Electrodisintegration of the Deuteron. I. Connection between the *n-p-d* Vertex Function and the Deuteron Wave Function*

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Higher-order relativistic corrections to the Durand theory for the electrodisintegration of the deuteron are considered, based on the Mandelstam representation for the transition matrix element. The analysis concentrates on the kinematical region corresponding to the broad quasi-elastic peak in the cross section $d^2\sigma/d\Omega_e de_0'$, where e_0' is the final electron energy. The presence of anomalous thresholds and the close connection between the nonrelativistic wave functions and the spectral functions in the anomalous region allow the relativistic expression to be recast in terms of ostensibly nonrelativistic wave functions. The calculation is facilitated by separating the neutron-proton-deuteron vertex function into angular momentum components corresponding to momentum-space wave functions. The result clarifies the role of certain off-mass-shell effects, particuarly those contributions which involve antiparticles in the intermediate state. The antiparticle contributions are found to be small for $q^2 \lesssim 0.8$ (BeV/c)^2. The cross sections $d^2\sigma/d\Omega_e de_0'$ and $d^3\sigma/d\Omega_e de_0' d(\cos\theta)$ are presented in first Born approximation but the detailed inclusion of the effects of final-state interactions on the cross sections is reserved for a subsequent paper.

I. INTRODUCTION

THE theory of the electrodisintegration of the deuteron has received a good deal of attention in past years because of the importance of the process for the determination of the electromagnetic form factors of the neutron.[1] The relativistic description of the process, based on the initial papers of Durand,[2,3] allows a clear understanding of the main features of the interaction. The theory uses dispersion relations to display the leading relativistic corrections and to assure that the dominant contributions to the cross section are calculated exactly. Remaining corrections from the effects of final-state interactions are included by developing a semirelativistic interaction Hamiltonian for use with nonrelativistic wave functions.

Specifically, the one-photon-exchange approximation is assumed,[4] so that the basic problem reduces to the calculation of the transition amplitude $\langle np|j_\mu|d\rangle$, where j_μ is the operator describing the electromagnetic current in the deuteron.[5] On the basis of singularities found in perturbation theory, this amplitude is expected to satisfy a Mandelstam representation with anomalous thresholds.[6] The leading contribution to the transition amplitude comes from the pole terms[7] and to a lesser extent from the single dispersion integrals which appear in the usual three variables: $s=-(d+q)^2=-(p+n)^2$, $t=-(d-n)^2$, $u=-(d-p)^2$. There are also double dispersion integrals in the three pairs of variables. The pole terms are easily shown to correspond to contributions from the asymptotic part of the deuteron wave function. Furthermore, a close connection exists between contributions of the single dispersion integrals in the anomalous region and the wave functions at intermediate ranges.[8,9] Once the detailed connection with the wave functions is known, the transition amplitude can be recast in terms of nonrelativistic wave functions provided the kinematic variables are interpreted correctly.

In this paper we explore several higher-order corrections to the Durand theory which could be important in view of the improved experimental accuracy available, and the extension of electrodisintegration experiments to higher energies. One important problem is the connection between the neutron-proton-deuteron vertex function with one of the nucleons off the mass shell and the deuteron wave function. Several authors have established the connection between the *n-p-d* vertex function with all particles on the mass shell and the asymptotic part of the deuteron wave function.[8,10] Their results relate on-mass-shell form factors of the *n-p-d* vertex to the wave-function normalization and asymptotic *D*-to-*S* ratio. However, the connection between the wave functions and the *n-p-d* vertex function has been fully explored only recently.[11–13]

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[1] A summary of experimental and theoretical work is given in *Nucleon Structure*, edited by R. Hofstadter and L. I. Schiff (Stanford University Press, Stanford, California, 1964). More recent calculations are quoted in Refs. 16, 18, and 38. The present experimental status is summarized in J. R. Dunning *et al.*, Phys. Rev. **141**, 1286 (1966); E. B. Hughes *et al.*, *ibid.* **146**, 973 (1966); Ref. 17.

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In the present paper we obtain a unique correspondence between the vertex function and the wave functions including the antiparticle contributions[11] from such processes as $\bar{p}+d \rightarrow n$, in addition to the usual S and D components of the deuteron wave function. These results for the vertex function have been used in the calculation of the transition amplitude to ascertain the relative contribution of the antiparticle states to the electrodisintegration. The calculation shows that the largest antiparticle contributions to the cross section appear in terms of order $\mathbf{n}\cdot\mathbf{q}/m^2$ ($\mathbf{n}$, $\mathbf{q}$ are spectator momentum and momentum transfer in the lab frame, and m is the nucleon mass). Such terms are quite small and can safely be dropped relative to the leading terms.

The remaining corrections to the cross sections at the quasi-elastic peak arise from small final-state interactions between the outgoing nucleons. These corrections are associated with the Mandelstam double dispersion integrals, and would be calculated from first principles in a full dispersion theoretic calculation. Again, however, one can make use of the close connection between the nonrelativistic wave functions for the neutron-proton system and the double spectral function.[14] Hence, to calculate the small final-state interaction corrections, we construct from the basic relativistic form, a semirelativistic interaction Hamiltonian correct to order $m^{-2}(v^2/c^2)$ for use with approximate wave functions for the initial and final two-nucleon system.

The fact that we are able to use wave functions in the relativistic formalism simplifies the calculation enormously. At the same time, it permits maximum use of the accumulated knowledge of the nucleon-nucleon interaction via the introduction of semiphenomenological wave functions. Other dispersion-theoretic calculations of the electrodisintegration of the deuteron have been made using different approaches.[15–19] However, the calculations use various approximation schemes for determining the spectral functions. They are therefore less direct than the wave-function approach which uses the known connection between the relativistic spectral functions in the anomalous region and nonrelativistic wave functions.

The interaction Hamiltonian obtained in this analysis contains several interesting terms. In particular, it is found that a few higher-order terms arise from small spin-rotation corrections for the nucleons in the deuteron. The origin of these terms is demonstrated by constructing a simple model for the deuteron out of two free-nucleon states. The resulting "deuteron" wave function is fully relativistic but unbound.

In Sec. II we discuss the form of the Mandelstam representation for the transition amplitude. This discussion provides the motivation for the wave-function decomposition of the n-p-d vertex function containing an off-mass-shell nucleon given in Sec. IIB. An effective interaction, correct to order m^{-2}, is developed in Sec. III for calculation of the small final-state interaction corrections using approximate wave functions in the initial and final state. In Sec. IV, we present the usual unpolarized cross sections $d^3\sigma/d\Omega_e de_0' d(\cos\theta)$ and $d^2\sigma/d\Omega_e de_0'$ for the scattering of an electron into an element of solid angle $d\Omega_e$ and energy interval de_0' about the final energy e_0'. The former cross section involves the detection of the proton in coincidence with the final electron. The origin and magnitude of individual terms is discussed and compared with previous results. However, the calculation of the effects of final-state interactions and the azimuthal dependence of the cross section are reserved for a subsequent paper. Results of the calculation are summarized in Sec. V. Appendix I contains the general connection between the angular momentum components of wave functions and the n-p-d vertex function invariants described in Sec. II. Appendix II indicates the algebraic details of the effective interaction expansion to order m^{-2} using free-nucleon electromagnetic currents. In Appendix III, a model "deuteron" wave function is constructed from free-nucleon wave functions. The model demonstrates validity of the interaction expansion in Appendix II, and indicates the origin of a small spin-rotation correction for the nucleons's spin in a moving deuteron.

II. DISPERSION RELATIONS FOR THE TRANSITION AMPLITUDE

A. Introduction

We will be concerned primarily with the electrodisintegration of the deuteron near the peak region of the inelastic continuum. The peak occurs near final electron energies corresponding to elastic electron-nucleon scattering. It results essentially from the quasi-elastic scattering off the individual nucleons in the deuteron but is broadened by the Fermi momentum of the nucleons in the bound state. Because the nucleons are, on the average, rather far apart for the peak condition, the analysis depends mainly on the long-range structure of the deuteron rather than its lesser-known short-range structure. In addition, one expects fewer uncertainties due to meson-exchange effects. The nucleon pole terms, Fig. 1 and to a lesser extent the single dispersion integrals in t and u account for the

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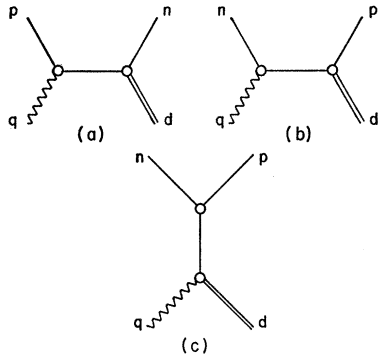

FIG. 1. Diagrams corresponding to the single-particle pole terms in the Mandelstam representation for the transition amplitude $\langle np|j_\mu|d\rangle$.

main contribution to the peak cross sections,[7] while the s-channel contributions affect only transitions to the final ${}^3S_1$ and ${}^3D_1$ states of the nucleons.[20] Accordingly, we consider in detail the t-channel scattering; the analysis of the u channel is essentially the same.

In the t channel, the one-particle singularities of the matrix element $\langle np|j_\mu|d\rangle$ arise in perturbation theory from diagrams which separate into two parts by cutting a single proton line. Figure 2 gives some sample graphs of this type, each containing a single proton as the intermediate state. Figures 2(a), 2(b) and 3 represent off-mass-shell corrections to the electromagnetic vertex function $\Gamma_p{}^\mu$, Fig. 2(c) is a correction to the n-p-d vertex function $\Gamma_{npd}(t)$, and Fig. 2(d) is one of the diagrams contributing to the complete proton propagator $S_F'(d-n)$. The total contribution from all such graphs may be expressed as the product of three factors: the n-p-d vertex function with the off-mass-shell proton, the complete proton propagator, and the electromagnetic vertex function for the off-mass-shell proton.

The leading terms from this set of graphs will of course be the pole terms, where all quantities are evaluated on the mass shell. The residue at the proton

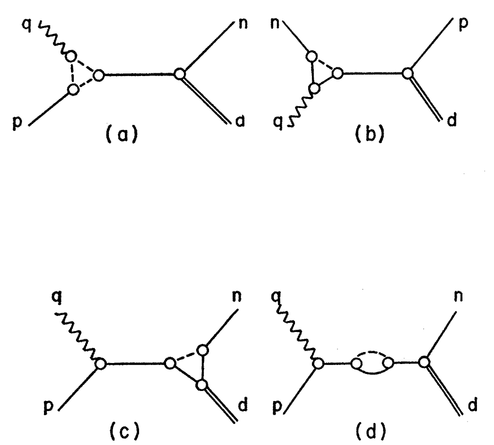

FIG. 2. Diagrams containing a single nucleon in the intermediate state which contribute to the absorptive part of the single dispersion relations in t. Dashed lines represent pions.

pole, $t=m^2$ is

$$J_\mu{}^p=\bar{u}(p)\Gamma_p{}^\mu(q,m)[-i\gamma\cdot(d-n)+m]\times\Gamma_{npd}(m^2)u^c(n),\quad(1)$$

where the truncated electromagnetic vertex function for the on-mass-shell proton at the pole is

$$\Gamma_p{}^\mu(q,m)=-ie\left[\gamma_\mu F_{1p}(q^2)+\frac{\kappa_p}{2m}F_{2p}\sigma_{\mu\nu}q_\nu\right].\quad(2)$$

This pole term contribution involving free-nucleon form factors can be calculated exactly. It is expressible in terms of the on-mass-shell form factors of the n-p-d vertex function which in turn are directly related to the asymptotic properties of the nonrelativistic deuteron wave function.

