

$$\delta(z_0)[j_0{}^i(z), j_\nu{}^j(0)] = i\epsilon^{ijk}j_\nu{}^k(0)\delta^{(4)}(z) \quad (3.4)$$

has been used, and a USDR has been taken for each invariant. The relation following from Eq. (3.3), namely,

$$\frac{2}{\pi}\int_0^\infty \mathrm{Im}A_1{}^{ij}(\nu',t,q_1{}^2,q_2{}^2)d\nu' = i\epsilon^{ijk}F_\pi{}^k(t), \quad (3.5)$$

has been discussed in great detail in Ref. 5. In particular, it was shown there how the derivative of Eq. (3.5) with respect to t, $q_1{}^2$, or $q_2{}^2$, at $t=q_1{}^2=q_2{}^2=0$ leads to the relation between the charge radius of the pion and the integral over scattering from pions of isovector photons derived by several authors[10]:

$$\langle r_\pi{}^2\rangle = \frac{1}{2\pi^2\alpha}\int_{\nu_0}^\infty \frac{d\nu'}{\nu'}[\sigma_{I=0}{}^V(\nu') + \sigma_{I=1}{}^V - (5/4)\sigma_{I=2}{}^V], \quad (3.6)$$

where $\sigma_I{}^V$ is the isovector-photon–pion total cross section in the I-isospin channel, and ν_0 is the threshold value of ν.

Preliminary Graphical Interpretation of the Sum Rule

Let us now see how the assumption of unsubtracted dispersion relations for the A's involved in Eq. (3.2) is different from the analogous assumption for the scalar amplitudes of Sec. II. To make the comparison clearer, let $j_\mu{}^i$ be the electromagnetic current apart from a factor e (the additional isoscalar current contained therein commutes with all components of the isotopic-spin currents and causes no complications) and let $j_\nu{}^j(0) \to j_\nu{}^+(0)$. The key to the argument is that $j_\nu{}^+$ is not the source of a strongly interacting particle as was J_π, but rather couples to a charged lepton pair through the vector current. The latter have no strong interactions so that to first order in the weak and electromagnetic interactions there are no graphs contained in $\tilde{T}_{\mu\nu}{}^{ij}$

analogous to those of Figs. 2(a) and 2(b), and therefore it is not inconsistent that the A's obtained by decomposing $\tilde{T}_{\mu\nu}{}^{ij}$ obey unsubtracted dispersion relations. We emphasize that we do not pretend to prove that these A's do obey unsubtracted dispersion relations, merely that assuming they do is not at the moment patently inconsistent.

To clarify these points, it is helpful to notice how $\tilde{T}_{\mu\nu}$ is related to the (physical) S matrix for the process

$$\gamma+\pi^+\rightarrow\pi^0+e^++\nu\,,$$

which we denote by

$$S=eG_F(2\pi)^4\delta^{(4)}(q_1+p_1-q_2-p_2)M\,.$$

G_F is the Fermi coupling constant, and M is given by

$$M=\epsilon_\mu V_\nu T_{\mu\nu}\,, \tag{3.7}$$

where ϵ_μ is the photon's polarization and V_ν is the lepton-pair's current. As always, M can be divided into a source-source term and an equal-time term

$$M=\epsilon_\mu V_\nu[\tilde{T}_{\mu\nu}+f_{\mu\nu}]\,. \tag{3.8}$$

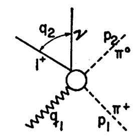
The graphs that contribute to $\tilde{T}_{\mu\nu}$ and $f_{\mu\nu}$ are shown in Figs. 3(a) and 3(b). As in Sec. II, the current-current term $\tilde{T}_{\mu\nu}$ contains all those graphs for which something happens before the currents join—i.e., all those graphs with discontinuities in s or u. M is of course gauge-invariant,

$$q_{1\mu}T_{\mu\nu}=0\,, \tag{3.9}$$

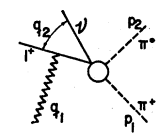
and it is very simple to verify that $V_\nu q_{1\mu}f_{\mu\nu}$ does in fact equal minus $V_\nu q_{1\mu}\tilde{T}_{\mu\nu}$. [Moreover, since $V_\nu q_{1\mu}f_{\mu\nu}=V_\nu P_\nu F_\pi(t)$, (3.9) is exactly the $i=3$, $j=+$ case of the sum rule (3.3a).] The graphical structure of the two pieces is almost exactly that found in the sum rule of Sec. II, except that now, to lowest order in weak and electromagnetic interactions, no graphs exist connecting one of the lepton lines and the strong-interaction "blob" of Fig. 3(a). These were the graphs that forced a subtraction on T_μ, the analog of $\tilde{T}_{\mu\nu}$, in Sec. II. The reason of course was that there the outgoing system was composed of a nucleon and a pion, and here it is composed of a nucleon plus a lepton pair. Thus, relative to this first simple criterion, there is no obvious inconsistency in taking USDR for the invariants of $\tilde{T}_{\mu\nu}$. It is worth emphasizing: *The fact that the vector currents couple to systems that only interact weakly (or electromagnetically) has shielded the Adler*[14]*-Cabibbo-Radicati-type sum rules from the criticism we were able to make of the sum rules in Sec. II.*

What about the situation relative to Reggeism? One can see easily that the gauge invariance of the S-matrix T [Eq. (3.9)], or the equivalent (for $q_1{}^2=q_2{}^2=0$) relation

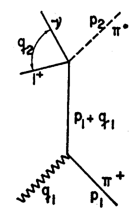
$$Q_\mu(P_\mu P_\nu A_1+P_\nu Q_\mu A_2)=P_\nu F(t)\,, \tag{3.10}$$



(a)



(b)



(c)

FIG. 3. (a) Current-current contribution to amplitude for $\gamma+\pi^+\rightarrow\pi^0+l^++\nu$. (b) Equal-time contribution to amplitude for $\gamma+\pi^+\rightarrow\pi^0+l^++\nu$. (c) Current-current contribution to amplitude for $\gamma+\pi^+\rightarrow\pi^0+l^++\nu$ in the absence of strong interactions.

