High-Energy Behavior in Quantum Field Theory. I. Strictly Localizable Fields

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We introduce the notion of a field which is strictly localizable within a region of space-time. We investigate what restrictions strict localizability imposes on the high-energy behavior of fields, and we find that it leads to an upper bound on the growth of a field in momentum space. This bound allows the off-mass-shell vacuum expectation values to grow in momentum space faster than any polynomial. Furthermore, it turns out that no maximum rate of growth exactly saturates our bound. In addition, strictly localizable fields need not be Schwartz distributions. However, the usual distribution fields are strictly localizable fields of a special type. We formulate a strictly local field theory in precise mathematical terms. Finally, we discuss simple examples of strictly localizable fields that are not distributions.

I. INTRODUCTION

In this paper we introduce the notation of a *strictly localizable field* (SLF). It is the first of a series of works on the properties of *strictly local field theory* (SLFT).

We shall study quantum field theories in which it is possible to incorporate the physically motivated requirements of: (a) a Hilbert space of states; (b) covariance of the fields under Lorentz transformations and space-time translations; (c) positive energy; (d) locality (as local commutativity of fields); and (e) a particle interpretation.

On the basis of the Hilbert space and covariance alone, it is known that a field $A(x)$ will not be a field of operators; rather it must be smoothly averaged over some space-time region in order to yield an operator.[1,2] In fact, using covariance one can write a spectral representation for the two-point vacuum-expectation value of a field, which for a scalar field has the form[3–6]

$$(\psi_0,A^*(x)A(y)\psi_0)=\int e^{-ip(x-y)}\rho(p)dp. \quad (1)$$

Here $\rho(p)$ is Lorentz invariant:

$$\rho(\Lambda p)=\rho(p). \quad (2)$$

From the positive metric in Hilbert space, it follows that $\rho(p)$ is a positive measure. If we assume that $A(x)$ is an operator applicable to the vacuum state ψ_0, then

$$\|A(x)\psi_0\|^2=(\psi_0,A^*(x)A(x)\psi_0)=\int\rho(p)dp<\infty. \quad (3)$$

Combining the facts that $\rho(p)$ is positive, Lorentz invariant, and integrable, would lead to

$$\rho(p)=c\delta^4(p). \quad (4)$$

Hence, we conclude that the field $A(x)$ can be a field of operators only in the trivial case that the two-point function is a constant:

$$(\psi_0,A^*(x)A(y)\psi_0)=c. \quad (5)$$

A similar result holds for fields with higher spin.

In other words, we are forced to formulate a field as an operator-valued generalized function. A field must be averaged with a smooth test function in order to yield an operator

$$A(f)=\int A(x)f(x)dx. \quad (6)$$

Let us introduce the idea of *strict localizability*. Suppose that a field $A(x)$ can be averaged with some test function $f(x)$ which vanishes outside a certain region of space-time. Then we say that the field A is *strictly localizable* in that region. Such a notion is convenient for the statement of local commutativity, so we shall insist that our fields are *strictly localizable* within bounded open regions of space-time. Then locality of the field A will be expressed by the fact that $A(f)$ commutes, or anticommutes, with $A(g)$ whenever the test functions $f(x)$ and $g(x)$ vanish outside spacelike separated regions. (Later we shall specify more precisely exactly which test functions are allowed, and on what set of states the field operators can be applied and are expected to commute.)

In this series of papers we show that it is possible to fit strictly localizable fields into the framework of a local quantum field theory. We introduce new classes of test functions for fields. We show that these lead to fields which need not be Schwartz distributions; rather

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[1] A. S. Wightman, Ann. de l'Institut Henri Poincaré 1, 403 (1964).

[2] Wightman assumes locality and covariance of the field under translations. We use Lorentz covariance of the fields in place of locality, which shortens the proof.

[3] H. Umezawa and S. Kamefuchi, Progr. Theoret. Phys. (Kyoto) 6, 543 (1951).

[4] G. Källen, Helv. Phys. Acta 25, 417 (1952).

[5] M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954).

[6] H. Lehmann, Nuovo Cimento 11, 342 (1954).

they are operator-valued generalized functions which include the tempered fields as a special case.