The full expression for the single-particle graphs is

$$\begin{aligned}\langle np|j_\mu|d\rangle_{\text{single proton}}&=\bar{u}(p)\Gamma_p{}^\mu(q,d-n)S_F'\Gamma_{npd}(t)u^c(n)\\&=\bar{u}(p)\Gamma_p{}^\mu(q,d-n)S_F\langle n|f_p|d\rangle\\&=\bar{u}(p)\Gamma_p{}^\mu(q,d-n)\langle n|\psi_p|d\rangle.\end{aligned}\quad(3)$$

The vertex function $\Gamma_p{}^\mu(q,d-n)$ reduces to the simple form Eq. (2) only at the pole. S_F, S_F' are the free and interacting propagators, respectively, and $\langle n|f_p|d\rangle$ is the full n-p-d vertex with the off-mass-shell proton.[21] ψ_p represents the proton field operator, hence $\langle n|\psi_p|d\rangle$ is just the momentum-space representation of the proton in the deuteron. Equations (3) are equivalent forms for the transition amplitude with no approximation, except for the absence of the double dispersion relation contributions, which are discussed separately below.

The free propagator has only a pole at $t=m^2$ and the vertex function $\Gamma_p{}^\mu$ is analytic in the t-plane cut from the normal threshold at $t=(m+\mu)^2$ to $t=\infty$ (μ is the pion mass). On the other hand, the n-p-d vertex function $\langle n|f_p|d\rangle$ has an anomalous threshold beginning at $t_0=m^2+2\mu(\mu+2\alpha)$, where α is related to the deuteron binding energy ϵ, $\alpha=(m\epsilon)^{1/2}$. Hence the transition amplitude is analytic in the cut t plane from the anomalous threshold at t_0 to $t=\infty$, and we may write for it a once-subtracted dispersion relation. If, however, we restrict the calculation to include only on-mass-shell behavior of the electromagnetic vertex function, we obtain an immediate factorization of the dispersion relation for the transition amplitude. That is, the electromagnetic vertex function, which depends in this approximation only on q and not on t, appears as an overall factor multiplying both the pole term and the single dispersion integral. As a consequence, the calculation of the single-particle contributions to the transition amplitude requires only the evaluation of the

[20] Since the deuteron pole affects only transitions to final ${}^3S_1$ and ${}^3D_1$ states of the two nucleons, Durand (Ref. 2) has included these contributions in final-state interaction effects.

[21] The vertex functions in Eq. (3) are discussed in more detail by Durand (Ref. 2, Sec. IV). For present purposes, it suffices to know merely the general character of the Mandelstam representation for $\langle np|j_\mu|d\rangle$.

dispersion relation for the wave function, $\langle n|\psi_p|d\rangle$. It is known, however, that a close connection exists between the nonrelativistic deuteron wave function and the spectral function of the single dispersion integrals for the n-p-d vertex in the anomalous region.[8,10] This is the motivation for our analysis in Sec. IIb of the n-p-d vertex function in terms of angular momentum components of momentum-space wave functions. The possibility of using wave functions in the relativistic formalism offers enormous simplification since, as will be seen, no dispersion integrals need actually be calculated.

The double dispersion integrals in the Mandelstam representation for the transition amplitude describe the effects of final-state interaction (FSI) between the nucleons and the contributions from meson exchange currents, Fig. 3(b). The latter corrections have not been explored in detail but are expected to be small near the quasi-elastic peak. An example of the FSI correction is shown in Fig. 3(a). The contributions from such a diagram can be written for the s and t channel as

$$\frac{1}{\pi^2}\int\int\frac{\eta(s',t')ds'dt'}{(s'-s)(t'-t)}, \tag{4}$$

where $\eta(s',t')$ represents the double spectral function. It is easily verified that the diagram in Fig. 3(a) has a normal threshold in s at $s=4m^2$, and an anomalous threshold in t at $t_0=m^2+2\mu(\mu+2\alpha)$.

If now one neglects off-mass-shell behavior at the electromagnetic vertex, a factorization again occurs in the dispersion relation, since the electromagnetic vertex function can be extracted from the integral over t and s. Moreover, the resulting double dispersion integral has the form characteristic of the nonrelativistic wave functions for the initial and final two-nucleon state. This is more evident if the denominator is recast in terms of momentum variables in the center-of-mass frame of the outgoing nucleons.

$$(s'-s)(t'-t)=8(p'^2-p^2)[\sigma^2+(\mathbf{p}-\tfrac{1}{2}\mathbf{q})^2-\tfrac{1}{4}q_0^2]. \tag{5}$$

The new dispersion variables σ^2 and p'^2 are associated with the deuteron wave function and the final neutron-proton wave function, respectively. The result is practically identical to the form given in the nonrelativistic analysis of Martin *et al.*[14] who showed that for a dispersion integral of this type, there is an equality between the nonrelativistic wave functions and the double spectral functions in the anomalous region. For the correspondence with wave functions here, we note that p is the relativistic nucleon momentum and q_0 is essentially zero at the quasi-elastic peak, hence $|\mathbf{q}|^2=q^2$.

This close connection as well as the smallness of FSI effects is a compelling reason to include such corrections in a semirelativistic approximation. We do this in Sec. III by developing an effective interaction Hamiltonian for use with approximate wave functions for the initial and final state.

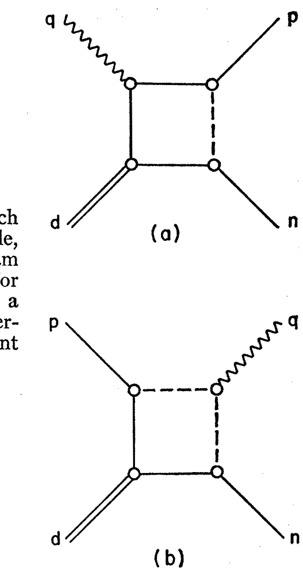

FIG. 3. Typical diagrams which are not single-particle reducible, and contribute to the Mandelstam double spectral functions for $\langle np|j_\mu|d\rangle$. These represent (a) a correction for final-state interactions, and (b) a meson-current term.

B. Angular Momentum Classification of the n-p-d Vertex Function

Blankenbecler and Cook[8] showed that the n-p-d vertex function is expressible in terms of four form factors when one of the nucleons is off the mass shell. For an off-mass-shell proton, the vertex function has the form[22]

$$\langle n|f_p|d\rangle=\left\{F(t)i\gamma\cdot\xi+G(t)\frac{n\cdot\xi}{m}+\frac{[i\gamma\cdot(d-n)+m]}{2m}\times\left[H(t)i\gamma\cdot\xi+I(t)\frac{n\cdot\xi}{m}\right]\right\}u^c(n). \tag{6}$$

Here ξ_μ is the complex polarization vector describing the deuteron spin, $\xi\cdot d=0$, and $u^c(n)=[\bar{u}(n)C]^T$, where $C=\gamma_2\gamma_4$ is the charge-conjugation matrix.[23] F, G, H, and I are form factors depending on the momentum-transfer variable $t=-(d-n)^2$, and satisfy dispersion relations with anomalous thresholds:

$$\begin{aligned}F(t)&=F(m^2)-\frac{(m^2-t)}{\pi}\int_{t_0}^{\infty}\frac{\mathrm{Im}F(t')dt'}{(t'-t)(t'-m^2)},\\ G(t)&=G(m^2)-\frac{(m^2-t)}{\pi}\int_{t_0}^{\infty}\frac{\mathrm{Im}G(t')dt'}{(t'-t)(t'-m^2)},\\ H(t)&=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{\mathrm{Im}H(t')dt'}{t'-t},\\ I(t)&=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{\mathrm{Im}I(t')dt'}{t'-t}.\end{aligned} \tag{7}$$

[22] Equation (6) corresponds to Eq. (2.23) of Ref. 8 with extra factors of m^{-1} inserted so that the form factors are all dimensionless.

[23] Spinors are normalized as $\bar{u}(p)\gamma_\mu u(p)=-2ip_\mu$, $\bar{u}(p)u(p)=2m$, where $\bar{u}(p)=u^\dagger(p)\gamma_4$. We use the standard representation for the Dirac matrices, $\boldsymbol{\gamma}=-i\beta\boldsymbol{\alpha}$, $\gamma_4=\beta$.

At $t=m^2$, only the terms $F(m^2)$ and $G(m^2)$ survive in the n-p-d vertex function Eq. (6). These constants are expressible in terms of the deuteron binding energy, the asymptotic D-to-S ratio,[8,10] and the effective range for n-p scattering. The results are given in Eq. (24).

For the case of scalar particles, the connection between the n-p-d vertex function and the deuteron wave function has been studied by Blankenbecler and Cook,[8] and Bertocchi *et al.*[9] These authors observe that a deuteron wave function corresponding to a sum of Yukawa potentials may be written in the form

$$\tilde{u}(p)=N\int_0^\infty \frac{\sigma(p'^2)}{p'^2+p^2}dp'^2=\frac{N}{\alpha^2+p^2}-\int_{\lambda^2}^\infty \frac{\eta(p'^2)}{p'^2+p^2}dp'^2, \quad (8)$$

where $\sigma(p'^2)=\delta(p'^2-\alpha^2)-\eta(p'^2)\theta(p'^2-\lambda^2)$.[24] $\mathbf{p}$ is the center-of-mass momentum of the nucleons and λ is equal to $\mu+\alpha$, where μ is the minimum decay constant which appears in the potential. The wave-function normalization N is given in terms of the deuteron effective range r_e by

$$N^2=2\alpha/(1-\alpha r_e). \quad (9)$$

The analytic structure of the function $(\alpha^2+p^2)\tilde{u}(p)$ is identical to that of the leading vertex form factor, $F(t)$. This is readily seen by changing to the variables $t=m^2-2(p^2+\alpha^2)$, $t'=m^2+2(p'^2-\alpha^2)$. Then the spectral weight functions $\eta(t)$ and $\mathrm{Im}F(t)$ are related as

$$\eta(t)=\mathrm{Im}F(t)[\pi F(m^2)(t-m^2)]^{-1}. \quad (10)$$

In fact Blankenbecler and Cook showed that the relativistic weight function $\pi^{-1}\,\mathrm{Im}F(t)/(t-m^2)$, which arises from the exchange of a single pion between the nucleons, is practically identical to that which would be obtained in a nonrelativistic calculation of $\sigma(t)$ using a Yukawa potential.

The connection between the form factors F, G, H, I in Eq. (6) and wave functions is more complicated when the spins of all particles are included. As noted in Eq. (3), the convenient starting point for the analysis is the matrix element $\langle n|\psi_p|d\rangle$:

$$\langle n|\psi_p|d\rangle=\left[\frac{-i\gamma\cdot(d-n)+m}{m^2-t}\right]\Gamma_{npd}(t)u^c(n). \quad (11)$$

This matrix element is just the momentum-space wave function for the proton in the deuteron. Initial attempts by Gross[12] and by Gourdin *et al.*[13] to extract the deuteron wave function from the vertex function led to ambiguous results because of the neglect of antiparticle contributions.[25] More recently, Gross[26] developed a consistent scheme for writing a relativistic deuteron wave function based on a one-channel coupling approximation. His technique lead to the observation that the propagator can be viewed as the superposition of physical nucleon and antinucleon states. The positive energy contributions, or nucleon states, are then identified with the deuteron wave function.

