of other channels in this latter type of calculation. In other words, from the many solutions of (2.8) (2.9), instead of restricting ourselves to the CDD-pole-free type of solution, with its undesirable energy dependence, we restrict ourselves to the type of form used to analyze experiments, and attempt to fix its parameters. We find that, if we insist on good solutions, the method achieves this, successfully predicting both the magnitude and energy dependence of the scattering width, and also both radiative widths.

We finally note that although the method has been chosen in such a way as to suppress the explicit appearance of higher channels, including the $SU(6)$ communicating channels $N^*\pi$ and $K\Sigma$, the approximate symmetry features of the experimental radiative widths are accounted for in a semidynamical way. It is hoped that the method may be used to throw some light on the symmetry properties in cases where data are not, nor are likely to be, so plentiful. In particular, it is hoped to establish, from a study of the $Y_1^*(1385)$, whether the approximate $SU(3)$ symmetry properties exhibited by the $\{10\}$ resonance pionic decay widths,[23] are also exhibited by the pionic coupling constants $g_{\Sigma\Lambda\pi}^2$, $g_{\Sigma\Sigma\pi}^2$, even though the kaon couplings[24,25] appear to violate $SU(3)$.

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Lorentz-Invariant Localization for Elementary Systems

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Philips has axiomatically defined sets of localized states which are Lorentz-invariant (while Newton-Wigner sets of localized states are not) and has divided these sets into three classes. Philips's conjecture concerning spin-zero localized states, namely, that the postulates define the sets of localized states in a unique way, is proved to be incorrect by finding a class-III set that satisfies Philips's postulates. Philips's work is discussed and his calculations (spin-zero case) are repeated without using some (explicit and implicit) unnecessary hypotheses. These calculations are also extended to the spin-$\frac{1}{2}$ case, for which it is proved that there are only class-III sets of localized states. The results are discussed. Incidentally, an explicit form of the effects induced by a Lorentz transformation on representation space is found for both the spin-zero and spin-$\frac{1}{2}$ cases.

I. INTRODUCTION

GENERAL invariance principles from which position operators for elementary systems could be found have been proposed by Newton and Wigner[1] (NW). They have chosen as postulates some properties that seem naturally associated with the notion of localization. These invariance principles actually lead to localized states which are eigenstates of the position observable.

However, the NW localization is not Lorentz-invariant,[2] that is, if an elementary system (for example an elementary particle) is localized at the space-time point x relative to an inertial frame of reference and if a Lorentz transformation relates this frame with a new one in such a way that the point x is invariant, then the system is no longer localized in the new frame. This is undesirable considering that Lorentz invariance is a property naturally associated with the notion of localization and also that NW localization is invariant under spatial rotations, and it seems that there are no physical reasons for privileging the subgroup of the spatial rotations.

Philips[3] has proposed another set of postulates as naturally associated with the notion of localization as the NW postulates, which include the Lorentz-invariance condition.

Let us denote by S_x a set of states localized at $x=(t,\mathbf{x})$. Then Philips's postulates are:

(a) The set S_x is a linear vector space which is invariant under all homogeneous Lorentz transforma-

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[1] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949); hereafter NW.
[2] This is true if the standard ideas about position are retained. However, another formulation of localization exists where the NW position operator is covariant but where position depends not only on the point of localization but also on a timelike vector [see G. N. Fleming, Phys. Rev. **137**, B188 (1965)].
[3] T. O. Philips, Phys. Rev. **136**, B893 (1964).

tions (continuous and inversions[4]) that leave invariant the point x of localization.

(b) The eigendifferentials formed by the superposition of localized states are normalizable to unity.[5]

(c) The set S_x contains no subset which satisfies the preceding conditions (irreducibility).

The orthogonality of states localized at different points, which is one of Newton and Wigner's postulates, is lost in this Lorentz-invariant localization. We are therefore faced with a choice between orthogonality and Lorentz invariance, but while there is no physical reason for giving up the covariance, the lack of orthogonality can be justified if the particle has a structure.[6]

The physical significance of the postulates is clear. We only want to emphasize that the construction of eigendifferentials[7] is the most realistic way to approach the actual state of experimental affairs in the sense that the position operator has a continuous spectrum of eigenvalues and an experiment to measure position values can only single out a portion of the range. Besides, any eigendifferential can be made to come arbitrarily close to a true solution of the eigenvalue equation (which may not belong to the Hilbert space) by taking the region of integration small enough.