and the USDR assumption for the invariants, requires that the high-ν behavior of A_1 be

$$A_1\underset{\nu\to\infty}{\sim}\frac{F(t)}{\nu} \tag{3.11}$$

to cancel the large-ν behavior of the equal-time term. Thus A_1 cannot be Reggeized in the usual sense (at least not its real part). This also means that the S matrix itself cannot Reggeize in the usual sense. [The $\tilde{T}$ term contributes $A_1\sim(F(t)/\nu)$ to the $P_\mu P_\nu$ invariant of T, and the equal-time term is powerless to subtract this off.] The reason for this non-Regge behavior is that $\tilde{T}_{\mu\nu}$ (or $T_{\mu\nu}$) contains a "hard core" of weak interactions, i.e., does not vanish when the strong interactions are imagined to be turned off. In this limit, only the purely weak Born term [Fig. 3(c)] contribution to $\tilde{T}_{\mu\nu}$ survives. With only this graph left, A_1 would have the (purely real) large-ν behavior

$$A_1\underset{\nu\to\infty}{\sim}\frac{1}{\nu}\,. \tag{3.12}$$

Thus one learns that the strong interactions have a dual function in the sum rule: When turned on they (a) change the large ν coefficient of A_1 from the "hard core" value of 1 to $F(t)$, and (b) create an imaginary part for A_1 (above the pion mass). In that the imaginary part has no "hard core" (is purely strong) one feels that it can be taken as Reggeized. Thus, *for these sum rules it seems consistent with Reggeology to have a Reggeized imaginary part, but non-Reggeized real parts for both T and $\tilde{T}$.* This situation should be contrasted with the case of Sec. II, in which, there being no "hard core" (all graphs vanish if strong interactions are turned off), S should Reggeize and T_μ must not.

B. Detailed Graphical Analysis of the Sum Rule

In this section, we want to see what becomes of certain subsets of Feynman graphs contributing to the 4-point ("Compton" scattering) side of the sum rule. To get our feet on the ground and to illustrate the techniques we will employ in more interesting cases, let us first consider the Born (pion pole) term. These are shown in Fig. 4(a). The external wiggly lines should be thought of for isospin index 3 as off-shell photons and for 1 and 2 as lepton-antilepton pairs. Note that seagull graphs (if such exist) are not found in the current-current term. (Just as in ordinary Compton scattering, these are always part of the equal-time term of the S matrix.) Omitting the photon polarization vector and/or the lepton current, these graphs have the form

$$\epsilon^{jl'm}\epsilon^{ill'}(2p_1+q_1)_\mu\frac{1}{(p_1+q_1)^2-\mu^2}(p_1+p_2+q_1)_\nu$$
$$+\epsilon^{il'm}\epsilon^{jll'}(2p_2-q_1)_\mu\frac{1}{(p_2-q_1)^2-\mu^2}(p_1+p_2-q_1)_\nu. \quad (3.13)$$

Thus their contribution to the three-point-side function of the Ward identity (3.3a) is

$$q_{1\mu}\tilde{T}_{\mu\nu}\cong[t^i,t^j]^{lm}(p_1+p_2)_\nu+[t^i,t^j]_+{}^{lm}q_{1\nu}, \quad (3.14)$$

where the three matrices ($i=1, 2, 3$) $(t^i)^{mn}=\epsilon^{imn}$ are the $I=1$ representation of the isospin generators. Just as we saw in Sec. II, these two terms are the negative of $q_{1\mu}$ times the graphs of Fig. 4(b). (The first of the figures in 4(b) represents the graph in which the two currents meet before going on to interact with the pions.) As far as the antisymmetric structure is concerned, the first term on the right-hand side of Eq. (3.14) is indeed exactly, to lowest order in all interactions, $\epsilon^{ijk}F_\pi{}^k(t)P_\nu$, that is, Fig. 4(c). In other words, to lowest order, it is the sum of the two terms of Fig. 4(a), with the isospin structure taken in opposite order, that reproduces the commutator structure.

The situation is not so simple in the case of the higher-order graphs with one-pion pole terms. This entire set of graphs is shown in Fig. 5. The study of these graphs is best begun in coordinate space, where the Ward identity takes the form

$$\frac{\partial}{\partial x_\mu}\langle p_2,m|T\{j_\mu{}^i(x)j_\nu{}^j(y)\}|p_2,l\rangle$$
$$=2i\epsilon^{ijk}\delta^4(x-y)\langle p_2,m|j_\nu{}^k(y)|p_1,l\rangle. \quad (3.15)$$

Parenthetically, note that if the pions had been off their mass shell, there would have been two more (3-point function) terms, corresponding to the folding together of $j_\mu{}^i(x)$ with each of the pions. Analytically, the one-pion pole graph's contribution to the current-current term is

$$\langle p_2,m|T\{j_\nu{}^i(y)j_\mu{}^j(x)\}|p_1,l\rangle\cong\int\int d^4zd^4z'[\langle p_2,m|T\{j_\nu(y)\pi^a(z')\}|0\rangle[\Delta_{FR}'(z'-z)]^{-1}\langle 0|T\{\pi^a(z)j_\mu{}^i(x)\}|p_1,l\rangle$$
$$+\langle p_2,m|T\{j_\mu{}^i(x)\pi^a(z')\}|0\rangle[\Delta_{FR}'(z'-z)]^{-1}\langle 0|T\{\pi^a(z)j_\nu{}^j(y)\}|p_1,l\rangle], \quad (3.16)$$

where π is the pion field. Taking the derivative of (3.16) with respect to x_μ, assuming the commutation relation (2.3), and going over to momentum space, we obtain

$$q_{1\mu}\langle p_2,m|T\{j_\nu{}^j(q_2)j_\mu{}^i(q_1)\}|p_1,l\rangle\cong2\epsilon^{il'l}\langle p_2,m|T\{j_\nu{}^j(q_2)\pi^{l'}(q_1+q_2)\}|0\rangle$$
$$\times[\Delta_{FR}'(q_1+q_2)]^{-1}+2\epsilon^{il'm}\langle 0|T\{\pi^{l'}(p_2-q_1)j_\nu{}^j(q_2)\}|p_1,l\rangle(\Delta_{FR}'(p_2-q_1))^{-1}. \quad (3.17)$$

The structures on the right are 3-point functions, but they each have one pion off-mass shell. Thus, they cannot contribute to the sum rule in this form. Evidently, other $\tilde{T}_{\mu\nu}$ graphs contribute 3-point functions which add to these, bringing the sum into the form $F_\pi(t)$. The failure of all the pion-pole graphs to go over directly into pieces of $F(t)$ should not be surprising: One knows that Ward identities relate all $(n+1)$-point function graphs of a given order in the coupling to all the (n)-point function graphs of that same order—and that there is no necessary connection between all graphs of a given *type* (to all *orders*) at the $(n+1)$-point level and graphs at the (n)-point level. Probably one would need to add all the other graphs of the theory to the pion-pole graphs in order to get (at arbitrary q_1, q_2) some part of $F_\pi(t)$. On the other hand, as $q_{1\mu}$, $q_{2\mu}\to 0$, it is clear from Eq. (3.17) that no other graphs are needed to get $F_\pi(t)$. That is to say, at $q_{1\mu}=q_{2\mu}=0$, the pion-pole graphs saturate the sum rule. This corresponds to the trivial case of the matrix element between one-pion states of the commutator of two isospin generators (integrated fourth components of currents) which is of course saturated by a one-pion intermediate state.