The approach in this paper parallels the recent work of Gross[26] but extends the calculation to include the identification of antiparticle contributions as well. For example, the vertex with the deuteron and neutron on the mass shell describes both the process $p+n\leftrightarrow d$ and $\bar{p}+d\leftrightarrow n$.[11] Specifically, we assume the wave functions for describing the vertex $d\leftrightarrow n+p$, ϕ_{deuteron}, and the vertex $n\leftrightarrow \bar{p}+d$, ϕ_{neutron}, are defined by the relation[27,28]

$$\langle n|\psi_p|d\rangle=\frac{1}{2p_0'}[u(\mathbf{d}-\mathbf{n})\phi_{\mathrm{deuteron}}+v(\mathbf{n}-\mathbf{d})\phi_{\mathrm{neutron}}], \quad (12)$$

where $u(p)$, $v(p)$ are Dirac spinors for the proton and antiproton. The overall factor $2p_0'$ is included for phase space. Since we are conserving three-momentum but not energy in this separation, p_0' is given by $m^2+(\mathbf{n}-\mathbf{d})^2$.[29] Each process is characterized by two

[24] The spectral weight function is subject to the subsidiary condition $\int_0^\infty \sigma(p'^2)dp'^2=0$, to insure the correct indicial behavior of the wave function.

[25] The problem centers on the fact that all four invariants of the n-p-d vertex contribute to the wave function consisting of only two angular-momentum components. It is tempting therefore to neglect the H and I invariants in Eq. (6) in making such a comparison since these do not appear at all for the on-mass-shell case. Durand (Ref. 2) and later Gross (Ref. 26), noted the drastic error in such an omission by showing that $F(t)$ and $H(t)(m^2-t)/4m^2$ are comparable in magnitude when the nucleons are slightly off the mass shell.

[26] Franz Gross, Phys. Rev. **136**, B140 (1964).

[27] Ian J. McGee, Ph.D. thesis, Yale University, 1965 (unpublished).

[28] An alternative scheme was used initially (Ref. 27, Appendix), to separate the particle and antiparticle components which amounted to writing the propagator in terms of off-mass-shell spinors. Although the electromagnetic form factors then appear as free form factors, the method is less appealing because one must deal with expressions for virtual nucleons and antinucleons, whereas we wish ultimately to use nonrelativistic wave functions as input.

[29] Some insight into the particular form of the separation, Eq. (12), may be gained by noting how the usual Feynman result for the disintegration of the deuteron by a virtual photon, $\gamma+d\to n+p$, can be viewed as a superposition of time-ordered graphs involving exchanges of nucleons and antinucleons. Assuming the reaction is mediated by virtual proton exchange, for example, we can separate the covariant result for the amplitude into two identifiable parts:

$$\langle np|j_\mu|d\rangle=\bar{u}(p)\Gamma_\mu{}^p(q,d-n)\left[\frac{-i\gamma\cdot(d-n)+m}{m^2-t}\right]\langle n|f_p|d\rangle$$
$$=\sum_{\text{intermediate spins}}\frac{1}{2p_0'}\left[\frac{\bar{u}(\mathbf{p})\Gamma_\mu{}^p(q,d-n)u(\mathbf{d}-\mathbf{n})\bar{u}(\mathbf{d}-\mathbf{n})}{d_0+q_0-E'}+\frac{\bar{u}(\mathbf{p})\Gamma_\mu{}^p(q,d-n)v(\mathbf{n}-\mathbf{d})\bar{v}(\mathbf{n}-\mathbf{d})}{d_0+q_0-E''}\right]\langle n|f_p|d\rangle,$$
$$E'=n_0+p_0'+q_0,\quad E''=d_0+p_0'+p_0.$$

The terms are recognized as contributions from time-ordered graphs involving the exchange of a physical proton and antiproton, calculated in second-order perturbation theory. The total energies of the intermediate states are E' and E''. The two expressions would be exact if the arguments of the electromagnetic vertex function were replaced by their appropriate off-mass-shell values; q^2 becomes $q^2+(m^2-t)p^2/2m$ for proton exchange, and q^2 becomes q^2-4mp_0 for antiproton exchange. However the approximation for the particle exchange term is excellent as it stands since t is close to m^2. It is a questionable approximation for the antiparticle exchange term, but since the energy denominator is so much larger for this contribution, one could safely ignore its contribution.

angular-momentum functions giving a total of four functions which can be related unambiguously to the four form factors of the n-p-d vertex function. The particle contributions are relativistic generalizations of the familiar S- and D-state wave functions in momentum space. The antiparticle components can be shown to be $j=\frac{1}{2}$ P-state wave functions for $\bar{p}d$ spins of $\frac{1}{2}$ and $\frac{3}{2}$.[30]

The wave functions are separated farther into angular-momentum components by reducing the spinors to two-component form and identifying the associated spin combinations in the appropriate rest frame. For example, the momentum-space deuteron wave function has the form

$$\phi_{\text{deuteron}}(\mathbf{p}) = {}^3S_1\,{}^3\mathcal{S}_1 + {}^3D_1\,{}^3\mathcal{D}_1. \tag{13}$$

The expressions ${}^3S_1$, ${}^3D_1$ are Fourier transforms of the S- and D-state wave functions, and ${}^3\mathcal{S}_1$, ${}^3\mathcal{D}_1$ are the related spin functions for the states. In the nonrelativistic limit, ${}^3S_1$ and ${}^3D_1$ correspond to the usual angular-momentum components of the deuteron, $\tilde{u}(p)$ and $\tilde{w}(p)$. These have the form,[31] cf. Eq. (8),

$$\tilde{u}(p) = N\int_0^\infty \frac{\sigma_s(p'^2)dp'^2}{p'^2+p^2}, \tag{14}$$

$$\tilde{w}(p) = -\rho N\int_0^\infty \frac{\sigma_d(p'^2)dp'^2}{p'^2+p^2}, \tag{15}$$

where ρ is the asymptotic D-to-S ratio. The spin functions ${}^3\mathcal{S}_1$, ${}^3\mathcal{D}_1$ can be determined by a comparison with the usual expressions for spin-angle functions in coordinate space, or by a direct Clebsch-Gordan decomposition of the available spin-function combinations in the deuteron rest frame. For the deuteron, they are

$$\begin{aligned} {}^3\mathcal{S}_1 &= (4\pi)^{-1/2}\chi_p^\dagger \boldsymbol{\sigma}\cdot\boldsymbol{\xi}(i\sigma_2/\sqrt{2})\chi_n, \\ {}^3\mathcal{D}_1 &= (4\pi)^{-1/2}\chi_p^\dagger[\tfrac{3}{2}\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\xi}\cdot\mathbf{p} - \tfrac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\xi}]i\sigma_2\chi_n, \end{aligned} \tag{16}$$

where χ_p, χ_n are two component spinors for the neutron and proton.

The corresponding spin-operator combinations for the antiparticle contribution are less well known. The spins and parities involved indicate the neutron wave function can be expressed as two $j=\frac{1}{2}$ P-state angular-momentum components for $\bar{p}d$ spins of $\frac{1}{2}$ and $\frac{3}{2}$. That is,

$$\phi_{\text{neutron}} = [{}^2P_{1/2}\,{}^2\mathcal{P}_{1/2} + {}^4P_{1/2}\,{}^4\mathcal{P}_{1/2}], \tag{17}$$

where the doublet and quartet functions ${}^2P_{1/2}$, ${}^4P_{1/2}$ are the analogs of ${}^3S_1$, ${}^3D_1$ of Eq. (13) and ${}^2\mathcal{P}_{1/2}$, ${}^4\mathcal{P}_{1/2}$ are the associated spin functions. They are identifiable from the available spin-function combinations in the neutron rest frame using a Clebsch-Gordan decomposition into combinations of normalized doublet and quartet spin states. The result is

$$\begin{aligned} {}^2\mathcal{P}_{1/2} &= (4\pi)^{-1/2}\,\chi_{-\lambda_{\bar{p}}}{}^\dagger\Big[-(\sqrt{\tfrac{1}{3}})\frac{M}{d_0}\boldsymbol{\xi}\cdot\hat{d} \\ &\qquad + (\sqrt{\tfrac{1}{3}})i\boldsymbol{\sigma}\cdot\hat{d}\times\boldsymbol{\xi}\Big]i\sigma_2\chi_n, \\ {}^4\mathcal{P}_{1/2} &= (4\pi)^{-1/2}\,\chi_{-\lambda_{\bar{p}}}{}^\dagger\Big[-(\sqrt{\tfrac{2}{3}})\frac{M}{d_0}\boldsymbol{\xi}\cdot\hat{d} \\ &\qquad + (\sqrt{\tfrac{1}{6}})i\boldsymbol{\sigma}\cdot\hat{d}\times\boldsymbol{\xi}\Big]i\sigma_2\chi_n, \end{aligned} \tag{18}$$

where $\chi_{-\lambda_{\bar{p}}}$ is the two-component spinor for the antiproton with helicity $-\lambda_{\bar{p}}$. In obtaining Eq. (18) in the neutron rest frame, we have first re-expressed the deuteron polarization vector ξ_μ in terms of rest vectors.[32]

The reduction of the product of the n-p-d vertex function and propagator into angular-momentum components is now straightforward. Using Eqs. (12)–(18), we obtain the following results:

$$\begin{aligned} {}^3S_1 = (2n_0-M)^{-1}(2\pi)^{-1/2}\{&[2n_0+m]\tfrac{2}{3}F \\ &+[n_0^2-m^2](2/3m)G+[1+(n_0-m)/3m] \\ &\times(2n_0-M)H\}, \end{aligned} \tag{19}$$

$$\begin{aligned} {}^3D_1 = (2n_0-M)^{-1}&(4\pi)^{-1/2}\tfrac{4}{3}(n_0-m) \\ &\times\{F-(1+n_0/m)G-(2n_0-M)/(2m)H\}, \end{aligned} \tag{20}$$

$$\begin{aligned} {}^2P_{1/2} = (c/\sqrt{2})\{&[-2+d_0/M+(p_0+m)/M]F \\ &-[(p_0+m)/M]G+[-1+(d_0+p_0)/m \\ &-M/2+d_0/M]H-[d_0(p_0+m)-\mathbf{d}^2]/ \\ &(2mM)I\}, \end{aligned} \tag{21}$$

$$\begin{aligned} {}^4P_{1/2} = c\{&[-1-d_0/M-(p_0+m)/M]F \\ &+[(p_0+m)/M]G+[-1+(d_0-M \\ &+p_0+m)/2m-d_0/M]H \\ &+[d_0(p_0+m)+\mathbf{d}^2]/(2mM)I\}, \end{aligned} \tag{22}$$

[30] This identification appears naturally when the off-mass-shell effects are viewed as contributions from particle and antiparticle positive-energy states. Alternatively these terms can be viewed as arising from the P states of the full deuteron wave matrix when written as $\psi=\psi({}^3S_1)+\psi({}^3D_1)+\gamma_5\psi({}^3P_1)+\gamma_5\psi({}^1P_1)$. See J. Tran Thanh Van, Ann. Phys. (Paris) 9, 139 (1964).

[31] The spectral weights σ_s, σ_d in Eqs. (14), and (15) are subject to the subsidiary conditions

$$\int_0^\infty \sigma_s(z)dz=0,$$

$$\int_0^\infty \sigma_d(z)z^m dz=0, \quad m=-2,\,0,\,2.$$

These conditions guarantee that the wave functions are finite at the origin and have the correct indicial behavior.

