Current Commutators in Quantum Electrodynamics*

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The operator structure of the singular terms in the equal-time commutator of space and time components of the electromagnetic current is investigated in perturbation theory by establishing a connection with Feynman diagrams. It is made very plausible that the singular term is a c number. Some remarks are made about the same problem in the electrodynamics of a spinless particle.

I. INTRODUCTION

As first pointed out by Schwinger,[1] singular terms must be expected in the vacuum expectation values of equal-time commutators of space and time components of the electromagnetic current and gave an explicit proof of this for the case of the noninteracting Dirac field. Johnson[2] demonstrated that this was the case for interacting fields on the basis of Lorentz invariance and current conservation. Since current commutators have been applied widely with considerable success during the last two years, it is of more than academic interest to examine the structure of the singular (or Schwinger) terms.

In order to investigate whether the Schwinger term is an operator or a c number, we evaluate some off-diagonal matrix elements of current commutators in perturbation theory in quantum electrodynamics. In the latter case, the Schwinger term would have no physically observable effects. In the Appendix we discuss the electrodynamics of a spinless particle which is harder to interpret and of less interest than the spin-$\frac{1}{2}$ theory because the current is not analogous to a quark current.

II. FORMALISM

We can calculate the equal-time commutator by writing the current defined by $\Box A_\mu = j_\mu$ in terms of renormalized Heisenberg fields and employing the equal-time commutation rules in a straightforward manner. The current is $j_\mu(x)=(Z_1/Z_3)e\bar{\psi}\gamma_\mu\psi$ and this gives immediately

$$[j_\nu(\mathbf{x},0),\, j_0(0)]=0. \tag{1}$$

Therefore, we define the matrix element of the Schwinger term as $\langle\alpha^{(-)}|[j_\nu(\mathbf{x},0),\, j_0(0)]|\beta^{(+)}\rangle$ computed from the Feynman amplitude.

We can establish a connection between the matrix element $\langle\alpha^{(-)}|[j_\nu(\mathbf{x},0),\, j_0(0)]|\beta^{(+)}\rangle$ and the Feynman amplitude for the process $\beta\rightarrow\alpha+\gamma+\gamma$ in the following two different ways.[3]

The S-matrix element is

$$\langle\alpha;k_1,\epsilon_1;k_2,\epsilon_2{}^{(-)}|\beta^{(+)}\rangle = 1-i(2\pi)^4\delta^4(p_\alpha+k_1+k_2-p_\beta)\times\frac{\epsilon_1{}^\mu}{(2\omega_1)^{1/2}}\langle\alpha;k_2,\epsilon_2{}^{(-)}|j_\mu(0)|\beta^{(+)}\rangle.$$

We define the Feynman amplitude $\mathfrak{M}$ by

$$\begin{aligned}\mathfrak{M}&=-i\epsilon_1{}^\mu(2\omega_2\Pi 2E_\alpha 2E_\beta)^{1/2}\langle\alpha;k_2,\epsilon_2{}^{(-)}|j_\mu(0)|\beta^{(+)}\rangle\\&=-\epsilon_1{}^\mu\epsilon_2{}^\nu(\Pi 2E_\alpha 2E_\beta)^{1/2}\int d^4x\, e^{ik_2\cdot x}\Box\times\langle\alpha^{(-)}|T[A_\nu(x)\, j_\mu(0)]|\beta^{(+)}\rangle\\&\equiv\epsilon_1{}^\mu\epsilon_2{}^\nu\mathfrak{M}_{\mu\nu},\end{aligned}$$

and we break up $\mathfrak{M}_{\mu\nu}$ according to $\mathfrak{M}_{\mu\nu}\equiv\mathfrak{M}_{\mu\nu}{}^{(T)}+\mathfrak{M}_{\mu\nu}{}^{(P)}$, where

