Current-Algebra Determination of Low-Energy Pion-Nucleon Scattering

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The P-wave pion-nucleon scattering lengths and S-wave effective ranges are studied by means of the current-commutator algebra and the hypothesis of partially conserved axial-vector current (PCAC). This requires a model for the weak axial-vector production; but it is shown that the production of baryon resonances with $J=\frac{3}{2}^-$ and $J>\frac{3}{2}$ do not contribute to these threshold quantities, so that only the nucleon and N^* contributions are included in the model for the weak amplitude. The predicted P-wave scattering lengths are in reasonable agreement with experiment, while a discrepancy for the isotopic even effective range implies a possible $T=0$ exchange in the weak amplitude. The other S-wave effective range gives a determination of the induced pseudoscalar coupling which is a factor 2 smaller than the PCAC value.

I. INTRODUCTION

THE initial success of the Adler-Weisberger[1] sum rule in correlating the axial-vector coupling constant g_A with pion-nucleon scattering by means of current-commutator rules and PCAC (partial conservation of the axial-vector current) has been repeated in many other applications of these assumptions. Of particular interest is the work of Weinberg[2], which recasts the relations of Ref. 1 into a threshold theorem for pion scattering which agrees well with experiment. A significant part of this result is that PCAC indicates that the weak amplitude plays no role in determining the S-wave scattering lengths. Unfortunately this situation does not persist away from threshold; the details of the weak amplitude must enter the discussion. In the absence of sufficient experimental information on weak axial production one must resort to model making. However, if we are only interested in the P-wave scattering lengths or S-wave effective ranges, then the task of the construction of such a model may be simplified. Furthermore, there is compensation in the interrelation of strong interactions with the details of the weak axial production amplitude and electromagnetic vertex. It is this particular feature of the current commutator which is so powerful. Since we dig deeper into the weak amplitude, we get both further tests of the basic assumptions and additional information on models for the weak production.

In this paper we discuss the pion-nucleon P-wave scattering lengths and S-wave effective ranges which are the terms which go as $|\mathbf{q}|^2$ at threshold. If we can identify all the terms of this order, then we should be able to predict these quantities reasonably accurately. In Sec. II we discuss the general approach to the problem, where we argue that the nucleon and N^* are essentially the only baryon states which contribute to the quantities we are calculating. In fact in an Appendix we prove that baryon states with $J=\frac{3}{2}^-$ and $J>\frac{3}{2}$ do not contribute, so that our approach is not just the saturation of a sum rule by the lowest-lying states. In Sec. III we construct the model for the weak amplitude. The P-wave scattering lengths are discussed in detail in Sec. IV, while Sec. V is devoted to the S-wave effective ranges. An interesting feature which emerges is a relation between the induced pseudoscalar coupling constant and an S-wave effective range. The over-all results suggest that the methods employed are sound; however, we find numerical disagreements for the isotopic even amplitudes which are fairly small for the P waves, but noticeable for the S-wave case. We interpret this to mean that an isotopic scalar-exchange contributes a term to the weak axial-vector amplitude.

II. OFF-SHELL PION-NUCLEON SCATTERING

The current-commutation relations and PCAC assumption provide a connection between pion-nucleon scattering and the elastic scattering of nucleons by a weak axial-vector current. The basic relation[1] is

$$\int d^4x d^4y\, e^{iq\cdot x}e^{-ik\cdot y}\langle p_2|T[\partial_\mu A_\mu{}^b(x),\partial_\nu A_\nu{}^a(y)]|p_1\rangle = q_\mu k_\nu \int d^4x d^4y\, e^{iq\cdot x}e^{-ik\cdot y}\langle p_2|T[A_\mu{}^b(x),A_\nu{}^b(y)]|p_1\rangle$$
$$-\sqrt{2}q_\mu\epsilon_{abc}\int d^4x\, e^{i(q-k)\cdot x}\langle p_2|V_\mu{}^c(x)|p_1\rangle - C\int d^4x\, e^{i(q-k)\cdot x}\langle p_2|\sigma_{ab}(x)|p_1\rangle \quad (1)$$

obtained from the equal-time commutation relations

$$\delta(x_0-y_0)[A_0{}^a(x),A_\mu{}^b(y)]=\sqrt{2}i\epsilon_{abc}\delta^4(x-y)V_\mu{}^c(x) \quad (2)$$

and

$$\delta(x_0-y_0)[A_0{}^a(x),\partial_\mu A_\mu{}^b(y)]=C\sigma_{ab}(x)\delta^4(x-y),$$

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[1] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); Phys. Rev. **140**, B736 (1965); W. I. Weisberger, Phys. Rev. Letters **14**, (1965); Phys. Rev. **143**, 1302 (1966).

[2] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966). Earlier papers which calculate the S-wave πN scatterings lengths from current algebras are Y. Tomozawa, Nuovo Cimento **46A**, 707 (1967); K. Raman and E. C. G. Sudarshan, Phys. Rev. Letters **21**, 450 (1966); A. P. Balachandian, M. G. Gunzik, and F. Nicodemi, Nuovo Cimento **44A**, 1257 (1966); however we prefer Weinberg's method because of its clarity.

where superscripts a, b, c are isospin indices, $A_\mu(x)$ the axial-vector current, $V_\mu(x)$ the vector current, $\sigma_{ab}(x)$ a scalar field, and p_1, (p_2) the initial (final) nucleon momentum. We *define* the off-mass-shell pion-nucleon scattering amplitude by the equation

$$T_{ba}(p_2,q;\,p_1,k)=-i\left(\frac{E_2}{m}\frac{E_1}{m}\right)^{1/2}(2\pi)^3\frac{(q^2-\mu^2)(k^2-\mu^2)}{\mu^4 f_\pi^2}\times\int d^4z\, e^{iq\cdot z}\langle p_2|T[\partial_\mu A_\mu{}^b(z),\partial_\nu A_\nu{}^a(0)]|p_1\rangle, \quad (3)$$

with $p_1+k=p_2+q$, μ the pion mass, and f_π given approximately by the Goldberger-Treiman relation

$$f_\pi\simeq i\sqrt{2}mg_A/g_{\pi N}(0)\,,$$

where m is the nucleon mass. Thus $T_{ba}(p_2,q;\,p_1,k)$ describes the scattering of a pion of momentum k and isospin a from a target nucleon with momentum p_1 to a pion of momentum q and isospin b and nucleon with momentum p_2. When the pions are on the mass shell, the S-matrix element for this process is given by

$$\langle p_2,q;\,b|S-1|p_1,k;\,a\rangle = \lim_{q^2\to\mu^2;\,k^2\to\mu^2}\frac{(2\pi)\delta^4(p_2+q-p_1-k)}{(2\omega_q)^{1/2}(2\omega_k)^{1/2}}\frac{(q^2-\mu^2)(k^2-\mu^2)}{\mu^4 f_\pi^2}\times\int d^4z\, e^{iq\cdot z}\langle p_2|T[\partial_\mu A_\mu{}^b(z),\partial_\nu A_\nu{}^a(0)]|p_1\rangle. \quad (4)$$