[32] The deuteron polarization vector ξ_μ is connected to the rest-frame vector $\boldsymbol{\xi}^r$ by the relation

$$\xi_\mu=[\boldsymbol{\xi},\xi_0]=[\boldsymbol{\xi}^r+(d_0/M-1)\boldsymbol{\xi}^r\cdot\hat{d}\hat{d},\ \boldsymbol{\xi}^r\cdot\mathbf{d}/M],$$

where M is the deuteron mass.

where

$$c=\frac{d}{(d_0+p_0-m)}\left[\frac{4m}{3(p_0+m)}\right]^{1/2},\quad d=|\mathbf{d}|,$$
$$p_0=(m^2+\mathbf{d}^2)^{1/2}.$$

The expressions ${}^3S_1$, ${}^3D_1$ are expressed entirely in terms of the variable $n_0=\frac{1}{2}M+(m^2-t)/2M$, the neutron energy in the deuteron rest frame. The corresponding neutron wave-function components ${}^2P_{1/2}$, ${}^4P_{1/2}$ are expressed in terms of the deuteron energy in the neutron rest frame, $d_0=(M^2+m^2-t)/2m$. Eqs. (19)–(22) are easily generalized to an arbitrary frame. The general forms are listed in Appendix I. If only terms through order $\mathbf{n}^2/m^2$ are retained in Eqs. (19) and (20), the results agree with those given by Gross.[12]

We have yet to discuss the question of normalization for these relativistic angular-momentum states. The question can be settled for the ${}^3S_1$ and ${}^3D_1$ components by requiring that these relativistic "wave functions" reduce to the nonrelativistic ones at low energies.[33] More directly, the normalization can be determined by using the n-p-d vertex function to calculate the deuteron pole term in low-energy neutron-proton scattering and comparing the result with that obtained by extrapolating the effective range formula,

$$p\cot\delta=-\alpha+\tfrac{1}{2}r_e(\alpha^2+p^2)\,,\tag{23}$$

to the pole at $p=i\alpha$. In Eq. (23), δ is the eigenphase shift and α, r_e are as defined earlier.

Such a calculation shows that $F(m^2)$ and $G(m^2)$ are given by[34]

$$\begin{aligned}F(m^2)&=N\frac{(1+\rho/\sqrt{2})}{(1+\rho^2)^{1/2}}\left(\frac{8\pi}{m}\right)^{1/2},\\ G(m^2)&=\frac{3m^2}{\sqrt{2}\alpha}\rho\frac{N}{(1+\rho^2)^{1/2}}\left(\frac{8\pi}{m}\right)^{1/2},\end{aligned}\tag{24}$$

where the normalization factor N is defined by Eq. (9). The components in Eqs. (19) and (20) are normalized in this way. The analogous normalization of the antiparticle components does not need such careful consideration since we ignore ultimately the small contribution from the antiparticle states in the electrodisintegration.

The knowledge of the components of the n-p-d vertex function gives us directly information on the form of the dispersion relations for the transition amplitude. This is especially true for conditions under which the extraction of the electromagnetic vertex function from the dispersion relations is a good approximation. We consider now the kinematical situation at the quasi-elastic peak where the on-mass-shell behavior at the electromagnetic vertex is most closely approximated.[35] In the c.m. frame of the nucleons, the pole term denominator has the form

$$m^2-t=2[\alpha^2+(\mathbf{p}-\tfrac{1}{2}\mathbf{q})^2-\tfrac{1}{4}q_0^2]\,,\tag{25}$$

where $\mathbf{p}$ is the proton momentum, and $\mathbf{q}=\mathbf{e}-\mathbf{e}'$ is the electron three-momentum transfer in this system. The timelike component of the momentum transfer is

$$q_0=(\alpha^2+p^2-\tfrac{1}{4}q^2)/E\,,\quad E=p^2+m^2.\tag{26}$$

The peak in the inelastic cross section occurs for the kinematic condition $|\mathbf{p}|=\frac{1}{2}|\mathbf{q}|$, where q_0 is essentially zero. The expression $[m^2-t]^{-1}$ is consequently strongly peaked for $\hat{p}\cdot\hat{q}\approx 1$ and the direction of the scattered proton is limited to a narrow cone about the direction of $\mathbf{q}$. The corresponding kinematic situation for the neutron pole term is obviously $\hat{p}\cdot\hat{q}\approx -1$, hence there is essentially no interference between processes in which the electron scatters from one nucleon, and those in which it scatters from the other.

The pole-term expressions for the transition amplitude at the quasi-elastic peak are therefore expressible directly in terms of the asymptotic part of the deuteron wave function. In addition, the results in this limit are essentially identical to the nonrelativistic results, except that $|\mathbf{q}|^2=q^2$, and $\mathbf{p}$ is the relativistic momentum of the nucleons. The inclusion of the single dispersion integrals introduces corrections from the intermediate structure of the deuteron and off-mass-shell effects in the nucleons and the propagator. However, it is easy to show (see below) that if we assume smooth (on-mass-shell) behavior of the electromagnetic vertex function, then the resulting dispersion relation for the transition amplitude factors, leaving simply the dispersion integrals for the relativistic "wave functions." The proximity of the anomalous threshold of these integrals to the physical region gives yet a further simplification of $\langle np|j_\mu|d\rangle$ using the results of Blankenbecler and Cook.[8] The form of the spectral weight function in the anomalous region is determined by the behavior of the wave function at intermediate distance and since the long-range structure is given exactly by the pole term result, one can immediately adopt the phenomenological theory of the deuteron as far as the calculation of the functions ${}^3S_1(t)$ and ${}^3D_1(t)$ is concerned.

[33] It is not obvious that the wave functions which are derived from semiphenomenological potentials using the Schrödinger equation correspond to the relativistic wave functions in any sense. However, the Blankenbecler and Cook result (Ref. 8) indicates that the spectral weights of relativistic wave functions which are dominated in the low-energy limit by the exchange of a single pion are identical to nonrelativistic wave functions obtained from the Schrödinger equation using a one-boson-exchange potential. One-boson-exchange potentials exist which reproduce the nucleon-nucleon scattering data very well up to several hundred MeV. Hence one can conclude that relativistic wave functions and semiphenomenological wave functions based on the Schrödinger equation are equivalent at low energies.

[34] These relations agree with those quoted by the authors in Ref. 10, but differ slightly from those of Gross [Eq. (3.26), Ref. 26].

[35] Durand argues (in Sec. IVb of Ref. 2), that a rough estimate of the difference between the free-nucleon form factors and those for a nucleon off the mass shell is on the order of $\epsilon/\mu\sim 1.6\%$ at the quasi-elastic peak for inelastic scattering.

Assuming the on-mass-shell behavior of the electromagnetic vertex function, we can write the transition amplitude as

$$\langle np|j_\mu|d\rangle=\frac{J_p{}^\mu}{m^2-t}+\frac{J_n{}^\mu}{m^2-u}\,, \tag{27}$$

where

$$J_p{}^\mu=\bar{u}(p)\left[\gamma_\mu F_{1p}(q^2)+\frac{\kappa_p}{2m}F_{2p}(q^2)\sigma_{\mu\nu}q_\nu\right]\times[-i\cdot\gamma(d-n)+m]\Gamma_{npd}(t)u^c(n)\,. \tag{28}$$

The neutron residue term $J_n{}^\mu$ has a similar form. The n-p-d vertex function in Eq. (28) is expressed in terms of the relativistic angular-momentum functions. For purposes of orientation we first write down the expression for $J_p{}^\mu$ in the laboratory frame to allow a direct comparison with the familiar Rosenbluth cross-section form for scattering of free nucleons. The standard terms are easily recognized and the smallness of the antiparticle contribution is more apparent in this frame. The result to order m^{-2} is given by

$$J_p{}^\mu=\chi_p{}^\dagger[{}^3S_1\Omega_\mu{}^s+{}^3D_1\Omega_\mu{}^d+{}^4P_{1/2}\Omega_\mu{}^{4p}+{}^2P_{1/2}\Omega_\mu{}^{2p}]i\sigma_2\chi_n\,, \tag{29}$$

where the functions $\Omega_\mu{}^i$ are combinations of spin functions and nucleon form factors which appear in the two-component reduction of (28).

$$\begin{aligned}\Omega_0{}^s=&F_{1p}(1+\mathbf{q}^2/8m^2)\boldsymbol{\sigma}\cdot\boldsymbol{\xi}-(\mathbf{q}^2/4m^2)\kappa_pF_{2p}\boldsymbol{\sigma}\cdot\boldsymbol{\xi}\\&+(1/2m^2)(F_{1p}+\kappa_pF_{2p})\mathbf{n}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\boldsymbol{\xi}+(1/4m^2)\\&\times(3F_{1p}+2\kappa_pF_{2p})(\mathbf{n}\cdot\boldsymbol{\xi}\boldsymbol{\sigma}\cdot\mathbf{q}-\boldsymbol{\xi}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{n}\\&-i\boldsymbol{\xi}\cdot\mathbf{q}\times\mathbf{n})\,,\\\boldsymbol{\Omega}^s=&-i(\mathbf{q}/2m)F_{1p}\boldsymbol{\sigma}\cdot\boldsymbol{\xi}+(1/2m)(F_{1p}+\kappa_pF_{2p})\\&\times(\boldsymbol{\sigma}\times\mathbf{q})\boldsymbol{\sigma}\cdot\boldsymbol{\xi}\,,\\\Omega_0{}^{2p}=&(1/2m^2)(F_{1p}+\kappa_pF_{2p})(\mathbf{n}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{n}\\&+\boldsymbol{\xi}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{q}+i\mathbf{q}\cdot\mathbf{n}\times\boldsymbol{\xi})\,,\\\boldsymbol{\Omega}^{2p}=&(F_{1p}/2m)(-i\mathbf{n}-\boldsymbol{\sigma}\times\mathbf{n})\boldsymbol{\sigma}\cdot\boldsymbol{\xi}\,.\end{aligned} \tag{30}$$

Similar forms are obtained for the coefficients of ${}^3D_1$ and ${}^4P_{1/2}$. Here $\mathbf{n}$ represents the laboratory momentum of the neutron or more generally of the spectator nucleon.

If the $\mathbf{n}$-dependent terms are neglected, the interaction current for the ${}^3S_1$ (and ${}^3D_1$) states reduces to that given by Durand,[36] and gives directly the Rosenbluth cross section.[37] The surviving terms in $\Omega_0{}^s$ of Eq. (30) correspond to the interaction of the effective proton charge with the electron field. [The $(\mathbf{q}^2/m^2)$ terms are relativistic corrections to the charge as noted by several authors.[2,38]] The $\mathbf{n}$-dependent terms in $\boldsymbol{\Omega}^s$ are easily recognized as the convection current and magnetic moment interaction, respectively. The largest contribution of the $\mathbf{n}$-dependent terms (including antiparticle contributions) to the cross section will be down by a factor $(\mathbf{n}\cdot\mathbf{q})/4m^2$ relative to the leading term. Since the cross section $d^2\sigma/(d\Omega_e de_0')$ involves the integration over all spectator nucleon directions, contributions from such terms will be further diminished to magnitudes of order $\mathbf{n}^2/4m^2=T_{\text{spec}}/m$. The average laboratory kinetic energy of the spectator nucleon, T_{spec} is less than 10 MeV for q^2 up to 0.8 $(\text{BeV}/c)^2$, and $T_{\text{spec}}/m<0.005$ in this region. We may consequently neglect $\mathbf{n}$-dependent terms in the integrated cross section $d\sigma^2/d\Omega_e de_0'$.[39] The single-nucleon contributions can be given therefore essentially by the Rosenbluth cross-section forms. However the factors $F(m^2)$, $G(m^2)$ are to be replaced by the more complicated functions $F(t)$, $G(t)$, $H(t)$, $I(t)$ in the particular combinations corresponding to the angular-momentum functions, ${}^3S_1(t)$ and ${}^3D_1(t)$.