II. CONVENTIONS AND FORMULAS

We will use the same metric tensor, index conventions, and Dirac's gamma matrices as those used in Messiah's book.[8] The "p representation" and the Heisenberg picture will be used. Also, following Newton-Wigner, we will describe the states of an elementary system by means of wave functions $\psi(\mathbf{p},\xi)$ of the variable $\mathbf{p}$ and a discrete variable ξ, these functions being defined on the positive mass shell. For spinless systems $\xi=1$ and it will be omitted, whereas for spin-$\frac{1}{2}$ systems ξ can take the four values 1, 2, 3, and 4.

The invariant scalar product reads[1]

$$(\psi,\varphi)=\int\frac{d^3p}{p_0}\psi^*(\mathbf{p})\varphi(\mathbf{p})\quad\text{for spin } s=0 \tag{1a}$$

and

$$(\psi,\varphi)=\sum_\xi\int\frac{d^3p}{p_0{}^2}\psi^*(\mathbf{p},\xi)\varphi(\mathbf{p},\xi)\quad\text{for spin } s=\tfrac{1}{2}, \tag{1b}$$

where as usual

$$p_0=+(p^2+\mu^2)^{1/2},\quad p=|\mathbf{p}|,\quad \mu=\text{mass}.$$

The generators of the homogeneous continuous Lorentz transformation are[9]

$$M_{\mu\nu}/i=p_\mu\partial/\partial p^\nu-p_\nu\partial/\partial p^\mu\quad\text{for spin } s=0$$

and (2)

$$M_{\mu\nu}/i=p_\mu\partial/\partial p^\nu-p_\nu\partial/\partial p^\mu+\tfrac{1}{2}\gamma_\mu\gamma_\nu\quad\text{for spin } s=\tfrac{1}{2}.$$

As in Ref. 3, since any homogeneous Lorentz transformation may be written as the product of two rotations and a pure Lorentz transformation (Lorentz acceleration) along, e.g., the third axis, it will be sufficient to consider Lorentz accelerations along the third axis, the generators of this transformation being

$$K_3=M_{03}/i=p_0\partial/\partial p^3\quad\text{for spin } s=0$$

and (3a)

$$K_3=M_{03}/i=p_0\partial/\partial p^3-\tfrac{1}{2}\gamma_0\gamma_3\quad\text{for spin } s=\tfrac{1}{2},$$

where in polar coordinates

$$p_0\partial/\partial p^3=p_0(\cos\theta\,\partial/\partial p-\sin\theta\, p^{-1}\partial/\partial\theta). \tag{3b}$$

$\varphi_{0\lambda}$ will stand for a state which is localized in the origin of coordinates, and whose specification is completed with the (eventually existent) quantum number(s) λ. We shall call $\varphi_{\mathbf{x}\lambda}$ the state obtained from the former by a spatial translation. As it is well known,

$$\varphi_{\mathbf{x}\lambda}=e^{-i\mathbf{p}\cdot\mathbf{x}}\varphi_{0\lambda}. \tag{4}$$

III. CLASSES OF LOCALIZED STATES

The space S_0 can be spanned by an angular-momentum eigenfunction basis. Let us indicate the total, orbital, and spin angular momentum by j, l, and s, respectively; the symbols J, J' and L, L' stand for the maximum and minimum values taken on by j and l, respectively.

The classification into classes established by Philips for the spin-zero case is then obviously generalized for every spin in the following way:

Class I groups all sets of states for which a finite J exists; it can be said that for these sets a finite L exists owing to the relation between L and J.

Class II groups all sets of states for which $J'\neq s$ (or, equivalently, a nonvanishing L' exists).

Class III groups all sets of states which do not belong to the other two classes.

IV. SUMMARY OF PHILIPS'S RESULTS

Philips considers elementary systems of nonvanishing mass and spin. He uses a basis consisting of functions

$$\varphi_m{}^l(\mathbf{p})=f_l(p)Y_m{}^l(\theta,\varphi) \tag{5}$$

to span the set S_0 of states localized at the origin. One of these functions (which we indicate by $\varphi_{m_0}{}^{l_0}$) is chosen and the functions $(K_3)^n\varphi_{m_0}{}^{l_0}$ ($n=0, 1, \cdots$) are formed,

[4] It seems that invariance under inversions can be omitted as a postulate.

[5] Philips considers eigendifferentials formed by the superposition of states localized in a small region of space. See, however, remark (C) of Sec. V.

[6] A. J. Kálnay and B. P. Toledo, Nuovo Cimento **48**, 997 (1967); J. A. Gallardo, A. J. Kálnay, A. B. Stec, and B. P. Toledo, *ibid.* **48**, 1008 (1967); A. J. Kálnay, Bol. IMAF **2**, No. 5 (1966); J. A. Gallardo, A. J. Kálnay, A. B. Stec, and B. P. Toledo, Nuovo Cimento **49**, 393 (1967).