For convenience we write the amplitude for the axial-vector scattering by nucleons as

$$(2\pi)^4\delta^4(p_2+q-p_1-k)R_{\mu\nu}{}^{ba}=\int d^4x d^4y e^{iq\cdot x}e^{-ik\cdot y}\times\langle p_2|T[A_\mu{}^b(x),A_\nu{}^a(y)]|p_1\rangle. \quad (5)$$

As a result of Adler's[3] consistency condition we neglect the contribution of σ_{ab} in the following. For convenience in the subsequent discussion we set $k^2=q^2$. The matrix elements of the vector current together with Eqs. (3) and (5) enable us to write

$$T_{ba}(p_2,q;\,p_1,k)=-i\left(\frac{q^2-\mu^2}{\mu^2 f_\pi}\right)^2\left\{(2\pi)^3\left(\frac{E_2}{m}\frac{E_1}{m}\right)^{1/2}q_\mu k_\nu R_{\mu\nu}{}^{ba}-\epsilon_{abc}\times\bar{u}(p_2)\tau_c\left[-\frac{(p_1+p_2)\cdot q}{2m}F_2(t)+\gamma\cdot Q(F_1(t)+F_2(t))\right]u(p_1)\right\}, \quad (6)$$

where $F_{1,2}(t)$ are the Dirac isovector electromagnetic form factors, with $F_1(0)=\frac{1}{2}$, $F_2(0)=1.85$, $t=(p_2-p_1)^2$, and $Q=\frac{1}{2}(q+k)$. At threshold the PCAC assumption indicates that the coefficients of q^2, k^2, and $q\cdot k$ are small compared to those of $p\cdot q$ and $p\cdot k$; thus the term $q_\mu k_\nu R_{\mu\nu}$, which is $O(q^2,k^2,q\cdot k)$, may be neglected at threshold. One is then able to cast (6) into a threshold theorem for the S-wave pion-nucleon scattering lengths which agrees well with the experimental values.

The relationship (6) is supposed to be true for all energies and momentum transfers, although this has not been fully exploited. It is desirable to extend the results away from threshold to obtain further tests of the basic assumptions. One might be tempted to compute the P-wave scattering lengths by either neglecting $q_\mu k_\nu R_{\mu\nu}$ entirely since it is $O(q^2,k^2,q\cdot k)$, or perhaps by keeping only the nucleon-pole contribution to $q\cdot R\cdot k$. Unfortunately, one must dig deeper into the axial-vector scattering. This becomes clear if one separates the terms in Eq. (6) into invariant amplitudes:

$$T_{ba}(s,t,q^2)=A_{ba}(s,t,q^2)-\gamma\cdot QB_{ba}(s,t,q^2)\,, \quad (7)$$

with

$$A_{ba}=\delta_{ba}A^{(+)}+\tfrac{1}{2}[\tau_b,\tau_a]A^{(-)}\,,$$
$$B_{ba}=\delta_{ba}B^{(+)}+\tfrac{1}{2}[\tau_b,\tau_a]B^{(-)}\,,$$

where

$$A^{(+)}=\tfrac{1}{3}(A^{1/2}+2A^{3/2})\quad\text{and}\quad A^{(-)}=\tfrac{1}{3}(A^{1/2}-A^{3/2}).$$

Also

$$W^2=s=(p_1+k)^2,\quad u=(p_2-k)^2,\quad\text{and}\quad s+t+u=2m^2+2q^2.$$

The off-shell scattering amplitude in the center-of-mass system is

$$F(s,\cos\theta)=(m/4\pi W)T,\quad\text{where}\quad d\sigma/d\Omega=|F|^2.$$

Important kinematical relations are

$$F=f_1+\frac{(\boldsymbol{\sigma}\cdot\mathbf{q})(\boldsymbol{\sigma}\cdot\mathbf{k})}{|\mathbf{q}|\,|\mathbf{k}|}f_2\,,$$

$$f_1=(16\pi W^2)^{-1}[(W+m)^2-q^2][A-(W-m)B]\,,$$

and (8)

$$f_2=(16\pi W^2)^{-1}[(W-m)^2-q^2][-A-(W+m)B]\,,$$

where

$$|\mathbf{q}|^2=[(W+m)^2-q^2][(W-m)^2-q^2]/4W^2$$

is the pion 3-momentum squared in the c.m. system, and

$$f_{l\pm}=e^{i\delta_{l\pm}}\sin\delta_{l\pm}/|q| =\frac{1}{2}\int_{-1}^{1}dx[f_1(x)P_l(x)+f_2(x)P_{l\pm1}(x)]. \quad (9)$$

This then gives the connection between the partial-wave

[3] S. L. Adler, Phys. Rev. **137**, B1022 (1965); **139**, B1638 (1965).

amplitudes $f_{l\pm}$ and the invariant amplitudes

$$f_{l\pm}=(16\pi W^2)^{-1}\{[(W+m)^2-q^2][A_l-(W-m)B_l] \\ -[(W-m)^2-q^2][A_{l\pm1}+(W+m)B_{l\pm1}]\}, \quad (10)$$

where

$$A_l(s)=\frac{1}{2}\int_{-1}^{1}dx\,P_l(x)A(s,t),$$

and similarly for $B_l(s)$. One also expands the right-hand side of (6) in terms of the same set of invariants. However, even though $q\cdot R\cdot k$ is negligible at threshold, it does *not* mean that the contribution of the axial-vector scattering to A and B separately is negligible. Quite the contrary, the contribution to each of these terms from the axial-vector, scattering is large, but they cancel in the combination $A-\nu_tB$ at threshold when $q^2\rightarrow 0$ (where we have defined $\nu_t=p_1\cdot k/m$ at threshold for convenience).