One ω-meson Intermediate State

A more interesting set of graphs is that set which contains the one ω-meson pole (see Fig. 6). This pole has been found to contribute significantly to the Cabibbo-Radicati sum rule. Analytically, the one-ω graphs can be written in coordinate space

$$\langle p_2,m|T\{j_\mu{}^i(x)j_\nu{}^j(y)\}|p_1,l\rangle\cong\int\int d^4zd^4z'[\langle p_2,m|T\{j_\nu{}^j(y)\omega_{\lambda'}(z')\}|0\rangle[D_{\lambda'\lambda}'(z'-z)]^{-1}\langle 0|T\{\omega_\lambda(z)j_\mu{}^i(x)\}|p_1,l\rangle$$
$$+\langle p_2,m|T\{j_\mu{}^i(x)\omega_{\lambda'}(z')\}|0\rangle[D_{\lambda'\lambda}'(z'-z)]^{-1}\langle 0|T\{\omega_\lambda(z)j_\nu{}^j(y)\}|p_1,l\rangle], \quad (3.18)$$

where $D_{\lambda'\lambda}(z'-z)$ is the ω propagator and $\omega_\lambda(z)$ is the ω field. *Because ω is an isotopic scalar*, i.e.,

$$\delta(x_0-z_0)[j_0{}^i(x),\omega_\lambda(z)]=0, \quad (3.19)$$

these graphs fail to contribute to the sum rule; that is, (3.18) is divergenceless in x. Physically, this had to happen, for, if the one-ω intermediate-state graphs had survived the $q_{1\mu}$, they would have implied the presence of absurd graphs at the 3-point function level, like those shown in Fig. 7. (A π changing to an ω like this is a violation of total isospin conservation.) It bears emphasizing then that *although Cabibbo and Radicati (and others) find with* USDR *that the ω contribution is sizeable, the sum of all Feynman graphs with a one-ω intermediate state cannot contribute to the sum rule.*

In the first place, it is easily seen that the same consideration applies relative to the N^* intermediate state (when the external states of the sum rule are nucleons): The sum of all N^* intermediate-state Feynman graphs is, in coordinate space,

$$\langle p_2,\beta|T\{j_\mu{}^i(x)j_\nu{}^j(y)\}|p_1,\alpha\rangle\cong\int d^4z\int d^4z'[\langle p_2,\beta|T\{j_\nu{}^j(y)S_{\sigma'}{}^{n'}(z')\}|0\rangle[\mathfrak{D}'_{\sigma'\sigma}{}^{n'n}(z'-z)]^{-1}\langle 0|T\{S_\sigma{}^n(z)j_\mu{}^i(x)\}|p_1,\alpha\rangle$$
$$+\langle p_2,\beta|T\{j_\mu{}^i(x)S_{\sigma'}{}^{n'}(z')\}|0\rangle[\mathfrak{D}'_{\sigma'\sigma}{}^{n'n}(z'-z)]^{-1}\langle 0|T\{S_\sigma{}^n(z)j_\nu{}^j(y)\}|p_1,\alpha\rangle], \quad (3.20)$$

where $\mathfrak{D}_{n'/n}{}^{\sigma'/\sigma}$, $S_\sigma{}^n$ are the N^* propagator and field respectively, and $|p_1,\alpha\rangle$ (say) is a nucleon with isospin index α. The x divergence of the N^* graphs is then proportional to structures like (where $[\;]_{\text{e.t.}}$ means equal-time commutator)

$$\langle 0|[j_0{}^i(x),S_\sigma{}^n(z)]_{\text{e.t.}}|p_1,\alpha\rangle =\delta^{(3)}(\mathbf{x}-\mathbf{z})(\mathcal{T}_i)^{nn''}\langle 0|S_\sigma{}^{n''}(z)|p_1,\alpha\rangle, \quad (3.21)$$

where the matrices $\mathcal{T}_i$ are the 4×4 representation of the isospin generators. Because the matrix element on the right of Eq. (3.21) vanishes by total isospin conservation, we learn that (in apparent contradiction to Cabibbo and Radicati) none of the N^* graphs survive the $q_{1\mu}$ of the sum rule. Again, had they survived, it would have implied the presence of three-point graphs in which N changed abruptly to N^*, violating total isospin conservation. Notice that we have not used spin conservation here, or in the case of the ω graphs. Spin conservation has nothing to do with, e.g., the vanishing of Eq. (3.21); that is, an off-mass shell N^* has a spin-$\frac{1}{2}$ component, so that, as far as spin is

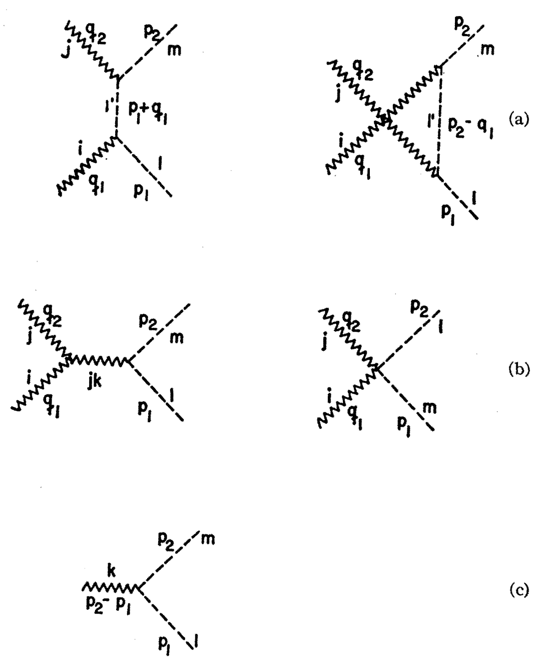


FIG. 4. (a) Direct and crossed pion pole terms. (b) Lowest-order contributions to $f_{\mu\nu}$. (c) Pion form factor to lowest order.

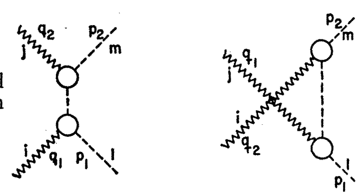


FIG. 5. Direct and crossed pion pole term with full vertices.