[7] G. L. Trigg, *Quantum Mechanics* (D. Van Nostrand Company, Inc., New York, 1964).

[8] A. Messiah, *Mécanique Quantique* (Dunod Cie., Paris, 1960).

[9] V. Bargmann and E. P. Wigner, Proc. Nat. Acad. Sci. (U. S.) **34**, 211 (1948).

so that linear manifolds are built up by means of their combinations.

In order to find the functions $(K_3)^n \varphi_{m_0}{}^{l_0}$, the following formula, deduced from Eq. (3b), is used:

$$K_3 Y_m{}^l f_l(p) = N_m{}^l p_0 [f_l' + (l+1) f_l p^{-1}] Y_m{}^{l-1} + N_m{}^{l+1} p_0 (f_l' - l f_l p^{-1}) Y_m{}^{l+1}, \quad \text{(6a)}$$

where

$$N_m{}^l = [(l^2 - m^2)/(4l^2 - 1)]^{1/2}. \quad \text{(6b)}$$

Then, classification into classes of the linear manifolds follows immediately from the form of the functions.

Now, the eigendifferentials centered at point $\mathbf{x}$ are constructed and the manifolds which do not satisfy the postulate of normalizability are disregarded. The expression for the eigendifferentials reads

$$\varphi_{\mathbf{x}}{}^{\epsilon} = \left(\frac{3}{4\pi\epsilon^3}\right)^{1/2} \int_{S(\epsilon,\mathbf{x})} d\mathbf{x}' \varphi_{\mathbf{x}'}, \quad \text{(7)}$$

where $\varphi_{\mathbf{x}'}$ is a state localized at $\mathbf{x}'$ [Eq. (4) is used but no additional quantum number λ is assumed] and $S(\epsilon,\mathbf{x})$ stands for a sphere of radius ϵ centered at $\mathbf{x}$. It follows that

$$\varphi_{\mathbf{x}}{}^{\epsilon} = N(\epsilon p) \varphi_{\mathbf{x}}, \quad \text{(8a)}$$

where

$$N(z) = 3z^{-1} j_1(z), \quad \text{(8b)}$$

and $j_1(z)$ is a spherical Bessel function.

Philips's final results are:

(i) If the manifold is to be a class-I set then it must include a state of the form

$$\varphi_m{}^J \approx p^J Y_m{}^J, \quad J = 0, 1, 2, \cdots. \quad \text{(9)}$$

Then by using the postulate of normalization of the eigendifferentials it is shown that $J > 0$ is to be disregarded, so that there is only one set S_0, with the single localized state

$$\varphi_0{}^{\text{Philips}}(\mathbf{p}) = (2\pi)^{-3/2}. \quad \text{(10)}$$

(ii) In a similar way it is proved that there are no class-II sets of localized states.

(iii) Philips conjectures that all class-III sets violate (b) (Sec. I). He supports this assumption by showing that this happens if the manifold is spanned by, e.g., the Lorentz transform of

$$\psi(\mathbf{p}) \approx p^m e^{-\gamma p}, \quad m \text{ an integer}, \quad \gamma > 0. \quad \text{(11)}$$

If it were true, Philips's conjecture would imply that his postulates uniquely define this state [Eq. (10)] localized at the origin [and then, by using Eq. (4), at every point] for the spin-zero case.

(iv) The state Eq. (10) is different from the Newton-Wigner localized state:

$$\varphi_0{}^{\text{NW}}(\mathbf{p}) = (2\pi)^{-3/2} p_0{}^{1/2}. \quad \text{(12)}$$

(v) Two eigendifferentials formed with Eq. (10) centered at $\mathbf{x}$ and $\mathbf{y} \neq \mathbf{x}$ are not orthogonal:

$$(\varphi_{\mathbf{x}}{}^{\epsilon}, \varphi_{\mathbf{y}}{}^{\epsilon}) = (2\pi)^{-3} m^2 \int_{S(\epsilon,\mathbf{x})} d\mathbf{x}' \times \int_{S(\epsilon,\mathbf{y})} d\mathbf{y}' K_1(m|\mathbf{x}' - \mathbf{y}'|)/m|\mathbf{x}' - \mathbf{y}'|.$$

(vi) There is no self-adjoint position operator (see Ref. 6, especially the fourth article).