From Eq. (10) we see that the P-wave scattering lengths and S-wave effective ranges depend on A_0, B_0, A_1, and B_1 evaluated at threshold, but not on A_2 or B_2, since these terms vanish like $|\mathbf{q}|^4$ or faster. Thus a discussion of these quantities requires that we include all low-lying S and P states which contribute to the axial-vector scattering (or weak axial production of strongly interacting states if we use unitarity). If we take this to mean only single-particle baryon states, then only the nucleon and N^* states qualify. This is more than just a simple saturation argument, since the higher spin resonances (with s and u poles taken together) do not contribute to these threshold quantities (see Appendix). Exchange poles in the t channel could also contribute to the axial scattering; however, we will not include them in our calculation. The validity of this last assumption will be examined in the light of our results.

III. THE WEAK AMPLITUDE

In the previous section we discussed the connection between the weak axial-vector scattering amplitude and off-shell pion-nucleon scattering. It is clear that to determine the P-wave scattering lengths[4] and S-wave effective ranges we must find an accurate model for $q_\mu R_{\mu\nu}k_\nu$. The first term we consider is the one-nucleon intermediate state which can be computed from the nucleon matrix elements of the axial-vector current,

$$\langle p_2|A_\mu{}^b(0)|p_n\rangle=\frac{1}{(2\pi)^3}\left(\frac{m}{E_2}\frac{m}{E_n}\right)^{1/2}\bar{u}(p_2) \\ \times[g_A(q^2)i\gamma_\mu\gamma_5+h_A(q^2)iq_\mu\gamma_5]\frac{\tau^b}{\sqrt{2}}u(p_n), \quad (11)$$

with $q=(p_n-p_2)$. The principle of PCAC enables us to write

$$(q^2-\mu^2)[-2mg_A(q^2)+q^2h_A(q^2)] \\ =-i\sqrt{2}f_\pi\mu^2 g_{\pi N}(q^2), \quad (12)$$

where $g_{\pi N}(\mu^2)=g_{\pi N}$ is the conventional pion-nucleon coupling constant $[g_{\pi N}{}^2/4\pi\simeq 14.6]$. It is now straightforward to evaluate the nucleon contribution to the right-hand side of Eq. (6):

$$A_N{}^{(-)}(s,t,q^2)=0, \quad (13\text{a})$$

$$A_N{}^{(+)}=-2\left(\frac{q^2-\mu^2}{\mu^2 f_\pi}\right)^2[mg_A(q^2)-q^2h_A(q^2)]g_A(q^2), \quad (13\text{b})$$

$$B_N{}^{(-)}=\frac{g_{\pi N}{}^2(q^2)}{s-m^2+i\epsilon}+\frac{g_{\pi N}{}^2(q^2)}{u-m^2+i\epsilon} \\ -\left(\frac{q-\mu^2}{\mu^2 f_\pi}\right)^2 g_A{}^2(q^2), \quad (13\text{c})$$

and

$$B_N{}^{(+)}=\frac{g_{\pi N}{}^2(q^2)}{s-m^2+i\epsilon}-\frac{g_{\pi N}{}^2(q^2)}{u-m^2+i\epsilon}, \quad (13\text{d})$$

where the subscript N indicates that these terms come from the nucleon poles.

The contribution of the (33) resonance intermediate state is more complicated since there is no unique model for the resonance production. Here we choose a particularly simple model, namely the direct production of a narrow N^* state. With this choice of production mechanism the remaining calculations are lengthy but straightforward. There are four linearly independent form factors[5] for the N^*-N axial-vector vertex with both baryons on their mass shell, which we choose as follows (suppressing isospin indices):

$$\langle N^*(p)|A_\nu(0)|p_1\rangle \\ =\frac{1}{(2\pi)^3}\left(\frac{m}{E_1}\frac{M}{E}\right)^{1/2}\bar{\psi}_\sigma(p)\{g_1(k^2)k_\sigma(k^2p_{1\nu}-k_\nu p_1\cdot k) \\ +ig_2(k^2)k_\sigma(\gamma_\alpha\epsilon_{\alpha\rho\lambda\nu}p_{1\rho}k_\lambda\gamma_5)+ig_3(k^2)\epsilon_{\sigma\alpha\beta\gamma} \\ \times\epsilon_{\gamma\rho\tau\nu}p_{1\alpha}p_{1\rho}k_\beta k_\tau+ig_A{}^*(k^2)\delta_{\sigma\nu}\}u(p_1), \quad (14)$$

with $k=p-p_1$, and $\bar{\psi}_\sigma(p)$ is the Rarita-Schwinger wave function for a particle of spin $\frac{3}{2}+$ and mass M. It is clear that the coefficients of g_1, g_2, and g_3 are transverse to k, and hence do not contribute to Eq. (6), while the coefficient of $g_A{}^*$ is longitudinal. If we assume that $R_{\mu\nu}$ satisfies an unsubtracted dispersion relation for each of its invariant amplitudes, and recall that the spin-$\frac{3}{2}+$ projection operator is

$$\mathcal{P}_{\mu\nu}=\left[g_{\mu\nu}-\frac{2}{3M^2}p_\mu p_\nu-\tfrac{1}{3}\gamma_\mu\gamma_\nu-\frac{1}{3M}(p_\mu\gamma_\nu-\gamma_\mu p_\nu)\right]\frac{\gamma\cdot p+M}{2M},$$

[4] See also S. L. Adler (unpublished). K. Raman, Phys. Rev. Letters 17, 983 (1966), studied the P-wave scattering lengths, but did not examine the weak amplitude in sufficient detail. [However, see Errata (to be published).] A. P. Balachandian, M. G. Gundzik, and F. Nicodemi (to be published) study the same question with a different philosophy and obtain very different results.