III. EFFECTIVE HAMILTONIAN FOR CALCULATION OF FINAL-STATE INTERACTION CORRECTIONS

The double dispersion integrals of the transition amplitude give rise to final-state interaction effects, (and small meson-exchange corrections which are ignored in this paper). Again assuming on-mass-shell behavior at the electromagnetic vertex, we are able to use the results of Martin *et al.*[14] to observe the near equality at the quasi-elastic peak between the double spectral function in the anomalous region and the nonrelativistic initial- and final-state wave functions. Furthermore we showed in Sec. II that simultaneous contributions from both nucleons are kinematically inhibited at the quasi-elastic peak. Hence FSI corrections are minimal at the peak and can be reliably included in the calculation using a semirelativistic Hamiltonian with approximate wave functions for the initial and final state.

The natural frame in which to make the evaluation is the center-of-mass system of the outgoing nucleons. Here effects of FSI can be introduced naturally in a partial-wave series. On the basis of arguments in Sec. II, we ignore all antiparticle contributions and recast the results for the transition amplitude, Eq. (27), in this frame. The result to order m^{-2} is

$$\langle np|j_\mu|d\rangle=\chi_p{}^\dagger\Omega_\mu{}^p(i\sigma_2/\sqrt{2})\chi_n\,{}^3S_1(t)+\chi_p{}^\dagger\Omega_\mu{}^n(i\sigma_2/\sqrt{2})\chi_n\,{}^3S_1(u)\,, \tag{31}$$

[36] Reference 2, Eqs. (64.1) and (64.2).

[37] M. Rosenbluth, Phys. Rev. **79**, 615 (1950).

[38] K. Hölzl, G. Saller, and P. Urban, Phys. Letters **10**, 120 (1964); P. Breitenlohner, K. Hölzl, and P. Kocevar, *ibid.* **19**, 54 (1965).

[39] The above argument does not hold, of course, for polarized cross sections where these contributions would in principle manifest themselves. However, conditions most favorable for their detection (large q^2 and nucleon momenta away from the quasi-elastic peak), would be most complicated to analyze theoretically and involve extremely small cross sections.

where

$$\Omega_0{}^p = -2mi\{[F_{1p} - (q^2/4m^2)\kappa_p F_{2p}]\boldsymbol{\sigma}_p\cdot\boldsymbol{\xi} + (1/4m^2) \times(F_{1p}+2\kappa_p F_{2p})[i\mathbf{q}\times\boldsymbol{\xi}\cdot\mathbf{p} - (\boldsymbol{\xi}\times\boldsymbol{\sigma}_p) \cdot(\mathbf{p}\times\mathbf{q})] + (1/4m^2)i\mathbf{p}\cdot\mathbf{q}\times\boldsymbol{\xi}F_{1p}\}, \quad (32)$$

$$\boldsymbol{\Omega}^p = 2m\{\boldsymbol{\sigma}_p\times\mathbf{q}(F_{1p}+\kappa_p F_{2p})/(2m) + i(2\mathbf{p}-\mathbf{q})F_{1p}/(2m)\}\boldsymbol{\sigma}_p\cdot\boldsymbol{\xi}. \quad (33)$$

The arguments t, u are replaced by their appropriate values in this frame. Similar terms are obtained for the neutron term $\Omega_\mu{}^n$, with neutron form factors replacing the proton form factors and neutron momentum $-\mathbf{p}$, replacing $\mathbf{p}$. The D-state contribution is essentially identical to the S-state form except for the usual spin-function replacement, cf. Eq. (16).

The higher-order terms in this expression for the effective interaction are evidently small at the quasi-elastic peak, $|\mathbf{p}| = \frac{1}{2}|\mathbf{q}|$. The convection current term in Eq. (33) involves the vector $(2\mathbf{p}-\mathbf{q})/2m$ and furthermore does not interfere with the leading term in the unpolarized cross sections $d^2\sigma/d\Omega_e de_0'$ and $d^3\sigma/d\Omega_e de_0' d\Omega_p$; hence its maximum contribution to the cross section is of order $(\mathbf{p}-\frac{1}{2}\mathbf{q})^2/m^2$. In $\Omega_0{}^p$, the terms $(\boldsymbol{\xi}\times\boldsymbol{\sigma})\cdot(\mathbf{p}\times\mathbf{q})$, $i\mathbf{q}\times\boldsymbol{\xi}\cdot\mathbf{p}$ are also small because the momentum $\mathbf{p}$ is predominantly along the direction of $\mathbf{q}$ at the peak. The m^{-2} terms can be associated with the triplet and singlet combinations of the outgoing nucleons, respectively.[40]

We are interested in using coordinate-space wave functions for the initial and final state. Accordingly it is desirable to recast this effective interaction in a form suitable for use with wave functions. More directly, we can observe that the effective interaction Eq. (31) is the same to order m^{-2} as that obtained using a sum of free-nucleon currents, with the exception of the final singlet term in $\Omega_0{}^p$. This term would be absent in any case in cross sections correct to order m^{-2}, since it does not interfere with the leading interaction terms. Specifically, we can show the following correspondence to order m^{-2}:

$$\langle np|j_\mu|d\rangle \sim \langle np|j_\mu{}^p + j_\mu{}^n|[n'p']_r\rangle, \quad (34)$$

where the notation $[\;]_r$ around the initial n-p state is to indicate that the integration over the relative coordinates must be performed. This connection depends on the factorization of the electromagnetic vertex function from the dispersion relation for the transition amplitude as stated earlier. It yields an effective interaction in coordinate space which to order m^{-2} agrees with the momentum-space n-p-d vertex reduction, Eq. (31). The result, demonstrated in Appendix II, is

$$J_0 = \sum_{i=p,n} \{F_{1i} - (-F_{1i}+2\kappa_i F_{2i})(q^2/8m^2) + (F_{1i}/2m^2)\overleftarrow{\partial}_i\cdot\overrightarrow{\partial}_i + (1/4m^2) \times(F_{1i}+2\kappa_i F_{2i})\boldsymbol{\sigma}_i\cdot\mathbf{q}\times\overrightarrow{\partial}_i\}, \quad (35)$$

$$\mathbf{J} = \sum_{i=p,n} \{-(F_{1i}/2m)(\mathbf{q}-2i\overrightarrow{\partial}_i) \times(F_{1i}+\kappa_i F_{2i})\boldsymbol{\sigma}_i\times\mathbf{q}\}.$$

The derivatives ∂_i act only on the coordinate of particle i in the wave functions. The result depends on the replacement of the deuteron wave function by a product of free-nucleon wave functions, integrated over their relative coordinates. The validity of this replacement is demonstrated in Appendix III where we construct a model "deuteron" wave function out of free-nucleon wave functions. The resulting wave function is fully relativistic but unbound. It also shows the origin of the omitted singlet term in the above reduction as due to a spin-rotation correction for the nucleon's spin in the moving deuteron.

IV. CROSS SECTIONS NEGLECTING FINAL-STATE INTERACTIONS

In this section, we use the previous results to present the cross sections for electrodisintegration, $d^3\sigma/d(\cos\theta)de_0'd\Omega_p$ and $d^2\sigma/(d\Omega_e de_0')$, including the main D-state contributions but neglecting the effects of FSI between the outgoing nucleons. Estimates are given for the magnitude of the higher-order corrections which suggest that several are insignificant even when FSI are included. The contributions arising from interference between electrons scattered off the proton and those scattered off the neutron are explored using a realistic wave-function model. Their contribution is shown to vary considerably away from the quasi-elastic peak.

Using the n-p-d vertex reduction of the transition amplitude, Eq. (31), we obtain the following expression for the coincidence cross section at the peak:

$$d^3\sigma/d(\cos\theta)de_0'd\Omega_e = \sigma_{\text{Mott}}\frac{mp}{2\pi}\cdot\frac{m}{E}(1+\tau)^{-1}\{\Lambda^L(p,q)+\tau[1+2(1+\tau)\tan^2(\tfrac{1}{2}\vartheta)]\Lambda^T(p,q)\}, \quad \tau=(q^2/4m^2), \quad (36)$$

where σ_{Mott} is the Mott scattering cross section for an electron scattering of an external field. The angle ϑ is the electron scattering angle measured in the laboratory frame. E, p, q are variables defined in Sec. II and are measured in the c.m. system of the outgoing nucleons. The angle θ, also measured in this system, is defined by the

[40] The singlet term involving $i\mathbf{p}\cdot\mathbf{q}\times\boldsymbol{\xi}/m^2$ does not contribute to the cross sections to $O(m^{-2})$ since it does not interfere with the leading charge terms.

relation $\cos\theta=\hat{p}\cdot\hat{q}$. The angular distribution functions Λ^L, Λ^T are given by

$$\Lambda^L(p,q)=[F_{1p}-\tau\kappa_pF_{2p}]^2\{[{}^3S_1(k_p)]^2+[{}^3D_1(k_p)]^2\}-4\tau F_{1p}{}^2[{}^3S_1(k_p)\ {}^3S_1''(k_p)]+[F_{1n}-\tau\kappa_nF_{2n}]^2 \times\{[{}^3S_1(k_n)]^2+[{}^3D_1(k_n)]^2\}-4\tau F_{1n}{}^2[{}^3S_1(k_n)\ {}^3S_1''(k_p)]+2[F_{1p}-\tau\kappa_pF_{2p}][F_{1n}-\tau\kappa_nF_{2n}] \times[{}^3S_1(k_p)\ {}^3S_1(k_n)+{}^3D_1(k_p)\ {}^3D_1(k_n)P_2(\hat{k}_p\cdot\hat{k}_n)], \quad (37)$$

$$\Lambda^T(p,q)=(F_{1p}+\kappa_pF_{2p})^2\{[{}^3S_1(k_p)]^2+[{}^3D_1(k_p)]^2\}+(8p^2/q^2)F_{1p}{}^2[{}^3S_1'(k_p)]^2+(F_{1n}+\kappa_nF_{2n})^2\{[{}^3S_1(k_n)]^2 +[{}^3D_1(k_n)]^2\}+(8p^2/q^2)F_{1n}{}^2[{}^3S_1'(k_n)]^2+\tfrac{2}{3}(F_{1p}+\kappa_pF_{2p})(F_{1n}+\kappa_nF_{2n})\{{}^3S_1(k_p)\ {}^3S_1(k_n) +{}^3D_1(k_p)\ {}^3D_1(k_n)[3P_2(\hat{k}_n\cdot\hat{k}_p)+P_2(\hat{k}_n\cdot\hat{q})+P_2(\hat{k}_p\cdot\hat{q})-1]\}+\tfrac{1}{3}\sqrt{2}(F_{1p}+\kappa_pF_{2p})(F_{1n}+\kappa_nF_{2n}) \times[{}^3S_1(k_n)\ {}^3D_1(k_p)P_2(\hat{k}_p\cdot\hat{q})+{}^3S_1(k_p)\ {}^3D_1(k_n)P_2(\hat{k}_n\cdot\hat{q})], \quad (38)$$

where

$$\mathbf{k}_p=\tfrac{1}{2}\mathbf{q}-\mathbf{p},\quad k_p=|\mathbf{k}_p|=(p^2+\tfrac{1}{4}q^2-pq\cos\theta)^{1/2}, \qquad \mathbf{k}_n=\tfrac{1}{2}\mathbf{q}+\mathbf{p},\quad k_n=|\mathbf{k}_n|=(p^2+\tfrac{1}{4}q^2+pq\cos\theta)^{1/2}. \quad (39)$$

The arguments t, u of the angular-momentum functions have been replaced by their values in the c.m. frame of the outgoing nucleons,[41] $t=m^2-2(\alpha^2+k_p{}^2)$, $u=m^2-2(\alpha^2+k_n{}^2)$. The leading S- and D-state contributions correspond to the exact results obtainable from the pole terms of the single dispersions relations, and reproduce the Rosenbluth cross section for scattering off the individual nucleons, cf. Eqs. (45) and (46). However, the other, smaller contributions are only correct to terms of order (q^2/m^2).