V. GENERAL REMARKS

(A) Considering that S_0 is not necessarily an irreducible representation of the (3-dimensional) rotation group, it cannot be assumed *a priori* that vectors of the type Eq. (5) are always included in S_0. All that can be said is that S_0 is a (reducible) representation of the rotation group and therefore, for the spin-0 case, any vector $\varphi_{0\lambda}{}^m \in S_0$ may be represented as

$$\varphi_{0\lambda}{}^m = \sum_{l=L'}^{L} f_l(p) Y_m{}^l. \quad \text{(13)}$$

Analogously, if $s = \frac{1}{2}$ we can write

$$\varphi_{0\lambda}{}^m = \sum_{j=J'}^{J} \sum_{l=j-1/2}^{j+1/2} \sum_{m'=-1/2}^{1/2} \langle l, \tfrac{1}{2}, m-m', m' | j, m \rangle \times f_l(p) Y_{m-m'}{}^l V_{m'}, \quad \text{(14)}$$

where, as in Ref. 1,

$$V_{m'} = E(p) v_{m'}, \quad E(p) = (\gamma^\nu p_\nu + \mu)\gamma^0, \quad \gamma^0 v_{m'} = v_{m'}.$$

The brackets are Clebsch-Gordan coefficients.

(B) It follows from Philips's postulates that S_0 is a representation of the homogeneous Lorentz group. This representation is unitary, relative to the scalar product of Eq. (1).

(C) In the general case, the impossibility of specifying the state of a particle by means of the position quantum numbers alone should not be disregarded *a priori;* i.e., a set of indices, which we indicate by λ [see Eq. (4)] may have to be used in order to have a complete set $(\mathbf{x},\lambda)$ of quantum numbers. (Actually, a $\varphi_{\mathbf{x}\lambda}$ function which depends on a continuous index λ will be introduced in Sec. IX.) However, if λ is (or includes) a continuous variable, then Eq. (7), or more specifically

$$\varphi_{\mathbf{x}\lambda}{}^{\epsilon} = \left(\frac{3}{4\pi\epsilon^3}\right)^{1/2} \int_{S(\epsilon,\mathbf{x})} dx' \varphi_{\mathbf{x}'\lambda}, \quad \text{(15)}$$

is an incomplete eigendifferential because the complete eigendifferential must include superposition of states in a small range of the parameter λ, that is,

$$\varphi_{\mathbf{x}\lambda}{}^{\epsilon\delta} = C(\epsilon,\delta) \int_{\lambda}^{\lambda+\delta} d\lambda' \int_{S(\epsilon,\mathbf{x})} d\mathbf{x}' \varphi_{\mathbf{x}'\lambda}. \quad \text{(16)}$$

Then, even if class-III sets are such that the norm of φ_x^ϵ is infinite, as was assumed by Philips, this would be no reason to disregard them because, as we have stated, the true eigendifferentials are given by Eq. (16).

(D) The physical requirement of equivalence between all inertial frames refers to physically realizable states which, if we deal with a continuous spectrum such as position, are the eigendifferentials formed by a superposition of pure eigenstates. Then, instead of requiring that S_0 be invariant under all homogeneous Lorentz transformations (see Sec. I), the physical requirement should be that the set of eigendifferentials be invariant under all homogeneous Lorentz transformations. However, to avoid great mathematical complexity, we shall use the original form of Philips's postulates. His procedure is in this case similar to the one used by NW for 3-dimensional rotations. Rigorous treatment of the latter case (rotations) is given by Wightmann.[10]

(E) It can be proved that result (vi) (Sec. IV) is not an unsatisfactory consequence of the postulates. Actually, the special properties of position require the use of nonorthodox operators, and a binary position operator is found to exist. Philips's localization [(i), Sec. IV] corresponds to a limiting case (see Ref. 6, especially the fourth article).

VI. EXPLICIT FORM OF THE EFFECTS INDUCED BY A LORENTZ TRANSFORMATION ON REPRESENTATION-SPACE FUNCTIONS

A. Spin-0 Case

Let us write $\mathbf{q}=\mathbf{q}(\mathbf{p})$ such that

$$p^1=q^1,\quad p^2=q^2,$$

and

$$p^3=[\mu^2+(q^1)^2+(q^2)^2]^{1/2}\sinh q^3 \tag{17}$$

and let us call

$$\tilde{\varphi}(\mathbf{q})=\varphi[\mathbf{p}(\mathbf{q})].$$

Then Eq. (3a) implies that

$$K_3\varphi(\mathbf{p})=\frac{\partial}{\partial q^3}\tilde{\varphi}(\mathbf{q}). \tag{18}$$

We can now see that the effects induced on representation space by a Lorentz transformation of coordinates correspond to a translation in $\mathbf{q}$ space, that is,

$$e^{\lambda K_3}\varphi(\mathbf{p})=\tilde{\varphi}(q^1,q^2,q^3+\lambda), \tag{19}$$

where λ stands for the (real) parameter of the transformation.