$$\mathfrak{M}_{\mu\nu}{}^{(T)}=-(\Pi 2E_\alpha 2E_\beta)^{1/2}\int d^4x\, e^{ik_2\cdot x}\times\langle\alpha^{(-)}|T[j_\nu(x)j_\mu(0)]|\beta^{(+)}\rangle,$$

$$\mathfrak{M}_{\mu\nu}{}^{(P)}=-(\Pi 2E_\alpha 2E_\beta)^{1/2}\int d^3\mathbf{x}\, e^{-i\mathbf{k}_2\cdot\mathbf{x}}\times\langle\alpha^{(-)}|[\dot{A}_\nu(\mathbf{x},0)+i\omega_2A_\nu(\mathbf{x},0),\, j_\mu(0)]|\beta^{(+)}\rangle.$$

Now let k_2 remain fixed as $\omega_2\rightarrow\infty$ and use the identity

$$\begin{aligned}j_\nu(x)&=e^{iHx_0}j_\nu(\mathbf{x},0)e^{-iHx_0}\\&=j_\nu(\mathbf{x},0)+ix_0[H,\, j_\nu(\mathbf{x},0)]+\frac{1}{2!}(ix_0)^2[H,[H,\, j_\nu(\mathbf{x},0)]]+\cdots\end{aligned}$$

to perform the integration over x_0 in $\mathfrak{M}_{\mu\nu}{}^{(T)}$. A series of decreasing integral powers of ω_2 results with the leading term $(1/\omega_2)X_{\mu\nu}$,

$$X_{\mu\nu}=-i(\Pi 2E_\alpha 2E_\beta)^{1/2}\int d^3\mathbf{x}\, e^{-i\mathbf{k}_2\cdot\mathbf{x}}\times\langle\alpha^{(-)}|[j_\nu(\mathbf{x},0),\, j_\mu(0)]|\beta^{(+)}\rangle.$$

Inverting the Fourier transform, we find

$$\langle\alpha^{(-)}|[j_\nu(x,0),\, j_\mu(0)]|\beta^{(+)}\rangle=\frac{i}{(\Pi 2E_\alpha 2E_\beta)^{1/2}}\int\frac{d^3k_2}{(2\pi)^3}e^{i\mathbf{k}_2\cdot\mathbf{x}}X_{\mu\nu}(\mathbf{k}_2,p_\alpha,p_\beta). \tag{2}$$

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[1] J. Schwinger, Phys. Rev. Letters 3, 296 (1959).
[2] K. Johnson, Nucl. Phys. 25, 431 (1961).
[3] See, e.g., J. D. Bjorken, Phys. Rev. 148, 1467 (1966).

The other relation between the commutator and the amplitude is based on current conservation. By translational invariance

$$\mathfrak{M}_{\mu\nu}{}^{(T)}=-(\Pi 2E_\alpha 2E_\beta)^{1/2}\int d^4x\, e^{-ik_1\cdot x}\times\langle\alpha^{(-)}|T[j_\nu(0)j_\mu(-x)]|\beta^{(+)}\rangle.$$

Therefore,

$$\frac{k_1{}^\mu\mathfrak{M}_{\mu\nu}{}^{(T)}}{(\Pi 2E_\alpha 2E_\beta)^{1/2}}=-\int d^4x(i\partial^\mu e^{-ik_1\cdot x})\langle\alpha^{(-)}|T[j_\nu(0)j_\mu(-x)]|\beta^{(+)}\rangle$$

$$=i\int d^3x\, e^{i\mathbf{k}_1\cdot\mathbf{x}}\langle\alpha^{(-)}|[j_\nu(0),\, j_0(-\mathbf{x},0)]|\beta^{(+)}\rangle$$

$$=i\int d^3x\, e^{-i\mathbf{k}_2\cdot\mathbf{x}}\langle\alpha^{(-)}|[j_\nu(\mathbf{x},0),\, j_0(0)]|\beta^{(+)}\rangle.$$