We can go one step further in this relativistic calculation by invoking the results of Blankenbecler and Cook to approximate the angular-momentum functions, ${}^3S_1$, ${}^3D_1$. This is justified only at the peak where one has a well-defined connection between the spectral functions in the anomalous region and the nonrelativistic wave functions. In this approximation, the functions ${}^3S_1$, ${}^3D_1$ are replaced by $\tilde{u}(k_i)$, $\tilde{w}(k_i)$ defined in Eqs. (14) and (15). In terms of the more familiar coordinate space wave functions $u(r)$, $w(r)$, the functions $\tilde{u}$, $\tilde{w}$ are[42]

$$ {}^3S_1(k_i)\rightarrow\tilde{u}(k_i)=\int_0^\infty j_0(k_ir)u(r)rdr, \qquad {}^3D_1(k_i)\rightarrow\tilde{w}(k_i)=\int_0^\infty j_2(k_ir)w(r)rdr. \quad (40)$$

Included also in the cross section, Eq. (36), are smaller matrix elements which are multiplied by the momentum factors $k_p{}^2$ or $k_n{}^2$. When such terms are recast in terms of coordinate space wave functions, the additional momenta transform to give derivatives acting on the wave functions, cf. Eq. (35). It is easy to show that the particular forms obtained are given by

$${}^3S_1'(k_i)\rightarrow\tilde{u}'(k_i)=(\sin\theta/2k_i)\int_0^\infty j_1(k_ir)r^2(d/dr) \times[u(r)/r]dr, \quad (41)$$

$${}^3S_1''(k_i)\rightarrow\tilde{u}''(k_i)=(1/q^2)\int_0^\infty j_0(k_ir)r(d^2/dr^2) \times u(r)dr. \quad (42)$$

Because of the smooth long-range behavior of the deuteron wave function, these expressions containing derivatives are sensitive mainly to the shorter, lesser known region of the wave function. Hence, the use of crude deuteron wave functions, or the neglect of FSI effects in evaluation of expressions of this type can be misleading. However, the particular integrals in Eqs. (41) and (42) are relatively small in any case. Because of the curvature of the deuteron wave function, significant contributions to $\tilde{u}''(k_i)$ come only from $r\lesssim2$ F, whereas the bulk of the wave function lies outside this range. Hence, the integral, $\tilde{u}''(k_i)$ is negligible compared to the main integral $\tilde{u}(k_i)$ even if FSI effects are included.

The expression $\tilde{u}'(k_i)$ arises from the convection current term in the interaction. It is not obviously a negligible term especially away from the quasi-elastic peak.[43] The apparent possibility for a significant contribution is much smaller, however, since the inclusion of FSI corrections to the matrix element Eq. (42) reduces the final-state wave function considerably in the sensitive short-range region.

Contributions from the last term in Eq. (33), the so-called spin-orbit term[44] in the interaction, have been omitted completely in writing the cross section. This term contributes only to m^{-2} terms, and then only to pure D-state and n-p, S-D interference terms; hence its contribution is negligible even including FSI effects.

[41] Except for small relativistic corrections, $\mathbf{k}_p$ and $\mathbf{k}_n$ are simply the neutron and proton lab momenta, respectively.

[42] $\tilde{u}(k_p)$ and $\tilde{w}(k_p)$ are identical to the expressions $F(\theta)$, $G(\theta)$ [Eqs. (II.3) and (II.5) of Ref. 3], introduced by Durand.

[43] However, a numerical calculation showed that $u'(k_i)/u(k_i)$ is very small ($\sim$0.5%) at the quasi-elastic peak for $q^2=1.0$ (BeV/c)^2.

[44] The presence of such terms has been noted previously [K. Hölzl, G. Saller, and P. Urban, Acta Phys. Austriaca 19, 168 (1964) and Ref. 37] but have always been neglected for reasons stated in the text.

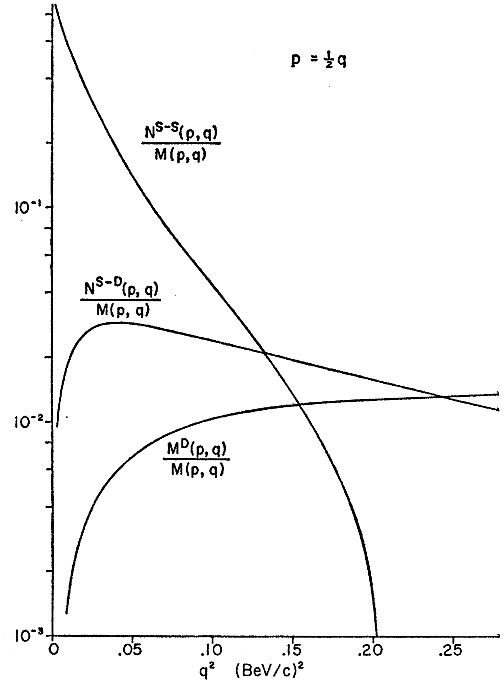

FIG. 4. Relative magnitudes of the S-state n-p interference, S-D n-p interference, and pure D-state terms, defined by Eqs. (47) and (48), for the quasi-elastic peak condition $p=\frac{1}{2}q$. Final-state interactions were neglected. Results were obtained using the repulsive-core deuteron wave function of Ref. (49).

We note finally that the Legendre polynomials appearing in Eq. (36) are easily expressed in terms of θ; for example,

$$P_2(\hat{k}_p\cdot\hat{q})=k_p^{-2}[p^2P_2(\cos\theta)+\tfrac{1}{4}q^2-pq\cos\theta], \\ =1-3p^2(\sin^2\theta)/2k_p^2. \tag{43}$$

Since the main terms in the angular distribution functions are pure S-state and pure D-state terms with the same coefficients, it is clear that the ratio method[3,45] for determining neutron form factors is a cleaner technique than measurements in which the final nucleons are not observed. The complicated structure of the deuteron and, to a lesser extent, the FSI corrections are effectively cancelled with this method.[3]

By integrating the cross section Eq. (36) over all directions of the outgoing nucleons, we obtain the cross section $d^2\sigma/d\Omega_e de_0'$ in which only the final electron is observed. The result is

$$d^2\sigma/d\Omega_e de_0'=\sigma_{\mathrm{Mott}}\left(\frac{mp}{\pi}\right)\left(\frac{m}{E}\right)I(\vartheta), \tag{44}$$

where the angular distribution function $I(\vartheta)$, depending on the final electron energy e_0' and electron scattering angle ϑ, (both measured in the lab system) is given by

$$I(\vartheta)=[M(p,q)+M^D(p,q)](G_p+G_n)+N^{S-S}(p,q) \\ \times[2F_{1n}F_{1p}-2\tau(\kappa_nF_{2n}F_{1p}+\kappa_pF_{2p}F_{1n}) \\ +(2\tau/3)[1+2\tan^2(\tfrac{1}{2}\vartheta)](F_{1p}+\kappa_pF_{2p}) \\ \times(F_{1n}+\kappa_nF_{2n})]+N^{S-D}(p,q)\tfrac{4}{3}\sqrt{2}\tau[1+2\tan^2(\tfrac{1}{2}\vartheta)] \\ \times(F_{1p}+\kappa_pF_{2p})(F_{1n}+\kappa_nF_{2n}). \tag{45}$$

G_p, G_n are convenient form-factor combinations which appear in the Rosenbluth cross sections,[46]

$$G_i=F_{1i}^2+\tau(\kappa_iF_{2i})^2+2\tau(F_{1i}+\kappa_iF_{2i})^2 \\ \times\tan^2(\tfrac{1}{2}\vartheta),\quad i=p,\,n. \tag{46}$$

The functions $M(p,q)$ and $N^{S-S}(p,q)$ are defined as usual[2]:

$$M(p,q)=\frac{1}{2}\int_{-1}^{1}[{}^3S_1(k_p)]^2d(\cos\theta), \\ N^{S-S}(p,q)=\frac{1}{2}\int_{-1}^{1}{}^3S_1(k_p)\,{}^3S_1(k_n)d(\cos\theta), \tag{47}$$

and represent scattering from the S-state part of the deuteron alone. The remaining functions, $M^D(p,q)$ and $N^{S-D}(p,q)$, are defined by

$$M^D(p,q)=\frac{1}{2}\int_{-1}^{1}[{}^3D_1(k_p)]^2d(\cos\theta), \\ N^{S-D}(p,q)=\frac{1}{2}\int_{-1}^{1}{}^3S_1(k_n)\,{}^3D_1(k_p)p_2(\hat{k}_p\cdot\hat{q})d(\cos\theta), \tag{48}$$

and represent, respectively, the dominant scattering from the D state of the deuteron alone, and the n-p interference scattering between the S and D components of the deuteron. Durand has given explicit expressions for $M(p,q)$, $N^{S-S}(p,q)$ for a Hulthén deuteron[47] and other authors have calculated them for a variety of wave-function models.[48] Their results show uniformly that $q^2M(p,q)$ is practically independent of momentum transfer for $q^2\gtrsim0.04$ (BeV/c)^2, while the neutron-proton interference term $N^{S-S}(p,q)$ is much smaller and decreases rapidly with q^2 (cf. Table I of Ref. 16).

These results are easily extended and put on a more systematic basis by exploiting the technique introduced

[45] P. Stein *et al.*, Phys. Rev. Letters **9**, 403 (1962); J. R. Dunning *et al.*, *ibid.* **13**, 631 (1964).

[46] If one makes the crude assumption that q^2 is constant over the whole of the quasi-elastic peak, it is easy to show that one obtains the familiar impulse approximation result

$$\int de_0'\frac{d\sigma^2}{d\Omega_e de_0'}=\sigma_{\mathrm{Mott}}[1+(2e_0/m)\sin^2(\vartheta/2)]^{-1}(G_p+G_n), \\ =d\sigma_p/d\Omega_e+d\sigma_n/d\Omega_e.$$

The above result is obtained, including all the relativistic kinematic factors, by evaluating $d^2\sigma/de_0'd\Omega_e$ in the limit that the deuteron binding energy goes to zero. The result is basic to the area method (Ref. 50) of determining neutron form factors.

[47] Reference 2, Eqs. (28), (29).