Turning back to the $\mathbf{p}$ variables, we obtain

$$e^{\lambda K_3}\varphi(\mathbf{p})=\varphi(\mathbf{p}'), \tag{20}$$

where

$$\mathbf{p}'=(p^1,p^2,p^3\cosh\lambda+p_0\sinh\lambda). \tag{21}$$

[10] A. S. Wightman, Rev. Mod. Phys. 34, 845 (1962).

We will later use the symbol p' by means of which we indicate

$$p'=[(p'^1)^2+(p'^2)^2+(p'^3)^2]^{1/2}. \tag{22}$$

B. Spin-½ Case

Using Eqs. (3a) and (20) and remembering that

$$e^{-\frac{1}{2}\lambda\alpha_3}=\cosh(\tfrac{1}{2}\lambda)-\alpha_3\sinh(\tfrac{1}{2}\lambda),$$

we obtain

$$e^{\lambda K_3}\varphi(\mathbf{p})=[\cosh(\tfrac{1}{2}\lambda)-\alpha_3\sinh(\tfrac{1}{2}\lambda)]\varphi(\mathbf{p}'). \tag{23}$$

VII. CLASS I

A. Spin-0 Case

We shall now introduce a method which leads to the result (i) (Sec. IV) found by Philips. We expect this method to have two advantages: One is that, although the postulate (b) of normalization by eigendifferentials is not used, the method does not lose simplicity (for this class, normalization is actually a theorem); the second is that the inclusion in S_0 of at least one vector of the form of Eq. (5) is not assumed, something which is implicitly done in Philips's work [see (A), Sec. V].

In the same way as in Ref. 3, only Lorentz accelerations along the third axis will be considered but we will apply K_3 to vectors of the form of Eq. (13) instead of to vectors like Eq. (5). We can always assume that the vector of Eq. (13), which has been chosen in order to build up the set S_0 by means of successive applications of the group operators, is such that $f_L\neq0$. It then follows from Eq. (6) and the definition of class-I sets that $f_L'-Lp^{-1}f_L=0$, so that

$$f_L=c_0p^L. \tag{24}$$

The successive application of $(K_3)^2$, $(K_3)^3$, $\cdots$ will specify the functions f_{L-1}, f_{L-2}, $\cdots$ up to constants of proportionality c_1, c_2, c_3, $\cdots$. The vectors belonging to S_0, i.e., the linear combinations of $\varphi_{o\lambda}$, $(K_3)\varphi_{o\lambda}$, $\cdots$ are therefore arbitrary linear combinations of no more than L vectors; from this result, recalling remark (B) (Sec. V), it follows that S_0 is a unitary finite representation of the Lorentz group. However, it is well known that, with the exception of the trivial unidimensional representation, the irreducible representations of the homogeneous Lorentz group are all infinite-dimensional. Our set S_0 consists therefore of a constant function because unidimensionality implies that the application of all generators $M_{\mu\nu}$ [Eq. (2)] gives zero. We can then write $\varphi_{o\lambda}=$ constant which is the result (i) (Sec. IV) found by Philips.

B. Spin-½ Case

We will now essentially follow the same steps as in the preceding case. We apply K_3 to Eq. (14), but instead of Eq. (6) the following formula [deduced from Eqs. (3)

and (6)] must be used:

$$K_3 \sum_{j=J'}^{J} \sum_{l=j-1/2}^{j+1/2} \sum_{m'=-1/2}^{1/2} \langle l, \tfrac{1}{2}, m-m', m' | j, m\rangle f_l(p) Y_{m-m'}{}^l V_{m'}$$

$$= \sum_{j=J'}^{J} \sum_{l=j-1/2}^{j+1/2} \sum_{m'=-1/2}^{1/2} \langle l, \tfrac{1}{2}, m-m', m' | j, m\rangle [K_3(f_l(p) Y_{m-m'}{}^l) V_{m'} + f_l(p) Y_{m-m'}{}^l K_3 V_{m'}], \quad (25a)$$

$$K_3 V_{1/2} = a V_{1/2} + b V_{-1/2}, \qquad K_3 V_{-1/2} = -b^* V_{1/2} + a V_{-1/2}, \quad (25b)$$

$$a = -p(p+2\mu)\cos\theta / 2p_0(p_0+\mu), \qquad b = p \sin\theta e^{i\varphi}/2(p_0+\mu). \quad (25c)$$