We derive for our more general class of fields, certain results obtained previously for tempered fields. These include the connection between spin and statistics,[7,8] the existence of CPT symmetry,[9,10] crossing symmetry,[11,12] the asymptotic condition,[13–15] and the proof of dispersion relations.[16–19]

The wider class of fields studied here is physically relevant, since it allows for the possibility that the off-mass-shell amplitudes can grow at large energies faster than any polynomial. Such behavior is ruled out by assumption in the study of tempered (Wightman) fields. Nevertheless, one believes that faster than polynomial growth at high energies is associated with fields which describe weak interactions, and possibly also strong interactions.

II. DISCUSSION

In the usual Wightman framework, one assumes that a field is an operator-valued tempered distribution.[20–22] Occasionally it was found convenient to relax that assumption and to only assume that fields are operator-valued Schwartz distributions.[23] Let us see why even this wider framework is inadequate for relevant field theories. Since the state space is a Hilbert space, vectors have a positive length:

$$\|A(f)\psi_0\|^2=\int(\psi_0,A^*(x)A(y)\psi_0)\bar{f}(x)f(y)dxdy\geq 0. \quad (7)$$

If A is a scalar field, this norm can be written in terms of the spectral representation (1) which gives

$$\|A(f)\psi_0\|^2=\int\rho(p)|\tilde{f}(p)|^2dp\geq 0. \quad (8)$$

Since (8) must be true for every $\tilde{f}(p)$ whose Fourier transform $f(x)\in\mathfrak{D}(\mathfrak{R}^4)$, the space of infinitely differentiable functions with compact support,[24] we infer that $\rho(p)$ is a positive, *tempered* measure.[25,26] In other words, there is a finite integer N such that

$$\int\frac{\rho(p)}{(1+\|p\|^2)^N}dp<\infty\,, \quad (9)$$

where $\|p\|^2=p_0^2+\mathbf{p}^2$ is the square of the Euclidean length of p. In particular, (9) shows that only a finite number of subtractions are necessary to define the time-ordered two-point function, or propagator.[27]

There are many indications that (9) is not true in relevant theories, and hence that some relevant fields cannot be operator-valued Schwartz distributions. For instance, in the study of Lagrangian field theory described by a nonrenormalizable interaction, perturbation calculations lead one to expect an infinite number of subtractions in defining the time ordered two-point function.[28] Secondly, certain exactly soluble models which come from nonrenormalizble Lagrangians have two-point functions in which $\rho(p)$ is not tempered.[29–31] For instance, if $\varphi(x)$ is a free, neutral scalar field, and $\psi(x)$ is a free spin-$\frac{1}{2}$ field, then $A(x)=:\exp\varphi(x):\psi(x)$ has a two-point vacuum expectation value

$$(\psi_0,A^\dagger(x)A(y)\psi_0)=(1/i)S^{(+)}(x-y)\exp\{-i\Delta^{(+)}(x-y)\}. \quad (10)$$

Here

$$(\psi_0,\psi^\dagger(x)\psi(y)\psi_0)=(1/i)S^{(+)}(x-y)\,,$$

and

$$(\psi_0,\varphi(x)\varphi(y)\psi_0)=(1/i)\Delta^{(+)}(x-y)\,.$$

Expression (10) is not a Schwartz distribution, but it can be defined as a generalized function[32,33] on all test functions which are Fourier transforms of functions in $\mathfrak{D}(\mathfrak{R}^4)$. Further evidence for the singular behavior of the two-point function comes from an approximate, but nonperturbative, calculation by Bardakci and Schroer[34]

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[20] A. S. Wightman, Phys. Rev. **101**, 860 (1956).
[21] R. F. Streater and A. S. Wightman, *CPT, Spin and Statistics, and All That* (W. A. Benjamin, Inc., New York, 1964).
[22] R. Jost, *The General Theory of Quantized Fields* (American Mathematical Society, Providence, Rhode Island, 1965).
[23] L. Schwartz, *Théorie des Distributions* (Hermann & Cie., Paris, 1950).