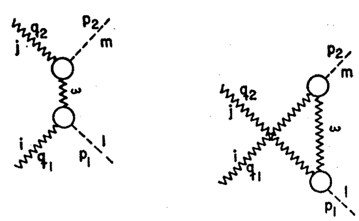


FIG. 6. Direct and crossed graphs with an ω pole.

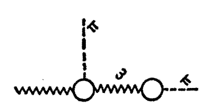


FIG. 7. A set of 3-point function graphs which does not exist.

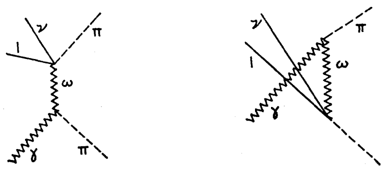


FIG. 8. Lowest-order direct and crossed graph ω-pole contributions.

concerned, the nucleon could go directly to an M^*. Note in fact that a similar analysis of the Adler-Weisberger relation would involve axial-vector currents and the commutator analogous to Eq. (3.21) would in general have nonvanishing matrix elements.

We can be somewhat more explicit about what is happening to these pole terms by looking directly at, e.g., the lowest-order ω pole, as shown in Fig. 8. This graph can be written using the effective Lagrangian density (summed over i)

$$\mathfrak{L}_I(x)=\epsilon_{\mu\nu\lambda\sigma}\omega_\mu W_{\lambda\nu}{}^i\partial_\sigma\phi^i, \tag{3.22}$$

where $\epsilon_{\mu\nu\lambda\sigma}$ is completely antisymmetric, ω_μ, ϕ^i are the ω and π fields, respectively, and

$$\begin{aligned} W_{\lambda\nu}{}^i(x)&\equiv G\partial_\lambda V_\nu(x), \quad i=1,2\\ &\equiv eF_{\lambda\nu}(x), \quad i=3. \end{aligned} \tag{3.23}$$

[$V_\nu(x)$ and $F_{\mu\nu}(x)$ are the leptonic current and the electromagnetic field tensor, respectively.] Note that because ω and π have different masses, etc., the photon must couple through $F_{\mu\nu}$—otherwise the electromagnetic current would not be conserved. Because of the form of (3.3a), only that part of the graph proportional to P_ν can contribute. One finds for these pieces [omitting $g_{\rho\omega\pi}{}^2$, $V_\nu(x)$, ϵ_μ], when $q_1{}^2=q_2{}^2=0$,

$$\tilde{T}_{\mu\nu}\bigg|_{\text{pole}}=\int\left\langle p_2,m\left|T\left\{j_\mu{}^i\!\left(\frac{x}{2}\right)j_\nu{}^j\!\left(-\frac{x}{2}\right)\right\}\right.\right.$$
$$\times\left.\left|p_1,l\right\rangle e^{-iQ\cdot x}d^4x\right|_{\text{pole}}\propto P_\mu P_\nu A_1+Q_\mu P_\nu A_2, \tag{3.24}$$
$$A_1=\frac{2t}{\nu-\nu_\omega},\quad A_2=\frac{-2\nu}{\nu-\nu_\omega}.$$

As should be the case, this graph is "transverse," i.e., $Q_\mu\tilde{T}_{\mu\nu}=0$. The important thing to notice is that, for the Born term it is *not* correct to assume a USDR for both A_1 and A_2; the invariant A_2 explicitly needs one subtraction. Suppose we forget we ever knew this and proceed formally by assuming USDR in the manner of the derivation of Eq. (3.3b). That is, suppose we assume, instead of (3.24), the form

$$\tilde{T}_{\mu\nu}\cong P_\mu P_\nu\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\operatorname{Im}A_1(\nu')d\nu'}{\nu'-\nu}$$
$$+Q_\mu P_\nu\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\operatorname{Im}A_2(\nu')d\nu'}{\nu'-\nu}+\cdots, \tag{3.25}$$

where the imaginary parts are calculated from (3.24). Then

$$\tilde{T}_{\mu\nu}\cong P_\mu P_\nu\frac{2t}{\nu-\nu_\omega}-Q_\mu P_\nu\frac{2\nu_\omega}{\nu-\nu_\omega}, \tag{3.26}$$

and $Q_\mu\tilde{T}_{\mu\nu}\neq 0$. In fact we have

$$Q_\mu\tilde{T}_{\mu\nu}=P_\nu\left(\frac{2t\nu}{\nu-\nu_\omega}-\frac{2t\nu_\omega}{\nu-\nu_\omega}\right)=2tP_\nu. \tag{3.27}$$

The crossed Born term shown in Fig. 8(b) is separately transverse, and the same discussion can be carried through for it.

The S-matrix theorist would say at this point that the USDR assumption is perhaps not so bad because, after all, one cannot necessarily trust the "bare" Feynman-Born graph to give the correct off-mass-shell behavior of the ω, and certainly the USDR assumption only changes that. On the other hand, we have shown that one cannot hope that the higher-order corrections to the ω will converge to (3.24)—because to all orders the ω pole projects to zero. Moreover, our arguments, being based as firmly as they are on isospin invariance, evidently do not depend on the ω being elementary, merely on its being a tightly enough localized state that a resonance approximation is a good one.

Historically, one tried to fix the subtractions in a dispersion relation in order to reproduce a field theory to all orders. In particular, if given the full field-theoretic pole term with *all* its off-mass-shell corrections, the S-matrix theorist would have chosen the subtractions in his dispersion relation so that his pole term reproduced this. If the assumption of USDR, as applied to the charge-radii sum rules is correct, and we cannot prove it is not, we are then witnessing a departure from this point of view.

Many-Particle Intermediate-State Contributions

If it turned out that $q_{1\mu}$ projected the entire $T=0$ wave (in the case of pion external states) to zero, the USDR assumption would be totally in contradiction to field theory—or any future theory, most probably. In fact however, as we shall see, the multi-particle states do continue to contribute in the $T=0$ wave; only the part that can be approximated by a pole, as just seen, is projected to zero. The USDR assumption (if correct) would imply a relation between the resonant and the nonresonant parts of the $T=0$ wave, so that feeding in the ω imaginary part would be just a shorthand for feeding in the nonresonant part of the $T=0$ wave.