[48] J. Nuttall and M. L. Whippman, Phys. Rev. **130**, 2495 (1963); D. Braess and G. Kramer, Z. Physik (to be published). See also Refs. 16, 18, and 38.

in the Appendix of Ref. 2. One expresses the relativistic vertex functions ${}^3S_1$, ${}^3D_1$ in the usual dispersion relation form, Eqs. (14) and (15). The Born term expressions, Eqs. (47) and (48), can then be written completely in terms of the spectral weight functions; for example

$$M(p,q)=\frac{N^2}{(2pq)}\int_0^\infty \sigma_s(z)dz\int_0^\infty \sigma_s(z')dz'\frac{1}{z'^2-z^2}$$
$$\times\ln\left[\frac{z'^2+(p-\frac{1}{2}q)^2}{z'^2+(p+\frac{1}{2}q)^2}\right]\left[\frac{z^2+(p+\frac{1}{2}q)^2}{z^2+(p-\frac{1}{2}q)^2}\right]. \quad (49)$$

The dependence of such expressions on q^2 can be investigated without regard to the particular model used for the deuteron wave function. Results are as follows:

For conditions corresponding to the quasi-elastic peak, $p=\frac{1}{2}q$, both $N^{S-S}(p,q)$ and $N^{S-D}(p,q)$ decrease as q^{-6}, while $q^2M(p,q)$ and $q^2M^D(p,q)$ are essentially constant for $q^2\gtrsim 0.04$ $(\text{BeV}/c)^2$ and 0.01 $(\text{BeV}/c)^2$, respectively. The error in the latter relations decreases as q^{-6} for $M(p,q)$ and as q^{-10} for $M^D(p,q)$ as q^2 increases.

By taking a particular form for the spectral weights, the cross section can be investigated further. The variation of the matrix elements is displayed in Figs. (4)–(6) using a realistic-model wave function.[49] The

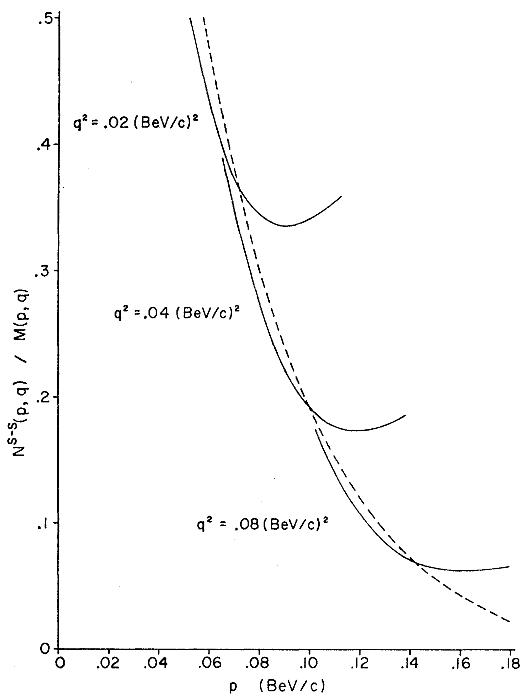

FIG. 5. Typical off-peak behavior of the S-state neutron-proton interference term neglecting final-state interactions. Solid curves are values of $N^{S-S}(p,q)/M(p,q)$ as functions of p for $q^2=0.02$, 0.04, and 0.08 $(\text{BeV}/c)^2$. The curves are terminated for values of p where the electrodisintegration cross section has decreased to roughly half its peak value. Dashed line is $N^{S-S}(p,q)/M(p,q)$ at $p=\frac{1}{2}q$. Results were calculated using the repulsive-core wave function for the deuteron given in Ref. (49).

[49] Ian J. McGee, Phys. Rev. **151**, 772 (1966).

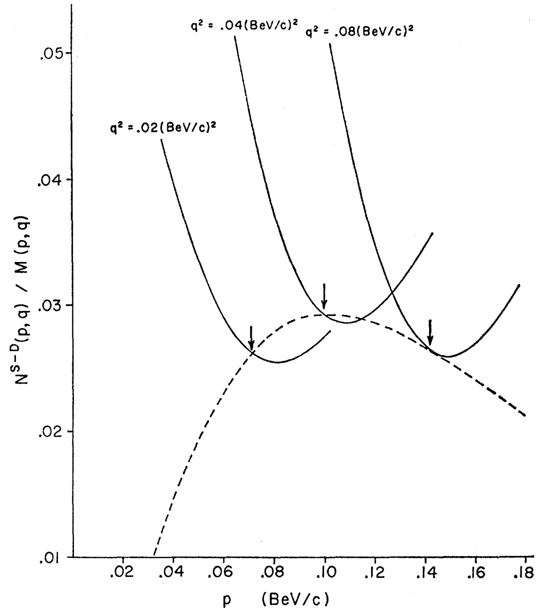

FIG. 6. Typical off-peak behavior of the S-D neutron-proton interference term neglecting final-state interactions. Solid curves are values of $N^{S-D}(p,q)/M(p,q)$ as functions of p for $q^2=0.02$, 0.04, and 0.08 $(\text{BeV}/c)^2$. The curves are terminated for values of p where the electrodisintegration cross section has decreased to roughly half its peak value. Dashed line is $N^{S-D}(p,q)/M(p,q)$ at $p=\frac{1}{2}q$, the corresponding points on the three solid curves being indicated by arrows. Results were calculated using the repulsive-core deuteron wave function given in Ref. (49).

wave function has the form given by Eq. (14) and (15) with the spectral weights $\sigma_i(z)$, $i=s$, d, approximated by a series of delta functions. Free parameters in the wave function were adjusted to reproduce the static properties of the deuteron and to give a reasonable fit to numerical wave functions.

The magnitude of the n-p interference terms in Fig. 4 appears artificially large at these low momentum transfers, i.e., $q^2<0.25$ $(\text{BeV}/c)^2$, because the small multiplicative factors such as $(q^2/4m^2)$ and $F_{1n}(q^2)$ which appear with them in the cross section have been omitted. However, the n-p interference matrix elements give negligible contribution to the quasi-elastic peak as soon as q^2 is at all appreciable. On the other hand, the pure D-state contribution, though small, maintains a constant fraction ($\sim$1.43%) of the dominant S-state contribution.

The off-peak behavior of the interference terms, shown in Figs. (5) and (6), is not uniform. As p increases, the effects of the internucleon repulsion depresses the magnitude of the matrix elements, whereas for $p<\frac{1}{2}q$, the n-p interference increases rapidly, as noted by Durand[2] when studying the threshold region for deuteron breakup. The changes shown in Figs. (5) and (6) should be taken only as indicative of the actual behavior since effects of FSI, which have been omitted, could alter their value appreciably. The results suggest, however, that the n-p S-state interference contribution should be included in analyzing experimental data using the area method.[50]

V. SUMMARY AND CONCLUSIONS

In the dispersion relation treatment of the transition amplitude $\langle np|j_\mu|d\rangle$, the leading contributions to the quasi-elastic peak cross section arise from the nucleon pole terms. These contributions are completely specified by the asymptotic properties of the deuteron wave function and the free-nucleon form factors from the electromagnetic vertex. The single dispersion integrals which correct the pole term result were obtained by assuming the smooth on-mass-shell behavior at the electromagnetic vertex, an approximation which is most reliable at the quasi-elastic peak. As a consequence the single-particle contributions to the transition amplitude are expressible simply in terms of the dispersion relation for the wave function $\langle n|\psi_p|d\rangle$. This result motivated the analysis in Sec. II of the n-p-d vertex function in terms of angular-momentum components of momentum-space wave functions. The antiparticle components of the wave function $\langle n|\psi_p|d\rangle$ were shown to contribute negligibly to electrodisintegration matrix element for q^2 less than ~ 0.8 (BeV/c)^2. The transition was therefore rewritten in terms of the vertex functions $^3S_1$, $^3D_1$. The dispersion integrals for these functions have anomalous thresholds, and furthermore the spectral functions are practically identical in the anomalous region to those for the nonrelativistic deuteron wave function. By interpreting the momentum variables properly, the identification gives an extremely convenient approximation to the single dispersion integrals.

A close connection is also known to exist between the Mandelstam double spectral function in the anomalous region and the wave functions for the two-nucleon system in the initial and final state. This connection, and the smallness of FSI effects, suggests that these corrections can be computed in a semirelativistic approximation. An effective Hamiltonian correct to order m^{-2} was developed for this purpose in Sec. III and will be used in a subsequent paper.

Using the wave-function approach of Durand[2] therefore, we were able to reformulate the theory in terms of ostensibly nonrelativistic wave functions and thereby make use of information on the well-studied nucleon-nucleon interaction. Bosco and others[15–17] have proposed an alternate procedure for calculation of the electrodisintegration matrix elements. The method applies dispersion relations to calculate the transition amplitude, first assuming S waves in the initial and final state. One free parameter appears in the formulation which is adjusted to the experimental data near threshold for deuteron breakup. The S-wave matrix elements can then be determined at other energies, for example, at the quasi-elastic peak. The method is then extended to higher angular momenta by an adjustment of the corresponding P-wave parameter from the experimental cross section somewhat above threshold and so on. This sequential adjustment of free parameters (which approximate integrals over cuts in the complex plane) is not very advantageous when more than a few partial waves are present.

Gourdin, Le Bellac, Renard, and Trân Thanh Vân[18,19] have recently presented a dispersion relation approach similar in principle to the Bosco treatment. In addition, the neutron, proton, and deuteron pole-term contributions are put on an equal footing to display explicitly the leading gauge-invariant terms. However, the presence of form factors precludes a full treatment of gauge invariance for the electrodisintegration. In a comparison with the low-energy data from Orsay[17] around $q^2=0.14$ (BeV/c)^2, both Bosco *et al.*[16] and Gourdin *et al.*[18] were able to obtain reasonable agreement between theory and experiment by inserting FSI corrections to the first few partial waves.

At very large momentum transfers, the current theories would require considerable modification because of absorptive processes in the final two-nucleon state.[27] We suggest an upper limit on the applicability of the present theory as $q^2\sim 0.8$ (BeV/c)^2.

ACKNOWLEDGMENT

This work is an outgrowth of the author's dissertation under the guidance of Professor L. Durand, III. His generous assistance in all phases of the calculation and careful reading of the manuscript are deeply appreciated.

APPENDIX I

The results of Eqs. (19)–(22) relating relativistic angular-momentum functions to form factors of the n-p-d vertex can be generalized to an arbitrary Lorentz frame by rewriting the kinematic variables in terms of invariants. For example in the deuteron rest frame, the neutron energy n_0 is given by $\frac{1}{2}M(1+\nu)$ where the invariant $\nu=(m^2-t)/M^2$ has been introduced for convenience. The states $^3S_1(t)$, $^3D_1(t)$ of Eqs. (19) and (20) are normalized by the method discussed in Sec. IIB. The results are:

$$(m^2-t)\,{}^3S_1(t)=\frac{2}{3}\left[\frac{2mM}{2\pi(1+\nu)}\right]^{1/2}{}^3s_1(t)\,, \tag{I1}$$

$$(m^2-t)\,{}^3D_1(t)=\frac{2}{3}\left[\frac{2mM}{4\pi(1+\nu)}\right]^{1/2}{}^3d_1(t)\,, \tag{I2}$$

where

$$\begin{aligned}{}^3s_1(t)=F(t)[M(1+\nu)+m]&\\ +G(t)[\tfrac{1}{4}M^2(1+\nu)^2-m^2]/m&\\ +H(t)[2m+\tfrac{1}{2}M(1+\nu)](m^2-t)/(2mM)\,,&\end{aligned} \tag{I3}$$

$$\begin{aligned}{}^3d_1(t)=[M(1+\nu)-2m]\{F(t)-G(t)[1+(1+\nu)&\\ \times(M/2m)]-H(t)(m^2-t)/(2mM)\}\,.&\end{aligned} \tag{I4}$$

[50] E. B. Hughes, T. A. Griffy, R. Hofstadter, and M. R. Yearian, Phys. Rev. 146, 973 (1966).