[5] J. D. Bjorken and J. D. Walecka, Ann. Phys. (N. Y.) 38, 35 (1966).

then it is easy to compute the N^* contribution. The results are

$$A_{N^*}{}^{(-)}(s,t,q^2)=-\frac{g_A{}^{*2}(q^2)}{6f_\pi{}^2}\left(\frac{q^2-\mu^2}{\mu^2}\right)^2\left\{\left(\frac{1}{s-M^2+i\epsilon}-\frac{1}{u-M^2+i\epsilon}\right)\times\left[(M+m)t+2q^{*2}\left[(M+m)+\frac{(M-m)(E^*+m)}{3(E^*-m)}\right]\right]+\frac{m(s-u)}{3M^2}\right\},\quad(15\text{a})$$

$$A_{N^*}{}^{(+)}=\frac{g_A{}^{*2}(q^2)}{3f_\pi{}^2}\left(\frac{q^2-\mu^2}{\mu^2}\right)^2\left\{\left(\frac{1}{s-M^2+i\epsilon}+\frac{1}{u-M^2+i\epsilon}\right)\left[(M+m)t+2q^{*2}\left[(M+m)+\frac{(M-m)(E^*+m)}{3(E^*-m)}\right]\right]+\frac{1}{3M^2}[2M^3+2(M+m)(M^2+2Mm-2m^2)+4(M+m)q^2+m(s+u)]\right\},\quad(15\text{b})$$

$$B_{N^*}{}^{(-)}=\frac{g_A{}^{*2}(q^2)}{6f_\pi{}^2}\left(\frac{q^2-\mu^2}{\mu^2}\right)^2\left\{\left(\frac{1}{s-M^2+i\epsilon}+\frac{1}{u-M^2+i\epsilon}\right)\left[t+2q^{*2}\left[1-\frac{(E^*+m)}{3(E^*-m)}\right]\right]+(3M^2)^{-1}[2(M^2+2Mm-2m^2+2q^2)+(s+u)]\right\},\quad(15\text{c})$$

and

$$B_{N^*}{}^{(+)}=-\frac{g_A{}^{*2}(q^2)}{3f_\pi{}^2}\left(\frac{q^2-\mu^2}{\mu^2}\right)^2\left\{\left(\frac{1}{s-M^2+i\epsilon}-\frac{1}{u-M^2+i\epsilon}\right)\left[t+2q^{*2}\left[1-\frac{(E^*+m)}{3(E^*-m)}\right]\right]+\frac{(s-u)}{3M}\right\}.\quad(15\text{d})$$

In these expressions q^* and E^* represent the c.m. 3-momentum and nucleon total energy evaluated at the πN resonance. In detail,

$$E^*+m=(2M)^{-1}[(M+m)^2-q^2]$$

and

$$q^{*2}=(E^*+m)(E^*-m)=[(M-m)^2(M+m)^2-2q^2(M^2+m^2)+q^4]/4M^2.$$

Thus Eqs. (13) and (15) together describe the N and N^* terms of the weak axial-vector contribution to the off-shell pion scattering. As we have argued above and in the Appendix, if t-channel exchanges can be neglected, these equations should provide all the information for the calculation of the P-wave scattering lengths and S-wave effective ranges.

For completeness we rewrite the contribution of the equal-time commutator:

$$A_c{}^{(-)}(s,t,q^2)=\frac{1}{f_\pi{}^2}\left(\frac{q^2-\mu^2}{\mu^2}\right)^2[s+\tfrac{1}{2}t-m^2-q^2]\frac{F_2(t)}{m},\quad(16\text{a})$$

$$A_c{}^{(+)}=0,\quad(16\text{b})$$

$$B_c{}^{(-)}=\frac{2}{f_\pi{}^2}\left(\frac{q^2-\mu^2}{\mu^2}\right)^2[F_1(t)+F_2(t)],\quad(16\text{c})$$

and

$$B_c{}^{(+)}=0.\quad(16\text{d})$$

Now we must relate $g_A{}^*$ to the strong-interaction πNN^* vertex by means of PCAC. We could do this directly by considering Eq. (14) and the $N^*N\partial_\mu A_\mu$ vertex in the standard way; however, we will take a more round-about route since (1) we will easily be able to take into account the large width of the N^*, and (2) our method may be useful for other applications. We begin by inserting Eqs. (13), (15), and (16) into (6), and by noting that

$$T^{(-)}(s,t,q^2)\underset{s\to\infty}{\longrightarrow}\left(\frac{q^2-\mu^2}{\mu^2 f_\pi}\right)^2\left\{\left[\frac{g_A{}^{*2}(q^2)m(u-s)}{18M^2}+\left(s+\frac{t}{2}-m^2-q^2\right)\frac{F_2(t)}{m}\right]+\gamma\cdot Q\left[g_A{}^2(q^2)-\frac{g_A{}^{*2}(q^2)}{18M^2}(2M^2+4Mm-4m^2+4q^2+s+u)-2(F_1(t)+F_2(t))\right]\right\}$$

$$+(\text{additional polynomial terms}).\quad(17)$$

The additional polynomial terms come from higher resonances and high-energy contributions, which although not important for near-threshold properties, do appear in (17). Their explicit form is unimportant for our purposes; all we need note is that if $R_{\mu\nu}^{(-)}$ satisfies unsubtracted dispersion relations, then the polynomial does not increase faster than s as $s\to\infty$. This becomes obvious when one generalizes our results for a $\frac{3}{2}^+$ resonance to arbitrary spin. The requirement that charge-exchange scattering vanish as $s\to\infty$ for off-shell pions requires the coefficient of s in (17) to vanish. For forward scattering this condition implies

$$1=g_A{}^2(q^2)-\frac{g_A{}^{*2}(q^2)}{9M^2}[(M+m)^2+3q^2] +(\text{additional terms}). \quad (18)$$

With a little thought one can see that the relation (coefficient of s)$=0$ in Eq. (17) is equivalent to the usual current-commutator sum rules derived by means of Fubini's method.[6] This is clear since the terms that appear in (17) are just the difference of the Hilbert transforms

$$q_\mu[\mathcal{H}r_{\mu\nu}]k_\nu-\mathcal{H}[q_\mu r_{\mu\nu}k_\nu], \quad (19)$$

where $r_{\mu\nu}$ is the absorptive part of the weak amplitude $[R_{\mu\nu}=\mathcal{H}r_{\mu\nu}]$. Our assumption for $T^{(-)}(s\to\infty)$ is then equivalent to a statement about the convergence properties of the dispersion relations $\mathcal{H}[q_\mu r_{\mu\nu}k_\nu]$. This then is sufficient to establish the equivalence of our condition and Fubini's method. In a similar way, if we require the off-shell total cross section to go to a constant as $s\to\infty$, then the constant term in the equation for $T^{(+)}(s\to\infty)$ yields a sum-rule for the isotopic even amplitude. One can now see that if we set $q^2=0$ in (18) we obtain the usual Adler-Weisberger sum rule

$$1=g_A{}^2-\frac{g_A{}^{*2}(0)}{9M^2}(M+m)^2+(\text{additional terms}), \quad (20)$$

which is to be compared with

$$1=g_A{}^2+\frac{2m^2g_A{}^2}{g_{\pi N}{}^2(0)}\frac{1}{\pi}\int\frac{d\nu}{\nu}[\sigma(\pi^-p)-\sigma(\pi^+p)], \quad (21)$$