Instead of looking at the $T=0$ wave of the 3-pion intermediate state, let us look at a much simpler model with all the same features. Suppose the external states are taken to be scalar "pions" of isospin 1 and even G-parity. Just as in the previous discussion, one can show that a $T=0$ resonance fails to survive the application of $q_{1\mu}$. Now we are interested in seeing if the $T=0$ wave of the 2-"pion" intermediate state, shown in Fig. 9, does survive the $q_{1\mu}$. Analytically the sum of all these graphs is

$$\langle p_2,m|T\{j_\nu^j(x')j_\mu^i(x)\}|p_1,l\rangle\sim\int d^4y\int d^4y'\int d^4z\int d^4z'\langle p_2,m|T\{j_\nu^j(x')\pi^u(y')\pi^v(z')\}|0\rangle \times[\Delta_{FR}'(y'-y)]^{-1}[\Delta_{FR}'(z'-z)]^{-1}\langle 0|T\{\pi^v(z)\pi^u(y)j_\mu^i(x)\}|p_1,l\rangle, \quad (3.28)$$

where π is the scalar pion field. The divergence of this sum with respect to x, after going to momentum space, is

$$\int d^4Q\int d^4\bar{Q}\langle p_2,m|T\{j_\nu^j(q_2)\pi^u(-\bar{Q})\pi^v(-Q)\}|0\rangle[\Delta_{FR}'(Q)\Delta_{FR}'(\bar{Q})]^{-1} \times[2\epsilon^{ivv'}\langle 0|T\{\pi^{v'}(Q-q_1)\pi^u(\bar{Q})\}|p_1,l\rangle+2\epsilon^{iuu'}\langle 0|T\{\pi^v(Q)\pi^{u'}(\bar{Q}-q_1)\}|p_1,l\rangle]. \quad (3.29)$$

Notice that this set of graphs contributes a "hash" to the three-point function. This time the result cannot even be interpreted Feynman-graphically, as, in each of the two terms, one of the two pions appears to arrive at the upper "blob" with more momentum than it took from the lower "blob" (although, of course, there is still over-all momentum conservation).

Now we study the $T=0$ wave in particular. This is obtained in Eq. (3.28) by setting $i=l$ and summing over i. If we then take the divergence, we obtain, in momentum space,

$$q_{1\mu}\sum_{i=1}^{3}\int e^{i(q_2\cdot x'-q_1\cdot x)}d^4xd^4x'\langle p_2,m|T\{j_\nu^j(x')j_\mu^i(x)\}|p_1,i\rangle \cong 4\int d^4Q\int d^4\bar{Q}\langle p_2,m|T\{j_\nu^j(q_2)\pi^u(-Q)\pi^u(-\bar{Q})\}|0\rangle[\Delta_{FR}'(Q)\Delta_{FR}'(\bar{Q})]^{-1} \times[f(Q,\bar{Q}-q_1;p_1)-f(Q-q_1,\bar{Q};p_1)], \quad (3.30)$$

where we have introduced the definitions

$$\langle 0|T\{\pi^{v'}(Q-q_1)\pi^u(\bar{Q})\}|p_1,l\rangle\equiv\epsilon^{v'ul}f(Q-q_1,\bar{Q};p_1), \quad (3.31a)$$

$$\langle 0|T\{\pi^v(Q)\pi^{u'}(\bar{Q}-q_1)\}|p_1,l\rangle\equiv\epsilon^{vu'l}f(Q,\bar{Q}-q_1;p_1). \quad (3.31b)$$

Now, because Eqs. (3.31) are explicitly antisymmetric in the isospin of the two pions, the function f must also be antisymmetric in the first two arguments (Bose statistics). Thus Eq. (3.30) does not vanish, as promised. In the same way, the $T=0$ wave of the more complicated 3-pion intermediate state in the situation of interest (ordinary pions) survives the projection.

One can go further and study the fate of the entire $T=0$ wave of $\tilde{T}_{\mu\nu}$ in the Ward identity—rather than worry further about individual intermediate states. It follows immediately that

$$q_{1\mu}\sum_{i=1}\int d^4x\int d^4x' e^{i(q_2\cdot x'-q_1\cdot x)}\langle p_2,m|T\{j_\nu^j(x')j_\mu^i(x)\}|p_1,i\rangle =(2\pi)^4\delta(q_2+p_2-q_1-p_1)2\epsilon^{ijk}\langle p_2,m|j_\nu^k(0)|p_1,i\rangle\neq 0. \quad (3.32)$$

Before going on to look at what possible support the USDR assumption might have, it is worth generalizing the conclusions of this section. If we repeated our analysis on the graphs of all one-particle intermediate states that contribute to $\tilde{T}_{\mu\nu}$, we would find in general that *only those resonances with the isospin of the external state* (pion or nucleon in the Cabibbo-Radicati cases) *survive the application of* $q_{1\mu}$—and contribute to $F(t)$. Thus, in e.g., the nucleon case, nonzero graphical contributions may come from the higher resonances in the nucleon channel as well as resonances with $I=\frac{1}{2}$ but higher spin (including the resonances on the nucleon Regge trajectory).

C. Off-Mass-Shell Extrapolations of Amplitudes

As mentioned in the Introduction, the USDR in the Adler-Weisberger relation leans on the assumption of USDR for the mass-shell π-N scattering amplitude, plus a smooth off-mass-shell continuation. Our point in

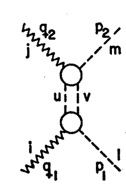


FIG. 9. Nonvanishing two-particle contribution to $T=0$ wave.

this section is that the situation with these conserved-current sum rules is very different. In fact, the Cabibbo-Radicati relations receive no support whatever from a USDR assumption for mass-shell ρ-N scattering plus a smooth continuation.

The reason for this is, as we saw in part A of this section, that the charge-radius–type sum rules depended critically on the essentially weak "hard core" present in the transition amplitude. That is, because of this "hard core," the current-current part of the amplitude went asymptotically like ν^{-1} for all values of t. It therefore seems evident that although

$$\lim_{q_1^2,\, q_2^2 \to m_\rho^2} (q_1^2 - m_\rho^2)(q_2^2 - m_\rho^2) \times \tilde{T}_{\mu\nu}{}^{ij}(\nu,t,q_1^2,q_2^2) = R_{\mu\nu}{}^{ij}(\nu,t) \quad (3.33)$$

is proportional to the ρ-π scattering amplitude (see Refs. 5 and 20 for a discussion of this limit), there will be no smooth off-mass shell extrapolation which will allow us to obtain the Cabibbo-Radicati sum rules. The reason is, of course, that any "smooth" continuation in q_1^2 and q_2^2 would not alter the presumably strong-interaction behavior of the amplitude at large ν. For instance, with Regge asymptotic behavior for $R_{\mu\nu}{}^{ij}$, any smooth off-shell extrapolation would behave as $\nu^{\alpha(t)}$, and not ν^{-1}. That is to say, a smooth continuation will not pick up the (weak) hard core that is missing on the mass shell.