Inverting the Fourier transform and using current conservation in the form $k_1{}^\mu\mathfrak{M}_{\mu\nu}=0$, which implies $k_1{}^\mu\mathfrak{M}_{\mu\nu}{}^{(T)}=-k_1{}^\mu\mathfrak{M}_{\mu\nu}{}^{(P)}$, we obtain

$$\langle\alpha^{(-)}|[j_\nu(x,0),\, j_0(0)]|\beta^{(+)}\rangle=\frac{i}{(\Pi 2E_\alpha 2E_\beta)^{1/2}}\int\frac{d^3k_2}{(2\pi)^3}e^{i\mathbf{k}_2\cdot\mathbf{x}}k_1{}^\mu\mathfrak{M}_{\mu\nu}{}^P(k_2,p_\alpha,p_\beta).\quad(3)$$

In using Eq. (2) or Eq. (3), $\mathfrak{M}_{\mu\nu}$ is written down from the Feynman rules and its asymptotic form is computed in the limit $\omega_2\to\infty$ with k_2 fixed. $\mathfrak{M}_{\mu\nu}{}^{(P)}$ is at most a linear polynomial in ω_2 and is easily isolated as the part of $\mathfrak{M}_{\mu\nu}$ which does not tend to zero in this limit.

Equation (3) may puzzle the reader for two reasons. First, it is not obvious that it agrees with Eq. (2) and second, the right-hand side appears to depend on ω_2, which the left-hand side clearly must not. Since $\mathfrak{M}_{\mu\nu}{}^{(P)}$ must have the form

$$\mathfrak{M}_{\mu\nu}{}^{(P)}(k_2,p_\alpha,p_\beta)=A_{\mu\nu}(k_2,p_\alpha,p_\beta)+\omega_2B_{\mu\nu}(k_2,p_\alpha,p_\beta),$$

we have as $\omega_2\to\infty$, k_2 fixed,

$$\mathfrak{M}_{\mu\nu}\to\omega_2B_{\mu\nu}+A_{\mu\nu}+\omega_2{}^{-1}X_{\mu\nu}+O(\omega_2{}^{-2}).$$

Now, using $k_1=p_\beta-p_\alpha-k_2\equiv q-k_2$ we can expand $k_1{}^\mu\mathfrak{M}_{\mu\nu}$ in powers of $\omega_2{}^{-1}$ to obtain[4]

$$0=k_1{}^\mu\mathfrak{M}_{\mu\nu}=-\omega_2{}^2B_{0\nu}+\omega_2(q_0B_{0\nu}-A_{0\nu}+k_1{}^iB_{i\nu})+(q_0A_{0\nu}-X_{0\nu}+k_1{}^iA_{i\nu})+O(\omega_2{}^{-1}).$$

This yields, of course,

$$B_{0\nu}=0,$$
$$q_0B_{0\nu}-A_{0\nu}+k_1{}^iB_{i\nu}=0,$$
$$X_{0\nu}=q_0A_{0\nu}+k_1{}^iA_{i\nu}.$$

But

$$k_1{}^\mu\mathfrak{M}_{\mu\nu}{}^{(P)}=-\omega_2{}^2B_{0\nu}+\omega_2(q_0B_{0\nu}-A_{0\nu}+k_1{}^iB_{i\nu})+(q_0A_{0\nu}+k_1{}^iA_{i\nu}).$$

Comparing this to the three equations above, we see that $k_1{}^\mu\mathfrak{M}_{\mu\nu}{}^{(P)}=X_{0\nu}$ and this resolves both of our apparent difficulties at the same time.

In order to apply our formalism, we must study reactions with two photons in the final state. Processes which can be obtained from these by crossing naturally would give no new information. The simplest cases to consider to lowest order are pair annihilation and photon-photon scattering. The latter would be expected to give a nonzero result for virtually any operator Schwinger term.