where the cross sections refer to zero-mass pions. From this comparison it follows that

$$\frac{g_A{}^{*2}(0)}{g_A{}^2}=\frac{12M^2}{(M+m)^2}\frac{m^2}{g_{\pi N}{}^2(0)}\frac{1}{\pi}\int\frac{d\nu}{\nu}\sigma(T=\tfrac{3}{2}), \quad (22)$$

where the integral is taken over the (33) resonance. If Eq. (22) is used to evaluate $g_A{}^*$, this allows us to take into account the large width of the N^* and hence is superior to estimates based on a narrow resonance. We can use Adler's and Weisberger's numerical computations[1] for the integral to evaluate (22). From Adler's value for the integral, which includes a correction for the zero mass of the pion, and hence presumably the more accurate, we find

$$g_A{}^{*2}(0)/g_A{}^2=1.4, \quad (23a)$$

while from Weisberger's results, which do not include this correction,

$$g_A{}^{*2}(0)/g_A{}^2=1.2; \quad (23b)$$

and finally the usual saturation arguments applied to the Adler-Weisberger sum rules generalized to $SU(3)$[7] imply that

$$g_A{}^{*2}(0)/g_A{}^2=1.7. \quad (23c)$$

IV. THE *P*-WAVE SCATTERING LENGTHS

In order to compute the P-wave scattering lengths, we have to use Eqs. (13), (15), and (16) in conjunction with (10). Our procedure is to evaluate $f_{1\pm}/|\mathbf{q}|^2$ at threshold for the off-shell scattering in the limit $q^2\to 0$. The basic assumption of PCAC is that the off-shell scattering length will extrapolate smoothly back to the pion mass shell. In practice this means replacing off-shell coupling constants by their mass-shell values, and replacing $\nu_t\to\mu$, in the final result [e.g., $g_{\pi N}{}^2(0)\to g_{\pi N}{}^2$]. At threshold one finds

$$f_{1-}/|\mathbf{q}|^2\to\frac{1}{4\pi(m+\nu_t)}\times\left\{\frac{m}{|\mathbf{q}|^2}(A_1-\nu_tB_1)-\frac{1}{4m}[A_0+(2m+\nu_t)B_0]\right\}, \quad (24a)$$

$$f_{1+}/|\mathbf{q}|^2\to\frac{m}{4\pi(m+\nu_t)}[A_1-\nu_tB_1]\frac{1}{|\mathbf{q}|^2}. \quad (24b)$$

The calculation is facilitated by noting that

$$A(t)=A(0)+t(\partial A/\partial t)+\cdots,$$

where $t=-2|\mathbf{q}|^2(1-\cos\theta)$ in the c.m. system, so that

$$A_0=A(0)-2|\mathbf{q}|^2\left(\frac{\partial A}{\partial t}\right)_{t=0},$$

and

$$A_1=\frac{2|\mathbf{q}|^2}{3}\left(\frac{\partial A}{\partial t}\right)_{t=0}.$$

This enables us to simplify Eq. (24) as follows:

$$\frac{f_{1-}}{|\mathbf{q}|^2}\to\frac{1}{4\pi(m+\nu_t)}\left\{\frac{2m}{3}\left[\frac{\partial A}{\partial t}-\nu_t\frac{\partial B}{\partial t}\right] -\frac{1}{4m}[A+(2m+\nu_t)B]\right\}_{t=0}, \quad (25a)$$

[6] S. Fubini, Nuovo Cimento **43A**, 475 (1966).

[7] I. S. Gerstein, Phys. Rev. Letters **16**, 114 (1966); H. J. Schnitzer, Phys. Letters **20**, 539 (1966).

TABLE I. The contributions of the various terms to the P-wave scattering lengths for $(g_A^*/g_A)^2=1.4$. The theoretical results are compared to the experimental analyses of Hamilton and Woolcock (Ref. 8) and Roper *et al.* (Ref. 9).

Amplitude	N	N^*	Commutator	Total	Experiment (HW)	Experiment (Roper)
$f_{1+}^{(-)}/\lvert\mathbf{q}\rvert^2$	−0.054	−0.014	−0.007	−0.075	−0.081	−0.081
$f_{1-}^{(-)}/\lvert\mathbf{q}\rvert^2$	−0.054	+0.016	+0.033	−0.005	−0.021	−0.001
$f_{1+}^{(+)}/\lvert\mathbf{q}\rvert^2$	+0.054	+0.060	0	+0.114	+0.134	+0.134
$f_{1-}^{(+)}/\lvert\mathbf{q}\rvert^2$	−0.108	+0.038	0	−0.070	−0.059	−0.039

and

$$\frac{f_{1+}}{|\mathbf{q}|^2}\to\frac{1}{4\pi(m+\nu_t)}\frac{2m}{3}\left[\frac{\partial A}{\partial t}-\nu_t\frac{\partial B}{\partial t}\right]_{t=0}\tag{25b}$$

evaluated at threshold with $q^2\to 0$. We will not display the complete results explicitly, but merely write an approximate expression for (25) keeping the numerically dominant part of the N^* contribution, and the complete N and commutator terms. It is interesting that the main part of the N^* term comes from the pole terms in Eq. (15). Notice also that in these expressions, because of an equivalence of $ps(ps)$ and $ps(pv)$ theories for the P-wave part of the nucleon poles, the nucleon term is identical to the usual Born approximation for π-N scattering. Finally we remark that we have already continued these expressions back to the mass shell by means of the replacements $\nu_t\to\mu$ and $g_{\pi N}^2(0)\to g_{\pi N}^2$. Our approximate scattering lengths are

$$\frac{f_{1+}^{(-)}}{|\mathbf{q}|^2}\to f^2\left\{-\frac{2}{3}-\frac{1}{9}\left(\frac{g_A^*}{g_A}\right)^2\frac{\mu^2(M+m)^2}{M^2q^{*2}}+\frac{4}{3g_A^2}\times\left[-\frac{\mu}{2m}F_2(0)+2\mu^2\frac{\partial F_1}{\partial t}(0)\right]\right\}\frac{1}{\mu^3},\tag{26a}$$

$$\frac{f_{1-}^{(-)}}{|\mathbf{q}|^2}\to f^2\left\{-\frac{2}{3}+\frac{1}{9}\left(\frac{g_A^*}{g_A}\right)^2\frac{\mu(M+m)^2}{M^2q^{*2}}[(M-m)-\mu]+\frac{4}{3g_A^2}\left[-\frac{\mu}{2m}F_2(0)+2\mu^2\frac{\partial F_1(0)}{\partial t}\right]+\frac{2}{g_A^2}\left(\frac{\mu}{m}\right)[F_1(0)+F_2(0)]\right\}\frac{1}{\mu^3},\tag{26b}$$

$$\frac{f_{1+}^{(+)}}{|\mathbf{q}|^2}\to f^2\left\{-\frac{4}{3}+\frac{2}{9}\left(\frac{g_A^*}{g_A}\right)^2\frac{\mu(M+m)^2(M-m)}{M^2q^{*2}}\right\}\frac{1}{\mu^3},\tag{26c}$$

TABLE II. Same as Table I, but for $(g_A^*/g_A)^2=1.7$.