The wave functions possess kinematic singularities in t in their overall coefficient; hence we have defined the new functions ${}^3s_1(t)$ and ${}^3d_1(t)$, which have no kinematic singularities and can be expected to satisfy dispersion relations in the variable t. For example,

$$ {}^3s_1(t)={}^3s_1(m^2)-\frac{(m^2-t)}{\pi}\int_{t_0}^{\infty}\frac{\eta(t')dt'}{(t'-t)(t'-m^2)}, \quad \text{(I5)} $$

and a similar form for the D state. The mass-shell value ${}^3s_1(m^2)$ corresponds exactly to the asymptotic part of the deuteron wave function with relativistic kinematics included, and the spectral weight $\eta(t')$ in the anomalous region $t'\gtrsim t_0$ corresponds identically to that which would be obtained in a nonrelativistic theory.[8,10]

The corresponding results for the antiparticle states are

$$ (m^2-t)\,{}^4P_{1/2}(t)=c\,{}^4p_{1/2}(t)\,, \quad \text{(I6)} $$

$$ (m^2-t)\,{}^2P_{1/2}(t)=(c/\sqrt{2})\,{}^2p_{1/2}(t)\,, \quad \text{(I7)} $$

where

$$ {}^4p_{1/2}(t)=F(t)[-1-(1+\nu)(M/2m)-b]+G(t)b+H(t)[b-\nu-2m/M+(1+\nu)(M/2m)](M/2m) \\ +I(t)[-1+b(1+\nu)(M/2m)+(1+\nu)^2(M/2m)^2](M/2m)\,, \quad \text{(I8)} $$

$$ {}^2p_{1/2}(t)=F(t)[-2+(1+\nu)(M/2m)+b]-G(t)b+H(t)[2b+\nu-4m/M+(1+\nu)(M/m)](M/2m) \\ -I(t)[-1+b(1+\nu)(M/2m)+(1+\nu)^2(M/2m)^2](M/2m)\,, \quad \text{(I9)} $$

and

$$ b=\{m+[m^2-M^2+(1+\nu)^2(M^2/2m)^2]^{1/2}\}/M\,, \quad \text{(I10)} $$

$$ c=2mM\,{}^3\nu[4m/(3M)]^{1/2}[1-2m/(Mb)]^{1/2}[M^2(1+\nu)-4m^2+2mMb]^{-1}. \quad \text{(I11)} $$

Again the kinematic singularities in the variable t occur only in the overall coefficient c.

APPENDIX II

The free-nucleon electromagnetic current operators taken between single-nucleon states are restricted to the general form

$$ \langle p|j_\mu{}^i|p'\rangle=\bar{u}(p)\left[\gamma_\mu F_{1i}(q^2)+\frac{\kappa_i F_{2i}(q^2)}{2m}\sigma_{\mu\nu}(p-p')_\nu\right]u(p')\,,\quad i=p,n. \quad \text{(II1)} $$

p', p are momentum operators which act on the initial and final wave functions, $\mathbf{p}'=-i\vec{\partial}$, $\mathbf{p}=i\overleftarrow{\partial}$. The vector and scalar components of this current have the following form, to order m^{-2}:

$$ \langle p|j_0{}^i|p'\rangle=-2mi\chi_p{}^\dagger[F_{1i}+(F_{1i}/8m^2)(q^2+4\mathbf{p}\cdot\mathbf{p}')-(\kappa_iF_{2i}/4m^2)q^2+(1/4m^2)(F_{1i}+2\kappa_iF_{2i})i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}']\chi_{p'}\,, \\ \langle p|\mathbf{j}^i|p'\rangle=2m\chi_p{}^\dagger[-iF_{1i}(\mathbf{p}+\mathbf{p}')/2m+(F_{1i}+\kappa_iF_{2i})i\boldsymbol{\sigma}\times\mathbf{q}/2m]\chi_{p'}\,,\quad i=p,n. \quad \text{(II2)} $$

The interaction Hamiltonian density is obtained by multiplying this current by $A_\mu(x)$, where A_μ is the usual Møller potential for the photon field generated by the scattered electron. We consider the sum of such a proton and neutron current as giving rise to the interaction of the deuteron current in $\langle np|j_\mu|d\rangle$ to order m^{-2}. By taking the matrix element of this operator between two-nucleon states consisting of direct products of free-neutron and proton spinors, we reproduce all the terms in the n-p-d vertex result, Eqs. (32) and (33), except for the small singlet term. The correspondence between the two forms of the result depends on the fact that the deuteron wave function in $\langle np|j_\mu|d\rangle$ is adequately described by the product of two free-nucleon wave functions. Such a description of the deuteron is expected to do quite well in this situation because the interaction to order m^{-2} contains no terms which depend on the deuteron's binding, that is the internucleon potential.[51]

APPENDIX III

In order to clarify the simple result for the effective interaction using free-nucleon spinors, and to demonstrate the origin of the small anomalous interaction term, we construct a "deuteron" wave function from two free-nucleon wave packets. The resulting model wave function is fully relativistic but has zero binding energy.

The construction is confined to the deuteron S state but results are easily extended to include the D state as well. We consider the direct product of a proton and neutron spinor multiplied by a spherically symmetric function $\phi(p)$ describing the momentum distribution of the nucleons in the "deuteron." The nucleons move in opposite

[51] In Appendix 1a of Ref. 3, Durand has considered the effect of the two-nucleon potential V, on the deuteron wave function. His estimate is consistent with the expectation that these effects will be smaller by roughly $V/2m$ than the effects of final-state interactions. However, the situation for elastic electron-deuteron scattering is more serious since the scattering is quite sensitive to the short-range structure of the deuteron. See F. Gross, Ref. 12.

directions (with momentum $\mathbf{p}$ say) in the "deuteron" rest frame. An angular-momentum state with $J=1$ is easily constructed from the two-nucleon plane-wave state using the helicity formalism of Jacob and Wick[52]:

$$|1M\lambda_1\lambda_2\rangle=[\tfrac{3}{4}\pi]^{1/2}\int_0^\infty \phi(p)p^2dp\int d\Omega\, \mathfrak{D}_{M,\lambda}^{1*}(\phi,\theta,-\phi)e^{i\mathbf{p}\cdot\mathbf{r}}\left[\frac{p_0+m}{2m}\right]\begin{bmatrix}1\\ \boldsymbol{\sigma}_p\cdot\mathbf{p}/(p_0+m)\\ -\boldsymbol{\sigma}_n\cdot\mathbf{p}/(p_0+m)\\ -\boldsymbol{\sigma}_p\cdot\mathbf{p}\boldsymbol{\sigma}_n\cdot\mathbf{p}/(p_0+m)^2\end{bmatrix}R\chi_{\lambda_1}R\chi_{-\lambda_2}. \quad \text{(III1)}$$

The direct product of nucleon four-spinors has been written as a column matrix and the rotations $R=R(\phi,\theta,-\phi)$, made to act on the two-component spinors directly. The effect of the rotation operators on the spinors in (III.1) can be expressed in terms of two $\mathfrak{D}^{1/2}$ rotation coefficients. These can be combined with the original $\mathfrak{D}^{1*}$ function to produce a single $\mathfrak{D}$ function (a $\mathfrak{D}^0$ function for the S state). When the intermediate sums on helicities are performed, one is left with the expression containing the spin-one spinor $\chi_{1,M}$ for the composite system:

$$|1,M,L=0,S=1\rangle=(4\pi)^{-1/2}\int_0^\infty \phi(p)p^2dp\frac{(p_0+m)}{2m}\int d\Omega\begin{bmatrix}1\\ \boldsymbol{\sigma}_p\cdot\mathbf{p}/(p_0+m)\\ -\boldsymbol{\sigma}_n\cdot\mathbf{p}/(p_0+m)\\ -\boldsymbol{\sigma}_p\cdot\mathbf{p}\boldsymbol{\sigma}_n\cdot\mathbf{p}/(p_0+m)^2\end{bmatrix}e^{i\mathbf{p}\cdot\mathbf{r}}\chi_{1,M}. \quad \text{(III2)}$$

This expression for the "deuteron" wave function in its rest frame is instructive. The function is spherically symmetric, has even parity, and triplet spin dependence. (This can be seen more clearly by performing the angular integration.) It is a wave packet for a relativistic two-particle system, which, unfortunately, spreads in time.

The deuteron moves with momentum $-\mathbf{q}$ in the center-of-mass frame of the outgoing nucleons. Thus, in order to use the above wave function in calculating the electrodisintegration matrix element, one must first apply a Lorentz transformation[53] to the state. The resulting wave function for the moving "deuteron" has the following form to order m^{-2}:

$$e^{-i\pi J_2}e^{-i\zeta K_3}|1,M,0,1\rangle=(4\pi)^{-1/2}\int_0^\infty \phi(p)p^2dp[1+(\mathbf{p}+\tfrac{1}{2}\mathbf{q})^2/(4m^2)]^{1/2}[1+(\mathbf{p}-\tfrac{1}{2}\mathbf{q})^2/(4m^2)]^{1/2}$$
$$\times\int d\Omega\, e^{i\mathbf{p}\cdot\mathbf{r}}\begin{bmatrix}1+\dfrac{i(\boldsymbol{\sigma}_p-\boldsymbol{\sigma}_n)}{2}\cdot\mathbf{q}\times\mathbf{p}/(4m^2)\\ -\boldsymbol{\sigma}_p\cdot(\mathbf{p}+\frac{1}{2}\mathbf{q})/2m\\ \boldsymbol{\sigma}_n\cdot(\mathbf{p}-\frac{1}{2}\mathbf{q})/2m\\ -\boldsymbol{\sigma}_p\cdot(\mathbf{p}+\frac{1}{2}\mathbf{q})\boldsymbol{\sigma}_n\cdot(\mathbf{p}-\frac{1}{2}\mathbf{q})/4m^2\end{bmatrix}(-i\sigma_2^p)(-i\sigma_2^n)\chi_{1,M}, \quad \text{(III3)}$$

where $\zeta=\tanh^{-1}(q/d_0)$. The wave function has a form similar to the initial state (a product of free-neutron and proton spinor) considered in Appendix II. Suppose we look at the matrix element of the proton current operator evaluated in the c.m. system of the final nucleons. In the final state, the neutron has momentum $-\mathbf{p}_{\text{c.m.}}$ and this momentum is not altered in the Born approximation by the interaction of the proton with the electron. Hence the initial momentum of the neutron in the (moving) deuteron is also $-\mathbf{p}_{\text{c.m.}}$. This requires $\mathbf{p}$ to be equal to $-\mathbf{p}_{\text{c.m.}}+\frac{1}{2}\mathbf{q}$ in Eq. (III.3) and eliminates the angular integration over $\hat{p}$. In addition, the initial proton momentum is specified as $\mathbf{p}_{\text{c.m.}}-\mathbf{q}$, as expected. Therefore the above wave function used in the matrix element of the proton current operator will reproduce the result indicated in Appendix II using a product of free-neutron and proton spinors. The only difference of course is the extra term with the singlet operator in the large component of the spinor, $\frac{1}{2}i(\boldsymbol{\sigma}_p-\boldsymbol{\sigma}_n)\cdot\mathbf{q}\times\mathbf{p}/(4m^2)$. Such a term has exactly the form required to reproduce the small singlet term in the n-p-d vertex expansion Eq. (32) and detailed calculations confirm that this is the case. The term results from a spin-rotation effect, i.e., a rotation of the nucleons's spin due to the motion of the deuteron. It says that, in a reference frame in which the deuteron is moving, small admixtures of the singlet spin state are present in order to preserve the overall quantum numbers of the deuteron.

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