III. RESULTS

For electron-positron annihilation, Eqs. (1), (2), and (3) all yield to second order

$$\langle 0|[j_\nu(\mathbf{x},0),\, j_0(0)]|p,s;\bar{p},\bar{s}\rangle=0.$$

For Delbrück scattering, Eq. (1) gives $\langle k_2,\epsilon_3;k_4,\epsilon_4|\times[j_\nu(\mathbf{x},0),\, j_0(0)]|0\rangle=0$ and Eq. (3) is much more convenient than Eq. (2) for comparison with the result expected from the Feynman amplitude. In spite of the fact that $\mathfrak{M}_{\mu\nu\lambda\sigma}(k_1,k_2,k_3,k_4)$ converges, regulators must be used to enforce gauge invariance.[5] This amounts to using the amplitude

$$\overline{\mathfrak{M}}_{\mu\nu\lambda\sigma}(k_1,k_2,k_3,k_4)\equiv\mathfrak{M}_{\mu\nu\lambda\sigma}(k_1,k_2,k_3,k_4)-\mathfrak{M}_{\mu\nu\lambda\sigma}(0,0,0,0).$$

Now in computing $\mathfrak{M}_{\mu\nu\lambda\sigma}{}^{(P)}(k_1,k_2,k_3,k_4)$, we may set $k_3=k_4=0$. Since[5]

$$\mathfrak{M}_{\mu\nu\lambda\sigma}(k_1,k_2,0,0)=\mathfrak{M}_{\mu\nu\lambda\sigma}(0,0,0,0)$$

we obtain immediately

$$\mathfrak{M}_{\mu\nu\lambda\sigma}{}^{(P)}(k_1,k_2,k_3,k_4)=\mathfrak{M}_{\mu\nu\lambda\sigma}(0,0,0,0).$$

Therefore, $\overline{\mathfrak{M}}_{\mu\nu\lambda\sigma}{}^{(P)}(k_1,k_2,k_3,k_4)=0$ and again we get no Schwinger term.

IV. CONCLUDING REMARKS

If we are prepared to ignore the fact that the introduction of regulators is a purely formal device whose effect on the current is unclear, we can postulate with confidence that the Schwinger term is a c number.

[4] Roman indices refer to space components only.

[5] K. Johnson, *Lectures on Particles and Field Theory* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964), Vol. 2, pp. 71, 72.

Boulware[6] reached the same conclusion by a completely different method.

We could also use Eq. (2) for Delbrück scattering, but since it necessitates finding the next to dominant terms of the amplitude, the calculation would be much more tedious and its independence of regulators only apparent because we must, in principle, regularize the complete amplitude before finding its limiting behavior. Pair annihilation to fourth order, besides being very messy, has an infrared divergence which cancels out only in the cross section and is not expected to do so in the commutator. It is very unlikely, in any case, that a Schwinger term which gives a null result in photon-photon scattering would show up here.

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APPENDIX

In the electrodynamics of a spinless particle, the current is

$$j_\mu(x)=(Z_1/Z_3)[ie(\varphi\partial_\mu\varphi^*-\varphi^*\partial_\mu\varphi)-2e^2A_\mu\varphi^*\varphi]$$

and therefore

$$[j_i(\mathbf{x},0),\,j_0(0)]=-2ie^2(Z_1/Z_3)^2[\delta^3(\mathbf{x})(\varphi^*\partial_i\varphi+\varphi\partial_i\varphi^*)-\varphi^*(0)\varphi(0)\partial_i\delta^3(\mathbf{x})]. \quad (4)$$

We now define the matrix element of the Schwinger term as $\langle\alpha^{(-)}|[j_i(\mathbf{x},0),\,j_0(0)]|\beta^{(+)}\rangle$, computed from Eq. (4) subtracted from the same quantity calculated from Eq. (2) or Eq. (3).