From the properties of spherical harmonics it can be easily seen that

$$Y_{m-m'}{}^l \cos\theta = \left\{\frac{[l+1+(m-m')][l+1-(m-m')]}{(2l+1)(2l+3)}\right\}^{1/2} Y_{m-m'}{}^{l+1} + \left\{\frac{[l+(m-m')][l-(m-m')]}{(2l+1)(2l-1)}\right\}^{1/2} Y_{m-m'}{}^{l-1}, \quad (26a)$$

$$Y_{m-m'}{}^l \sin\theta e^{i\varphi} = -(8\pi/3)^{1/2} \{[(2l+1)/\tfrac{4}{3}\pi(2(l-1)+1)]^{1/2} \langle l,1,0,0|l-1,0\rangle \langle l,1,m,1|l-1,m+1\rangle Y_{m+1}{}^{l-1}$$
$$+[(2l+1)/\tfrac{4}{3}\pi(2l+1)]^{1/2}\langle l,1,0,0|l,0\rangle\langle l,1,m,1|l,m+1\rangle Y_{m+1}{}^l$$
$$+[(2l+1)/\tfrac{4}{3}\pi(2l+3)]^{1/2}\langle l,1,0,0|l+1,0\rangle\langle l,1,m,1|l+1,m+1\rangle Y_{m+1}{}^{l+1}\}. \quad (26b)$$

Now we can arrive at our result through two different ways:

(1) We proceed via group-theoretical considerations. The successive applications of K_3, $(K_3)^2$, $\cdots$ to the $\varphi_{0\lambda}{}^m$ functions defined in Eq. (14), and the definition of class I (Sec. III), imply a set of differential equations from which the functions f_l are specified up to constants of integration.

We then have a finite (bidimensional) unitary representation of the Lorentz group. The two follows from the dimension of spinor space [see Eq. (14)]. Considering that the only finite unitary representation is the one-dimensional trivial one, it follows that no class-I sets of localized states exist. More specifically, the dimension of the representation implies

$$M_{\mu\nu}\varphi_{0\lambda}{}^m = 0 \quad \text{for all} \quad \mu\nu, \quad (27)$$

where $M_{\mu\nu}$ is given in Eq. (2), and this in turn implies that $j=l+s=0$, which is absurd.

(2) The preceding result can also be found in an elementary way. We recall that at least one of the functions $f_l(p)$ is a nonvanishing function. Equations (25) and (26) then show that the application of K_3 to a certain $\varphi_{0\lambda}{}^m$ generates factors with higher values of l. We could, in the same way as Philips, try to keep l from taking higher values by choosing $f_l(p)$ in such a way that, $f_l'(p)-p^{-1}f_l(p)=0$. However, we still have linearly independent terms, with l also taking higher values and whose addition can not be made to vanish unless all $f_l(p)=0$. We again conclude that no class-I sets of localized states exist.

VIII. CLASS II

A. Spin-0 Case

It can always be assumed that the vector of Eq. (13), by means of which the set S_0 is built up, is such that $f_{L'}\neq 0$. It is therefore found from Eq. (6) and the definition of class-II sets that

$$f_{L'}' + (L+1)p^{-1} f_L = 0,$$

so that

$$f_{L'} \approx p^{-L'-1}. \quad (28)$$

The orthogonality of the spherical harmonics implies that

$$\|\varphi_{0\lambda}{}^m\| \geq \|f_{L'} Y_m{}^{L'}\|, \quad (29)$$

where the norm has been calculated by using Eq. (1). Taking into account Philips's results, it now follows that $\|f_{L'}Y_m{}^{L'}\|$ diverges so that no class-II set of localized states exists. We have therefore arrived at Philips's result (ii) (Sec. IV).

B. Spin-½ Case

We use Eqs. (25) and (26). In the same way as in Sec. VII B, we could try to keep l from taking lower values (see Sec. III) by making $f_l'(p)+(l+1)p^{-1}f_l(p)=0$ vanish identically, but, as in the above mentioned case, it is seen that the addition of the remaining terms with lower values of l only vanishes if $f_l(p)=0$ for all l. This proves that, as for spin zero, no class-II sets of localized states exist for the spin-½ case.

IX. CLASS III

A. Spin-0 Case

By considering that the angular momentum generally has neither an upper nor a lower bound, it is found that when the group operators are applied to most functions of the form $f_l(p)Y_m{}^l$, and if the resulting states are localized, then the sets of such states are class-III sets. The point is therefore to see if those states are actually localized, i.e. if the corresponding sets satisfy Philips's postulates.