One can see just this sort of thing happening in various simple models of smooth off-shell continuation. E.g., consider (i not summed)

$$\tilde{T}_{\mu\nu}{}^{ij}(\nu,t,q_1^2,q_2^2) \sim \langle 0|T\{j_\mu{}^i(q_1)j_{\mu'}{}^i(q_1)\}|0\rangle \times\langle 0|T\{j_\nu{}^j(q_2)j_{\nu'}{}^j(q_2)\}|0\rangle R_{\mu'\nu'}{}^{ij}(\nu,t). \quad (3.34)$$

This $\tilde{T}_{\mu\nu}{}^{ij}$ will never give the Cabibbo-Radicati relations because it is transverse, i.e., $q_{1\mu}\tilde{T}_{\mu\nu}{}^{ij}=0$. Other simple attempts have similar difficulties. The point is that, to obtain the sum rules, one would have to rig up a complicated, certainly not smooth, continuation that would introduce in some way the essential weak hard core.

The procedure of trying to define the matrix element of two vector currents by coming off the ρ mass shell is really very different from that followed in the Adler-Weisberger relation. What actually occurs in this latter case is that the double divergence of the analogue of $\tilde{T}_{\mu\nu}{}^{ij}$, this time with two axial-vector currents (call it $\tilde{N}_{\mu\nu}{}^{ij}$), is evaluated, giving an equal-time term and a term involving the retarded product of two axial-vector current divergences, which we call[21] $\tilde{M}_{ij}$:

$$q_{1\mu}q_{2\nu}\tilde{N}_{\mu\nu}{}^{ij}(\nu,t,q_1^2,q_2^2) = G^{ij}(t) + \tilde{M}^{ij}(\nu,t,q_1^2,q_2^2), \quad (3.35)$$

where as usual, the equal-time term is a function only of t. This is very similar in form to, say, our Eq. (3.3a), except, of course, because the isospin current is conserved, we never have the analog of $\tilde{M}^{ij}$. On the other hand it is $\tilde{M}^{ij}$ for which USDR on the mass shell and a smooth continuation is assumed. The point is that *this term does not have a weak hard core.* (If the strong interactions were turned off, $\tilde{M}^{ij}$ would vanish.) Thus one can hope that a smooth off-shell extrapolation will pick up the essential features of the off-shell amplitude. The analogy to what is done in the derivation of the Cabibbo-Radicati sum rules would be to assume USDR for $\tilde{N}_{\mu\nu}{}^{ij}$ on an axial-vector meson mass shell, and then continue off. For exactly the reasons discussed above, namely that $\tilde{N}_{\mu\nu}{}^{ij}$ *has a hard core,* this procedure will fail to give the correct large-ν behavior of the off-shell $\tilde{N}_{\mu\nu}{}^{ij}$. The rule we are proposing is then this: *It is permitted to use a smooth-off-mass-shell continuation for structures which have no weak hard core.* In the Adler-Weisberger relation, one does just this; in the Cabibbo-Radicati relations, one has no terms that are free of hard cores.

As a final example of a type of sum rule in which the distinction between a strongly interacting particle being on the mass shell or not is crucial, we would like to consider a class of sum rules examined by Fubini, Furlan, and Rossetti[19] and more recently by Gasiorowicz.[22] These sum rules, which relate nucleon electromagnetic form factors to off-mass-shell pion photoproduction amplitudes, are obtained by considering the matrix element

$$M_\mu{}^i = -i\int d^4z\, e^{-iq_1\cdot z}\langle p_2|T\{j_\mu(z),\pi^i(0)\}|p_1\rangle, \quad (3.36)$$

where $q_2^2 \neq \mu^2$. The assumption of USDR for this structure appears, *a priori*, more similar to Sec. II than to the case we have just discussed. However, these relations really fall with the Adler–Cabibbo-Radicati relations, as we see by considering the structure $\epsilon_\mu \Gamma M_\mu$ where Γ is the vertex for off-mass-shell π-decay $\pi \to \mu + \nu$. M_μ is an off-pion-mass-shell function, including the pion propagator, so graphically ΓM_μ is as depicted in Fig. 10(a), i.e., it corresponds to the source-source term of

$$\gamma + N \to N + \mu + \nu$$

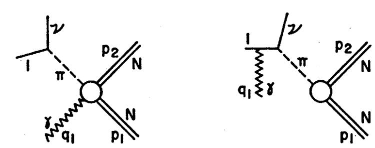


FIG. 10. (a) Current-current contribution to $\gamma+N \to l+\nu+N$ with lepton current coupled to strong interaction "blob" through a pion pole. (b) Equal-time contribution to $\gamma+N \to l+\nu+N$ with lepton current coupled to strong interaction "blob" through a pion pole.

[20] V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, Phys. Letters **21**, 576 (1966).

[21] This is the isospin antisymmetric part.

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with a one-pion pole between the strong-interaction blob and the lepton pair. It therefore falls into the category of sum rules we have just been discussing, the only difference being that we insist on having a one-pion pole between the lepton current and the strong-interaction "blob." As was previously the case, the equal-time term of the S matrix contains those graphs in which the photon is absorbed by the electron, as in Fig. 10(b). Here we are assuming USDR for the entire off-mass-shell amplitude for photoproduction, whereas in Sec. II, we needed to assume USDR for a part of the on-mass-shell amplitude to derive the sum rule. The two assumptions are clearly very different.

IV. CONCLUSION

We have discussed at great length the question of the validity of the USDR assumption as applied to the matrix elements of the retarded commutator of two currents, one of which was conserved. In principle of course, one could attempt to measure these matrix elements, by performing second-order weak-interaction experiments (by this we mean to include also one weak and one electromagnetic interaction) such as depicted in Figs. 3(a) and 3(b). Even then one would have to perform a complicated series of experiments to extract the contribution of the weak vector current from the total transition amplitude which includes (1) the unwanted equal-time term shown in Figs. 3(b) and (2) the contribution of the lepton-pair's coupling through both a vector and an axial-vector current (For a complete discussion, involving an analysis of the kinematics, see Ref. 14). In particular then, contrary to the Adler-Weisberger case, one cannot reasonably hope to avoid the USDR assumption in these cases through the relevant low-energy theorem[13] (for the current-current part).