For pair annihilation Eqs. (2), (3), and (4) all give

$$\langle 0|[j_i(\mathbf{x},0),\,j_0(0)]|q,\bar{q}\rangle=[2e^2/(4\omega_q\omega_{\bar{q}})^{1/2}][i\partial-(q+\bar{q})]_i\delta^3(\mathbf{x}).$$

Photon-photon scattering is complicated by the presence of seagull diagrams. Some of these are independent of k_2 and would therefore have to be evaluated exactly if we used Eq. (3). The calculation using Eq. (2) is considerably more tedious than the spin-$\frac{1}{2}$ case and occasionally it is necessary, in order to avoid spurious singularities, to break up the region of integration every Feynman parameters and to approximate the integrand differently in the two regions. As an example of this, we would write

$$\int_0^1 dz\int_z^1 dx\int_z^x dy \approx \int_\epsilon^1 dz\int_z^1 dx\int_z^x dy+\int_0^\epsilon dz\int_0^1 dx\int_0^x dy\,,$$

where $0<\epsilon\ll1$. A typical expression obtained after introducing Feynman parameters is

$$a^2=\mu^2-z(1-z)k_2{}^2-y(1-y)k_3{}^2-x(1-x)k_4{}^2-2z(1-y)k_2\cdot k_3-2z(1-x)k_2\cdot k_4-2y(1-x)k_3\cdot k_4.$$

In the first region $a^2\approx -z(1-z)\omega_2{}^2$, whereas in the second region $a^2\approx\mu^2-2y(1-x)k_3\cdot k_4-z\omega_2{}^2$. If we take $k_3\cdot k_4=0$ to save labor, we obtain

$$\langle k_3,\epsilon_3;\,k_4,\epsilon_4|[j_i(\mathbf{x},0),\,j_0(0)]|0\rangle=-(1/(4\omega_3\omega_4)^{1/2})(e^4/18\pi^2)[\epsilon_{3i}\epsilon_{40}(9\omega_3-\omega_4)+\epsilon_{4i}\epsilon_{30}(9\omega_4-\omega_3)]\delta^3(\mathbf{x}).$$

We have not succeeded in comparing this with Eq. (4) because evaluating the latter between the same states diverges. Consider, for example,

$$(e^2/(4\omega_3\omega_4)^{1/2})\epsilon_3{}^\mu\epsilon_4{}^\nu A_{\mu\nu}\equiv\langle k_3,\epsilon_3;\,k_4,\epsilon_4|\,\varphi(0)\varphi^*(0)|0\rangle=\sum_n\langle k_3,\epsilon_3;\,k_4,\epsilon_4|\,\varphi|n\rangle\langle n|\,\varphi^*|0\rangle.$$

To lowest order $|n\rangle=|q\rangle$ and $\langle q|\,\varphi^*|0\rangle=(1/(2\omega_q)^{1/2})$,

$$\langle k_3,k_4|\,\varphi|q\rangle=\frac{\langle k_3,k_4|J|q\rangle}{\mu^2-(k_3+k_4-q)^2}\,,$$

where $(\Box+\mu^2)\varphi(x)=J(x)$. Thus,

$$\langle k_3,k_4|J|q\rangle=\frac{-ie^2}{(8\omega_q\omega_3\omega_4)^{1/2}}\left[\frac{-i\epsilon_4\cdot(2q-2k_3-k_4)\epsilon_3\cdot(2q-k_3)}{(q-k_3)^2-\mu^2}+\frac{-i\epsilon_3\cdot(2q-2k_4-k_3)\epsilon_4\cdot(2q-k_4)}{(q-k_4)^2-\mu^2}+2i\cdot g_{\mu\nu}\right]$$

and $A_{\mu\nu}$ is given by the divergent integral

$$A_{\mu\nu}=\int\frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\frac{1}{2[q\cdot(k_3+k_4)-k_3\cdot k_4]}\times\left[\frac{(2q-2k_3-k_4)_\nu(2q-k_3)_\mu}{2q\cdot k_3}+\frac{(2q-2k_4-k_3)_\mu(2q-k_4)_\nu}{2q\cdot k_4}+2g_{\mu\nu}\right].$$

[6] D. G. Boulware, Phys. Rev. **151**, 1024 (1966).