It can be easily seen that they are linear manifolds satisfying postulates (a) and (c), but it is not so simple to verify whether postulate (b) is satisfied. In this respect we note that it is easy to find many $f_l(p)$ functions such that the eigendifferential derived from $f_l(p)Y_m{}^l$ has a finite norm, but we still must verify whether for the eigendifferentials derived from $(K_3)^n f_l(p)Y_m{}^l$ this condition is also satisfied. We are here faced with the difficulty that for most cases a single general expression for $(K_3)^n f_l(p)Y_m{}^l (n=1,2,\cdots)$ cannot be easily obtained. However, the explicit form of a Lorentz transformation which was found in Sec. VI is useful in the solution of this problem.

Let us take the function

$$\varphi_{00}=Q\exp[-(rp\mu^{-1}+sp^{-1}\mu)], \tag{30}$$

where Q, r, and s are non-negative constants. We shall now show that by the application of the group operators to φ_{00}, a set S_0 of states localized at the origin is generated. We indicate by $\varphi_{0\lambda}$ the transformed function resulting from an arbitrary Lorentz acceleration along the third axis. By means of Eq. (20) we obtain

$$\varphi_{0\lambda}=Q\exp[-(r\hat{p}\mu^{-1}+s\hat{p}^{-1}\mu)]. \tag{31}$$

By using Philips's result for the incomplete eigendifferential which was originally deduced for the special case of Eq. (5), and by applying a well-known test for the convergence of definite integrals,[11] we find that the norm [computed via Eq. (1)] of the incomplete eigendifferential is finite; that is,

$$\|\varphi_{0\lambda}{}^{\epsilon}\|<\infty. \tag{32a}$$

Now it follows from Eq. (16) and the mean-value theorem that this is also true for the complete eigendifferential, that is,

$$\|\varphi_{0\lambda}{}^{\epsilon\delta}\|<\infty. \tag{32b}$$

Surprisingly (see discussion in Sec. X) it is found that the norm of the states is also finite, so that we can write

$$\|\varphi_{0\lambda}\|<\infty. \tag{32c}$$

Indeed, we can compute the value of this norm by using the change of variables

$$q_1\cosh(q_3+\lambda)=x,$$
$$q_2\sinh(q_3+\lambda)=y,$$
$$\mu\sinh(q_3+\lambda)=z,$$
$$(x^2+y^2+z^2)^{1/2}=\rho,$$

from which it follows that

$$\|\varphi_{0\lambda}\|=\frac{4\pi|A|^2}{\mu}\int_0^\infty d\rho\,\rho^2\frac{\exp[-2(r\mu^{-1}\rho+s\rho^{-1}\mu)]}{(1+\rho^2/\mu^2)^{1/2}}. \tag{33}$$

Now the well-known formula

$$(K_3)^n\varphi_{00}=(\partial^n/\partial\lambda^n)e^{\lambda K_3}\varphi_{00}|_{\lambda=0} \tag{34}$$

leads us to prove, using the above-mentioned test, that the following inequalities:

$$\|(K_3)^n\varphi_{00}\|<\infty,\quad \|(K_3)^n\varphi_{00}{}^{\epsilon}\|<\infty,$$
$$\|(K_3)^n\varphi_{00}{}^{\epsilon\delta}\|<\infty, \tag{35}$$

are also true. For this proof it is useful to know that the successive derivatives of $\varphi_{0\lambda}$ are of the form

$$(\partial^n/\partial\lambda^n)\varphi_{0\lambda}=\sum_{a=-A}^{0}\sum_{b=0}^{B}\sum_{c=0}^{C}\sum_{d=0}^{D}\sum_{e=0}^{E}k^n{}_{a,b,c,d,e}p'^a(p_3)^b \times p_0{}^c(\sinh\lambda)^d(\cosh\lambda)^e, \tag{36}$$

where A, B, C, D, E are non-negative integers. Equation (36) follows from the analogous expression,

$$(\partial^n/\partial\lambda^n)p'=\sum_{a=-\bar{A}}^{0}\sum_{b=0}^{\bar{B}}\sum_{c=0}^{\bar{C}}\sum_{d=0}^{\bar{D}}\sum_{e=0}^{\bar{E}}\bar{k}^n{}_{a,b,c,d,e}p'^a(p_3)^b \times p_0{}^c(\sinh\lambda)^d(\cosh\lambda)^e, \tag{37}$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$ are also non-negative integers.