A relation "testable" by experiment is therefore only obtainable at the moment, making use of the USDR assumption. Sum rules, such as e.g.,

$$\tfrac{1}{3}\langle r^2\rangle=\left(\frac{\mu_p-\mu_n}{2M}\right)^2+\frac{2}{\pi\alpha^2}\int_{\nu_0}^{\infty}\left[2\sigma_{1/2}{}^V-\sigma_{3/2}{}^V\right]\frac{d\nu'}{\nu'},\quad(3.37)$$

(where r^2 is the nucleon's electric form-factor charge radius and $\sigma_{1/2}{}^V$ and $\sigma_{3/2}{}^V$ are total isovector-photon-photoproduction cross sections) are not, strictly speaking, testable by experiment, as there is only one photon, so $\sigma_{1/2}$ and $\sigma_{3/2}$ cannot be separately determined in Compton scattering. Another way of saying this is that only the isovector component of the photon contributes to our sum rule, and its contribution to Compton scattering cannot be separated by experiment from that of the isoscalar photon, as they both have $I_3=0$. In practice, however, sufficient familiarity has been acquired over recent years with isobar models and the like to allow us a fairly confident analysis of sum rules such as the above.

In concluding, we would like to emphasize that we have neither proven nor disproven the use of the USDR as applied to the matrix elements of the retarded commutator of two isospin currents. We have shown, however, that the contribution of intermediate resonant states to the scattering amplitude may be different if calculated using USDR or Feynman graphs, and that this applies to the standard low-energy theorems as well.

ACKNOWLEDGMENTS

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APPENDIX: THE WARD-TAKAHASHI IDENTITY

It appears to the authors that a mistake of two factors of Z_3, the photon wave-function renormalization, was made in the original derivation of the Takahashi identity. Taking care of these Z_3's, so as to obtain the usual identities (known to be true from perturbation theory) seems an amusing exercise in the relation between transverse and timelike vector-meson scattering amplitudes, which in a sense is the subject of this paper.[23]

The difficulty is this: If one intends to identify the zero-momentum-transfer limit of the matrix element of the current with the renormalized charge e_R, then the correct current to use is the source of the renormalized photon field,

$$j_\mu(x)=\frac{e_R}{Z_3}[\bar\psi_U\gamma_\mu,\psi_U],\quad(\mathrm{A1})$$

where ψ_U is the (Z_2) unrenormalized Heisenberg electron field. The correct commutation relation of this physical current with the electron field is then

$$[\psi(x),j_0(x')]_{\mathrm{e.t.}}=\frac{e_R}{Z_3}\delta^{(3)}(\mathbf{x}-\mathbf{x}')\psi(x).\quad(\mathrm{A2})$$

Instead, Takahashi used (A2) without the Z_3. If one is not very careful, this leads (erroneously) to an extra factor in the final identity itself. A correct derivation (with this extra Z_3) goes as follows:

Consider the function

$$\langle 0|T\{j_\mu(y)\psi_R(x)\bar\psi_R(x')\}|0\rangle,\quad(\mathrm{A3})$$

where $\psi_R(x)$ is the renormalized electron field. Taking the divergence of (A3) with respect to y, using (A2) and

[23] Throughout the derivation, we shall be concerned with the full details of the Z_3 renormalizations and shall assume that all Z_1 and Z_2 renormalizations are done (namely, $Z_1=Z_2$). In particular the proper vertex is meant to be completely Z_1 renormalized.

going to momentum space, we obtain

$$k^\mu S_{FR}'(p')\Gamma_{\mu R}{}^{(I)}(p',p)S_{FR}'(p) = \frac{1}{Z_3}[S_{FR}'(p')-S_{FR}'(p)], \quad \text{(A4)}$$

where $\Gamma_{\mu R}{}^{(I)}(p',p)$ is the *renormalized improper* vertex defined by

$$\langle 0|T\{j_\mu(y)\psi(x)\bar{\psi}(x')\}|0\rangle \equiv e_R \int d^4\xi d^4\eta S_{FR}'(x-\xi) \times \Gamma_{\mu R}{}^{(I)}(\xi-y;y-\eta)S_{FR}'(\eta-x'). \quad \text{(A5)}$$

By *improper*, we mean that this vertex contains not only the proper vertex, but also k^2 times the photon propagator graphs. The improper vertex is *renormalized* in that, when $k^2=0$, $\Gamma_{\mu R}{}^{(I)}$, as seen by a physical (transverse) photon, is equal to the renormalized proper vertex at $k^2=0$. Instead of

$$\epsilon^\mu \Gamma_{\mu R}{}^{(I)}(k^2=0) = \epsilon^\mu \Gamma_\mu{}^{(P)}(k^2=0), \quad \epsilon\cdot k=0, \quad \text{(A6a)}$$

we shall prefer the equivalent notation

$$\Gamma_{\mu R}{}^{(I)}(k^2=0)|_T = \Gamma_\mu{}^{(P)}(k^2=0). \quad \text{(A6b)}$$

The symbol $|_T$, meaning the part of a function "seen" by a transverse photon, will be contrasted with $|_0$ meaning that part seen by a timelike photon (in the sense of (A6a), that is a probe with an ϵ^μ parallel to k^μ). Of course, (A6) is a consequence of the definition (A5) of $\Gamma_{\mu R}{}^{(I)}$ in terms of e_R.

To learn more about this improper vertex, we study the structure of its (Z_3) unrenormalized form $\Gamma_\mu{}^{(I)}$. Graphically $\Gamma_\mu{}^{(I)}$ is the sum of the series shown in Fig. 11, where the blobs labelled by I, P are the (Z_3-unrenormalized) improper and the proper vertex; the Π^* blob is the photon's (proper) vacuum polarization tensor. The pictorial form will serve as a reminder that $\Gamma_\mu{}^{(I)}$, $\Gamma_\mu{}^{(P)}$ are defined without external electron or photon lines. Using the Feynman gauge for simplicity, this sum is analytically

$$\Gamma_{(I)}{}^{\mu}(p',p) = \Gamma_{(P)}{}^{\mu}(p',p) + \Gamma_\lambda{}^{(P)}(p',p)\left[g^{\lambda\mu} - \frac{k^\lambda k^\mu}{k^2}\right]\frac{1}{k^2}\Pi^*(k) + \cdots$$
$$\equiv k^2\left[\frac{g^{\mu\nu}-k^\mu k^\nu/k^2}{k^2-\Pi^*(k)}\right]\Gamma_\nu{}^{(P)}(p',p) + \frac{k^\mu k^\nu}{k^2}\Gamma_\nu{}^{(P)}(p',p). \quad \text{(A7)}$$