Amplitude	N	N^*	Commutator	Total	Experiment (HW)	Experiment (Roper)
$f_{1+}^{(-)}/\lvert\mathbf{q}\rvert^2$	−0.054	−0.018	−0.007	−0.079	−0.081	−0.081
$f_{1-}^{(-)}/\lvert\mathbf{q}\rvert^2$	−0.054	+0.019	+0.033	−0.002	−0.021	−0.001
$f_{1+}^{(+)}/\lvert\mathbf{q}\rvert^2$	+0.054	+0.074	0	+0.128	+0.134	+0.134
$f_{1-}^{(+)}/\lvert\mathbf{q}\rvert^2$	−0.108	+0.047	0	−0.061	−0.059	−0.039

and

$$\frac{f_{1-}^{(+)}}{|\mathbf{q}|^2}\to f^2\left\{-\frac{4}{3}+\frac{2}{9}\left(\frac{g_A^*}{g_A}\right)^2\times\frac{\mu(M+m)^2}{M^2q^{*2}}[(M-m)-\mu]\right\}\frac{1}{\mu^3},\tag{26d}$$

where we have set

$$(g_{\pi N}^2/4\pi)(\mu/2m)^2=f^2\simeq 0.08.$$

We have made our numerical evaluation of these scattering lengths using the complete expression [rather than Eq. (26)] with the results displayed in Tables I and II for $(g_A^*/g_A)^2=1.4$ and 1.7, respectively. Our predictions are compared with the analyses of Hamilton and Woolcock[8] and Roper *et al.*[9] There is certainly very reasonable agreement (particularly with the Roper phase shifts); however, the quantitative agreement is not quite as good as one has come to expect from current algebras. If we restrict our attention to the Roper *et al.* scattering lengths, then the disagreement is confined to the amplitudes $f_{1\pm}^{(+)}/|\mathbf{q}|^2$. There is the possibility that this situation would be improved if we included the Roper resonance $P_{11}(1480)$ in the weak amplitude. A second possibility is that a more accurate treatment of the N^* production, using a finite width from the beginning of the calculation, may be required. The explanation we prefer is that a $T=0$ exchange in the t channel should be added to the weak amplitude. The P-wave scattering lengths do not make this compelling, but in the next section we give evidence from the S-wave effective ranges which makes this the favored explanation.

V. THE S-WAVE EFFECTIVE RANGES

The calculation of the S-wave effective ranges is very similar to that of the P-wave scattering lengths, so we need only sketch the procedure. From Eq. (10) one finds

$$\frac{\partial f_{0+}}{\partial|\mathbf{q}|^2}\to\frac{1}{4\pi(m+\nu_t)}\left\{m\left[\frac{\partial A_0}{\partial|\mathbf{q}|^2}-\nu_t\frac{\partial B_0}{\partial|\mathbf{q}|^2}\right]-\frac{1}{4m}[A_1+(2m+\nu_t)B_1]-\frac{1}{2\nu_t}[A_0+mB_0]\right\}.\tag{27}$$

Again expanding the amplitudes as in the previous

[8] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963); abbreviated HW.

[9] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. 138, B190 (1965).

section, one can recast this as

$$\frac{\partial f_{0+}}{\partial|\mathbf{q}|^2}\to\frac{1}{4\pi}\left\{\frac{(m+\nu_t)}{\nu_t}\left[\frac{\partial A}{\partial s}-\nu_t\frac{\partial B}{\partial s}\right]-\left[\frac{\partial A}{\partial q^2}-\nu_t\frac{\partial B}{\partial q^2}\right]-2\left[\frac{\partial A}{\partial t}-\nu_t\frac{\partial B}{\partial t}\right]-\frac{1}{2(m+\nu_t)\nu_t}[A+mB]\right\}. \quad (28)$$

It is crucial to notice the term $[(\partial A/\partial q^2)-\nu_t(\partial B/\partial q^2)]$ which comes from the off-shell dependence of the amplitudes and $\partial q^2/\partial|\mathbf{q}|^2$. It makes an important contribution to the final result, and is a manifestation of the fact that we are doing an off-shell calculation. When Eqs. (13), (15), and (16) are inserted and the gentle continuation to the mass shell of PCAC assumed, one obtains for the effective ranges

$$\mathrm{Re}\frac{\partial f_{0+}^{(+)}}{\partial|\mathbf{q}|^2}\to 2f^2\left\{1-\left(\frac{g_A^*}{g_A}\right)^2\frac{(M+m)^2m\mu}{9M^4q^{*2}}\left[\frac{(M+m)^2}{16M^2q^{*2}}+[(M^2-m^2)^2+4m^2\mu^2]-m^2\right]\right\}\frac{1}{\mu^3}, \quad (29a)$$

and

$$\mathrm{Re}\frac{\partial f_{0+}^{(-)}}{\partial|\mathbf{q}|^2}\to f^2\left\{\frac{2\mu^2h_A(0)}{mg_A(0)}-\left(\frac{g_A^*}{g_A}\right)^2\frac{(M+m)^2m(M-m)}{9M^2q^{*2}}\cdot\times\left[\frac{(M+m)^2}{16M^4q^{*2}}[(M-m)^2(M+m)-4m\mu^2]-1\right]+\frac{2}{g_A^2}\times\left[5F_1(0)-\frac{\mu}{m}[F_1(0)+F_2(0)]-4\mu^2\frac{\partial F_1}{\partial t}\right]\right\}\frac{1}{\mu^3}. \quad (29b)$$