The set formed by means of the linear combinations of the functions resulting from the application of the group operators to the original function [Eq. (30)] is found to be a set S_0 of localized states. To see this, we note that postulates (a) and (c) are satisfied by construction. Now, considering that a general Lorentz transformation is the product of an acceleration along the third axis and spatial rotations, and that the action of rotation group operators can be found in an elementary way, then the above results allow us to see that postulate (b) is also satisfied by the set under consideration.

We have therefore found a set of states satisfying Philips's postulates which can be proved to be a class-III set by using the definition of class-III sets and the previous results for class-I and -II sets.

Incidentally we note that the state [Eq. (30)] which is localized at $\mathbf{x}=0$ is not completely specified by giving the position eigenvalue. [See remark (C), Sec. V.]

[11] Ch.-J. de la Vallée Poussin, *Cours d'Analyse Infinitésimale* (Gauthier-Villars, Paris, 1957).

The situation is similar for all the $\varphi_{0\lambda}$ functions we can form by using Eq. (31); the sole exception is the case where

$$K_i\varphi_{00}=0, \quad (i=1,2,3)$$

and this is only possible for Philips's localized state.

B. Spin-$\frac{1}{2}$ Case

According to Eq. (14), the analog of Eq. (30) for this spin-$\frac{1}{2}$ case is

$$\varphi_{00}=Q\exp[-(r\mu^{-1}p+sp^{-1}\mu)]E(\mathbf{p})v_{m'}. \qquad (38)$$

Now, the Lorentz-transformed function

$$\varphi_{0\lambda}=e^{\lambda K_3}\varphi_{00}, \qquad (39)$$

where K_3 is given by Eq. (3), can be written

$$\varphi_{0\lambda}=Q[\cosh(\tfrac{1}{2}\lambda)-\alpha_3\sinh(\tfrac{1}{2}\lambda)] \times\exp[-(r\mu^{-1}p'+\mu sp'^{-1})]E(\mathbf{p})v_{m'}. \qquad (40)$$

This latter result is obtained by using Eq. (23).

It can be proved that from Eq. (38) a set of states satisfying Philips's postulates for the spin-$\frac{1}{2}$ case can be obtained and that this is a class-III set. The proof is similar to the one we used for the spin-0 case and will therefore be omitted.

X. CONCLUSIONS

A more rigorous method which leads to Philips's results concerning class-I and class-II sets has been given, but our main result is that opposite to Philips's assumption for the spin-0 case, class-III sets of localized states are found to exist. We have also found that class-III sets exist for spin-$\frac{1}{2}$ systems.

We now face the following question: Considering the state found by Philips [see (i), Sec. IV] and the new states (Sec. IX), are all of them "true" localized states or must Philips's postulates be complemented with stronger requirements in order to uniquely define the localized states? If the latter point of view is correct, then one can ask whether the "true" localized states belong to class-I or class-III sets. As regards this last question we believe that the true localized states probably belong to class-III sets because if this were not so we would be faced with the difficulty that for spin-$\frac{1}{2}$ systems we would not have localized states. Indeed, our results show that Philips's localized state for spin-0 systems is not a typical example from which consequences for any other spin can be deduced.

On the other hand, if the true localized states belong to class-III sets, an accordance with the ideas expressed in Ref. 6 (especially in the fourth article, Sec. III) is obtained. Moreover, it is deduced from these ideas that the space S_0 of states localized at the origin is infinite-dimensional, the states depending on a continuous parameter which is found to be the relative velocity between inertial frames with which we deal in a pure Lorentz transformation. It is easily seen that due to the relation between λ and the relative velocity between inertial frames this is just what happens in our example of Sec. IX.

Let us mention here that Philips proposes an alternative set of postulates where the normalization postulate (b) is replaced by a new one. This new postulate requires the linear manifold S_0 of localized states to have a finite basis, leading therefore to the conclusion that no spin-$\frac{1}{2}$ elementary system could be localized. This is an undesirable consequence because there is no physical reason supporting the fact that spin-0 elementary systems can be localized while spin-$\frac{1}{2}$ systems cannot, so that we must probably disregard this alternative set of postulates.

As a last remark we recall that, not only the eigendifferentials derived from the localized states [Eq. (31)], but the states themselves also have a finite norm as mentioned in Sec. IX. This may seem strange because position has a continuous spectrum and the eigenfunctions of a continuous spectrum operator are usually normalizable in terms of Dirac's δ function. However, we note that this theorem of normalization cannot be applied to position eigenfunctions because it is based on the Hermiticity of the corresponding operator or at least on the normality, i.e. $[A,A^\dagger]=0$. Considering that we are not dealing with a normal operator, the localized states do not have to be normalizable in terms of Dirac's δ function.

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