To facilitate the summation, current conservation has

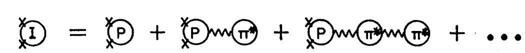

FIG. 11. Relation between unrenormalized improper vertex $\Gamma_\mu{}^{(I)}$ and the proper vertex $\Gamma_\mu{}^{(P)}$.

been used to write

$$\Pi_{\mu\nu}{}^*(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi^*(k). \quad \text{(A8)}$$

It is immediately clear from (A7) that

$$k_\mu \Gamma_{(I)}{}^\mu(p',p) = k_\mu \Gamma_{(P)}{}^\mu(p',p), \quad \text{(A9)}$$

that is, only the timelike part of the improper vertex survives in the Takahashi identity. This is obvious; in fact, Eq. (A4) could have been written

$$k^\mu S_{FR}'(p')\Gamma_{\mu R}{}^{(I)}(p',p)|_0 S_{FR}'(p) = \frac{1}{Z_3}[S_{FR}'(p')-S_{FR}'(p)]. \quad \text{(A10)}$$

Note that we cannot use (A9) to write a relation like (A10) for the proper vertex, because (A9) refers only to the unrenormalized $\Gamma^{(I)}$. To see the relation between the renormalized and unrenormalized improper vertex, we use (A7) to obtain [recall that $1-\Pi'^*(0)=Z_3{}^{-1}$]

$$\Gamma_\mu{}^{(I)}(k^2=0)|_T = Z_3\Gamma_\mu{}^{(P)}(k^2=0), \quad \text{(A11a)}$$

$$\Gamma_\mu{}^{(I)}(k^2=0)|_0 = \Gamma_\mu{}^{(P)}(k^2=0). \quad \text{(A11b)}$$

Thus we learn that, *in order to guarantee* (A6), we must divide the unrenormalized quantities $\Gamma_\mu{}^{(I)}$ through by Z_3 to obtain the renormalized $\Gamma_{\mu R}{}^{(I)}(p',p)$:

$$\Gamma_\mu{}^{(I)}(p',p) \equiv Z_3\Gamma_{\mu R}{}^{(I)}(p',p). \quad \text{(A12)}$$

Using (A11b) and (A12), we obtain finally the crucial relation

$$\Gamma_{\mu R}{}^{(I)}(p',p)|_0 = \frac{1}{Z_3}\Gamma_\mu{}^{(P)}(p',p), \quad \text{(A13)}$$

which allows us to rewrite (A10) as the familiar Takahashi identity

$$k^\mu S_{FR}'(p')\Gamma_\mu{}^{(P)}(p',p)S_{FR}'(p) = S_{FR}'(p')-S_{FR}'(p). \quad \text{(A14)}$$

A final word about the physics involved here is in order. Note that the charge of the universe [obtained from the current (A1)]

$$Q \equiv \int d\mathbf{x}\, j_0(x) \quad \text{(A15)}$$

has the commutation relation

$$[Q,\psi] = \frac{e_R}{Z_3}\psi. \quad \text{(A16)}$$

This means that Q "sees" a charge e_R/Z_3 in the state created by ψ acting on the vacuum:

$$Q\psi(x)|0\rangle=\frac{e_R}{Z_3}\psi(x)|0\rangle. \tag{A17}$$

On the other hand, this is not the *measured* charge in a (say) one-electron state

$$\langle 1|Q_{\text{exp}}|1\rangle=e_R, \tag{A18}$$

where

$$Q_{\text{exp}}=\int_{\text{experimental volume}} d\mathbf{x}j_0(x) \tag{A19}$$

Q_{exp} is measured say in some finite box, neglecting photons above some maximum wavelength, whereas $\psi|0\rangle$ contains arbitrary numbers of soft photons.

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Relation of Normal-Ordering Methods to Linked Diagrams*

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The normal-ordering method of Heffner and Louisell is extended to apply to anticommuting operators and systems with nonquadratic Hamiltonians. This extended method is shown to yield coupled ordinary differential equations for quantities which are sums of all linked-diagram contributions having the same external line configuration.

1. INTRODUCTION

CONSIDERABLE interest has been shown recently in methods of computation, sometimes called normal-ordering techniques,[1–3] in which the creation operators of Bose excitations $a_i^\dagger$ are replaced by c numbers α_i^* and the annihilation operators a_i by derivatives $\partial/\partial\alpha_i^*$. The subscript i refers to the state of the excitation. We wish to point out that this procedure is a specialization of the method of functional derivatives in quantum field theory,[4] and also that an explicit connection may be established between the trial functions involved in this method and linked-diagram theorems familiar in many-body theory.[5] It is hoped that explicit demonstration of these relationships will clarify and broaden the scope of the new techniques.

Recognizing the kinship between the derivatives mentioned above and the functional derivatives of quantum field theory, it is a simple matter to extend the normal-ordering techniques of Louisell, Walker, and Heffner to include anticommuting as well as commuting operators. This extension, described in Sec. 2, makes use of "anticommuting c numbers" γ_i^*, γ_i and their derivatives similar to those introduced by Schwinger for Fermi fields.[6,7]

The work of Louisell, Walker, and Heffner has made much use of a normal-ordered exponential trial function :exp$G$: for the various operator quantities under investigation. Here G is a finite polynomial in α_i^* and α_i. We have shown[8] that the time evolution and density operators, $U(t,t')$ and $\rho(t,\beta)$, corresponding to systems of Bose particles can have such a form only when the corresponding Hamiltonian is at most quadratic in the operators $a_i^\dagger$ and a_i. Similar results for systems of fermions and for mixed systems will be published elsewhere.[9] Section 3 includes a discussion of the method of Heffner and Louisell[3] for such systems.

For nonquadratic Hamiltonians this method may still be applied formally. In this case G is an infinite series in α_i^*, α_i, γ_i^*, and γ_i:

$$G=\sum_{stuv} G_{stuv}\alpha^{*s}\alpha^t\gamma^{*u}\gamma^v. \tag{1}$$

Here α^{*s} is an abbreviation for $\prod_i\alpha_i^{*s_i}$, a convention used throughout this paper. Notice that u_i and v_i may only equal 0 or 1 while $s_i, t_i=0, 1, 2, \cdots$. It is the principal aim of this paper to show that the coefficients G_{stuv} of G in the trial function $U(t,t')=$:exp$G$: are equal to sums of linked-diagram contributions having the same external-line configuration. Thus the normal-ordering method leads to a set of ordinary differential equations for sums of linked-diagram contributions.

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