These equations become more transparent when numbers are inserted for the kinematical quantities and electromagnetic form factors; then

$$\mathrm{Re}\frac{\partial f_{0+}^{(+)}}{\partial|\mathbf{q}|^2}\to 2f^2\left[1-\frac{2}{3}\left(\frac{g_A^*}{g_A}\right)^2\right]\frac{1}{\mu^3}, \quad (30a)$$

and

$$\mathrm{Re}\frac{\partial f_{0+}^{(-)}}{\partial|\mathbf{q}|^2}\to f^2\left[\frac{2\mu^2h_A}{mg_A}-0.656\left(\frac{g_A^*}{g_A}\right)^2+3.07\right]\frac{1}{\mu^3}. \quad (30b)$$

Because the experimental effective ranges are small, as an orientation one might examine the consequences of zero effective ranges, i.e., linear dependence of the S-wave phase shifts with c.m. momentum. This would then require $(g_A^*/g_A)^2=1.5$, an entirely reasonable value, which when combined with (30b) would give $[2\mu^2h_A/mg_A]=-2.1$. Using our other estimates of g_A^*, the resulting range of values would be $[2\mu^2h_A/mg_A]=-2.1\pm0.1$, which is to be compared with the theoretical prediction of PCAC:

$$[2\mu^2h_A/mg_A]=-4 \text{ (PCAC)}. \quad (31)$$

These numbers are not the best possible estimates, since the analysis of Hamilton and Woolcock[8] gives for the effective ranges

$$\mathrm{Re}(\partial f_{0+}^{(+)}/\partial|\mathbf{q}|^2)=-0.042(1/\mu^3)$$

and

$$\mathrm{Re}(\partial f_{0+}^{(-)}/\partial|\mathbf{q}|^2)=+0.010(1/\mu^3). \quad (32)$$

With these values we obtain from Eq. (30a)

$$(g_A^*/g_A)^2=1.9 \quad (33a)$$

and from (30b)

$$[2\mu^2h_A(0)/mg_A(0)]=-1.9, \quad (33b)$$

if we use our best value $g_A^*/g_A^2=1.4$. The value determined for the induced pseudoscalar coupling which is a factor of 2 smaller than that predicted by PCAC need not be in disagreement with experiments,[10] which themselves are far from definitive. On the other hand, there appears to be some discrepancy in Eq. (33a) since this value for g_A^{*2} is larger than any of our other estimates. This means that there may be another contribution to $\partial f_{0+}^{(+)}/\partial|\mathbf{q}|^2$ which we have not included. A very plausible term which could be reasonably added is an isotopic scalar exchange in the process $A_\mu+N\to\pi+N$. A particular example of this would be a 0^+ meson exchange in the axial production of mesons. The precise mechanism for the required $T=0$ term is not determined here, but we emphasize that some form of isotopic scalar contribution is indicated, the evidence being the curvature of the S-wave phase shifts.

VI. CONCLUSIONS

Our study of near-threshold pion-nucleon scattering by means of current-commutator rules and PCAC should certainly be considered successful. In particular, we have found reasonable agreement with experiment for the P-wave scattering lengths, although the agreement between our values for $f_{1\pm}^{(+)}/|\mathbf{q}|^2$ and the scattering lengths is not as good as it could be. The isotopic odd S-wave effective range gives a prediction for the induced pseudoscalar coupling constant which, although still experimentally reasonable, is smaller than that given by PCAC by a factor of 2. Furthermore, the curvature of the isotopic even S-wave phase shifts indicates that an additional $T=0$ exchange contribution is possibly required in the weak amplitude which could also improve the agreement of the P-wave scattering lengths. This point certainly deserves more study.

Our calculations have used the tenets of PCAC and current-commutation rules to relate off-shell pion-nucleon scattering to physical values. This procedure intertwines in a detailed and satisfactory way the

[10] T. D. Lee and C. S. Wu, Ann. Rev. Nucl. Sci. 15, 381 (1965).

strong interactions with the nucleon electromagnetic and weak vertices. In order to predict the P-wave scattering lengths and S-wave effective ranges we have required more detailed information on the weak amplitudes and electromagnetic vertex than needed for the Adler-Weisberger sum rule or the threshold theorem; yet the method works well as a calculation device, and gives new information on the weak production amplitude.

APPENDIX A: HIGHER RESONANCE CONTRIBUTIONS

In this Appendix we show that baryons with $J=\frac{3}{2}^-$ and $J>\frac{3}{2}$ do not contribute to our calculation. Consider the *normal* axial-vector transition

$$A_\nu+N\to R(J)\,,\quad J=\tfrac{3}{2}^+,\tfrac{5}{2}^-,\cdots;\quad k=p-p_1\,,$$

for nucleon with 4-momentum p_1 and a baryon resonance with spin J and momentum p. From the work of Bjorken and Walecka,[5] the vertex function can be written

$$\frac{1}{(2\pi)^3}\left(\frac{m}{E_1}\frac{M}{E}\right)^{1/2}\bar{\psi}_{\sigma_1\sigma_2\cdots\sigma_{J-1/2}}(p)k_{\sigma_2}\cdots k_{\sigma_{J-1/2}}[g_1(k^2)k_{\sigma_1}\times(k^2p_{1\nu}-k_\nu p_1\cdot k)+ig_2(k^2)k_{\sigma_1}(\gamma_\alpha\epsilon_{\alpha\rho\lambda\nu}p_{1\rho}k_\lambda\gamma_5)+ig_3(k^2)\epsilon_{\sigma_1\alpha\beta\gamma}\epsilon_{\gamma\rho\tau\nu}p_{1\alpha}p_{1\rho}k_\beta k_\tau+ig_4(k^2)\delta_{\sigma_1\nu}]u(p_1)\,.\quad\text{(A1)}$$

Just as in the $N^*(\frac{3}{2}^+)$ case, only the invariant proportional to $g_4(k^2)$ is longitudinal to k. Recall that the Rarita-Schwinger wave function is

(1) symmetric under permutation of any pair of indices,
(2) zero on contraction of any pair of indices,
(3) a solution of the Dirac equation, and
(4) orthogonal to γ_σ and p_σ.

We write for arbitrary J the projection operator

$$\mathcal{P}_{\mu\nu}(J)=\sum_s\psi_{\mu\mu_2\cdots\mu_{J-1/2}}(p)\bar{\psi}_{\nu\nu_2\cdots\nu_{J-1/2}}(p)\times q_{\mu_2}\cdots q_{\mu_{J-1/2}}k_{\nu_2}\cdots k_{\nu_{J-1/2}}\,,\quad\text{(A2)}$$

which can be re-expressed in terms of the spin-$\frac{3}{2}^+$ projection, using the above properties, as

$$\mathcal{P}_{\mu\nu}(J)=q\cdot kf(q\cdot k)\mathcal{P}_{\mu\nu}(\tfrac{3}{2}^+)\,,\quad\text{(A3)}$$

where $f(q\cdot k)$ is a monomial in $q\cdot k$ of degree $J-\frac{5}{2}$. We write for the contribution to the invariant amplitudes of a resonance with spin $J=\frac{5}{2}^-,\frac{7}{2}^+,\cdots$

$$A^J(s,t,q^2)=q\cdot kf(q\cdot k)A_{N^*}(s,t,q^2)\,,$$
$$B^J(s,t,q^2)=q\cdot kf(q\cdot k)B_{N^*}(s,t,q^2)\,,\quad\text{(A4)}$$

where $q\cdot k=\frac{1}{2}t-q^2$. Furthermore, it is important to recall that

$$[A_{N^*}-\nu_tB_{N^*}]_{q^2=0}\to 0\text{ at threshold}$$

From Eq. (25b), we see that we require

$$\left.\left(\frac{\partial A}{\partial t}-\nu_t\frac{\partial B}{\partial t}\right)\right|_{q^2=0}$$

at threshold. This is simply evaluated by noting

$$\left[\frac{\partial A^J}{\partial t}-\nu_t\frac{\partial B^J}{\partial t}\right]_{t=0,q^2=0}=f(0)[A_{N^*}-\nu_tB_{N^*}]_{t=0,q^2=0}\,.\quad\text{(A5)}$$

Thus

$$\left[\frac{\partial A^J}{\partial t}-\nu_t\frac{\partial B^J}{\partial t}\right]_{q^2=0}\to 0\text{ at threshold}.\quad\text{(A6)}$$

In addition

$$A^J(s,t,q^2)|_{q^2=0}\to 0\text{ and }B^J|_{q^2=0}\to 0\text{ at threshold}.\quad\text{(A7)}$$

Thus by comparing with Eqs. (25), we see that the P-wave scattering lengths do not receive contributions from normal transitions to baryon states with $J>\frac{3}{2}$.

To discuss the S-wave effective ranges we turn to Eq. (28). We have

$$\left[\frac{\partial A^J}{\partial s}-\nu_t\frac{\partial B^J}{\partial s}\right]=(t-\tfrac{1}{2}q^2)f(t-\tfrac{1}{2}q^2)\times\left[\frac{\partial A_{N^*}}{\partial s}-\nu_t\frac{\partial B_{N^*}}{\partial s}\right]\to 0\text{ at threshold},\quad\text{(A8)}$$

and

$$\left[\frac{\partial A^J}{\partial q^2}-\nu_t\frac{\partial B^J}{\partial q^2}\right]_{t=0,q^2=0}=-\tfrac{1}{2}f(0)[A_{N^*}-\nu_tB_{N^*}]\to 0\text{ at threshold}.\quad\text{(A9)}$$

Equations (A5)–(A9) together with (28) enable us to conclude that the normal transitions to states with $J>\frac{3}{2}$ do not contribute to the S-wave effective ranges.

To complete the proof, we turn to the abnormal transitions, i.e., axial-vector transitions to baryon resonances with $J=\frac{3}{2}^-,\frac{5}{2}^+\cdots$. This case is treated by replacing $u(p_1)\to\gamma_5u(p_1)$ in (A1), which means in the c.m. system we replace the Pauli spinors by

$$\chi\to i\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{(E+m)}\chi$$

and

$$\chi^\dagger\to -i\chi^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{(E+m)}\,.$$

Hence the abnormal transitions vanish at threshold by a factor $|\mathbf{q}|^2$ faster than the normal transition of the same spin, which rules them out for $J \geq \frac{3}{2}$. In conclusion, we have shown that baryon states with spins $J=\frac{3}{2}^-$ and $J>\frac{3}{2}$ do not contribute to the P-wave scattering lengths or S-wave effective ranges. This completes the justification of the choice of baryon states in our calculation.

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Model of the $N^*(1236)$

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A variational method is proposed, and is used to calculate the width and the radiative widths of the $N^*(1236)$. Good agreement with experiment is obtained.

I. INTRODUCTION

THE aim of this paper is to develop a dynamical scheme for relating the widths of the $\frac{3}{2}^+$ resonances of the $\{10\}$ representation of $SU(3)$ with the baryon-meson coupling constants. Such a method is developed, and used to calculate the width and the electric and magnetic radiative widths of the $N^*(1236)$. It is also inverted, and used to give a determination of the pion-nucleon coupling constant, and in a further paper will be applied to the $Y_1^*(1385)$ resonance.

The standard approach to this problem has been to use a one-channel N/D method, with the assumption of no Castillejo-Dalitz-Dyson (CDD) poles. The results of these calculations are in gross disagreement with experiment. For example, in the paper of Coulter and Shaw (G.L.),[1] the cutoff parameter is adjusted to give the correct resonance position, but the predicted width is too large, and the energy dependence totally wrong. (See Fig. 1. The Wigner condition[2] should be borne in mind.) Further, the fact that the phase shift begins to fall off immediately after resonance is characteristic of this type of calculation, whereas experimentally,[3] the phase shift rises towards 180° and stays there, suggesting the presence of a CDD pole. However, a one-channel CDD pole may not correspond to a CDD pole in a multichannel situation.[4–7] An explicit example of this for the $N^*(1236)$ has been given by Atkinson and Halpern,[8] who have shown that if the static $SU(6)$ bootstrap[9] is successful, then the p_{33} partial wave will contain a one-channel CDD pole. Unfortunately, in practical calculations, the incorporation of more (closed) channels adds to the number of adjustable parameters, and thus strips the method of all its predictive power—except in the restricted, and for detailed calculations, unreliable case of exact $SU(6)$. Also, the slowness of convergence of the integrals means that distant singularities, which are not amenable to any known approximation scheme, could well play an important role in these calculations. Thus a completely different approach is needed, one which

(a) is sensitive only to those regions of the complex plane around threshold, where one can make approximations with some confidence,

(b) in which the question of the presence of CDD poles is circumvented by a criteria making a more direct appeal to experiment,

(c) in which the minimum amount of input information is required.

A calculation which satisfies some of the above criteria is the variational calculation of Donnachie and Hamilton.[10] In this paper, parametric forms are fitted to a partial-wave dispersion relation in the low-energy region, via the minimization of a somewhat complicated function. The integrals are weighted with the threshold factor, so that the convergence is rapid, and the results insensitive to distant contributions. A Layson[11] formula is used for $f_{1+}^{(3)}$, and is found to give a good solution to the equation in excellent agree-

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