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[24] The support of a function is the smallest closed set outside which the function vanishes identically.
[25] A distribution T is said to be positive if for every positive test function f, $T(f)\geq 0$. A positive distribution must be a measure (see Ref. 23).
[26] A distribution T is positive definite (of positive type) if for every function f in $\mathfrak{D}$, $T(\bar{f}*f)\geq 0$, where $\bar{f}(x)=\bar{f}(-x)$. Every positive-definite distribution is the Fourier transform of a positive, tempered measure (see Ref. 23). This was applied to field theory by Wightman (see Ref. 20).
[27] O. Steinmann, J. Math. Phys. **4**, 583 (1963).
[28] N. N. Bogoliubov and D. V. Shirkov, *Introduction to Quanutm Field Theory* (Interscience Publishers, Inc., New York, 1959).
[29] W. Güttinger, Nuovo Cimento **10**, 1 (1958).
[30] T. Pradhan, Nucl. Phys. **43**, 11 (1963); B. Schroer, J. Math. Phys. **5**, 1361 (1964).
[31] B. Klaiber, Nuovo Cimento **36**, 165 (1965).
[32] I. M. Gelfand and G. E. Shilov, *Generalized Functions* (Academic Press Inc., New York, 1964), Vols. 1 and 2.
[33] I. M. Gelfand and N. Ya. Vilenkin, *Generalized Functions* (Academic Press Inc., New York, 1964), Vol. 4.
[34] K. Bardakci and B. Schroer, J. Math. Phys. **7**, 10 (1966); **7**, 16 (1966).

on vector mesons interacting with scalar mesons by a $\lambda A^\mu \partial_\mu \varphi^2$ coupling.

In all the cases described above, it is possible to use momentum-space test functions in $\mathfrak{D}$. In other words, it seems consistent to describe a field as an operator-valued distribution in momentum space, and this was proposed by Güttinger[29] and by Schroer.[30] However, the Fourier transform of $\mathfrak{D}$ contains no functions with compact support, so that fields defined on only those test functions may not be strictly localizable. Thus it is not clear how to formulate locality for such fields, and all the major results of local quantum field theory would not naturally carry over. A suggestion was made by Nguyen Van Hieu[35] and also by Güttinger[36] that a new class of test functions might be used to make a statement about locality.

We show here that it is possible to carry through the field-theory program for strictly localizable fields. In this work we shall assume that our fields are operator-valued Schwartz distributions in momentum space. While that considerably simplifies our analysis, and there is no known reason to believe that it is false, we shall remove that restriction in a later work.

III. TEST FUNCTIONS AND HIGH-ENERGY BOUNDS

A. Requirements on the Test Function (T.F.) Spaces

T.F.1. We denote the configuration-space test functions by $\mathfrak{C}(\mathfrak{R}^4)$ and their Fourier transform, the momentum-space test functions, by $\mathfrak{M}(\mathfrak{R}^4)$. Both $\mathfrak{C}$ and $\mathfrak{M}$ should be countably normed, complete, linear spaces in which the nuclear theorem holds.[37] They should be invariant under linear transformations and translations of the coordinates.

T.F.2. (Strict Localizability) Define $\mathfrak{L}(O)$ to be those configuration-space test functions, localized in the open space-time region O.

$$\mathfrak{L}(O)=\mathfrak{C}(\mathfrak{R}^4)\cap\mathfrak{D}(O). \tag{11}$$

We assume that $\mathfrak{L}(\mathfrak{R}^4)$ contains some function which is not identically zero.

T.F.3. (Momentum-Space Distributions) We assume that

$$\mathfrak{D}(\mathfrak{R}^4)\subset\mathfrak{M}(\mathfrak{R}^4).$$

T.F.4. (Topology) We assume that convergence in $\mathfrak{M}(\mathfrak{R}^4)$ is defined by the following family of norms:

$$\|f\|_{n,m,A}=\sup_{p\in\mathfrak{R}^4} g(A\|p\|^2)(1+\|p\|^2)^n|D^m f(p)|. \tag{12}$$

Here n and A are integers,

$$D^m=\frac{\partial^{|m|}}{\partial p_0^{m_0}\partial p_1^{m_1}\cdots\partial p_3^{m_3}},\quad |m|=m_0+m_1+\cdots+m_3, \tag{13}$$

and $g(t)$ is an entire function which will characterize the momentum-space growth of the off-mass-shell amplitudes

$$g(t^2)=\sum_{r=0}^{\infty} c_{2r}t^{2r},\quad c_{2r}\geq 0,\quad c_0\neq 0. \tag{14}$$

Then

$$\mathfrak{M}(\mathfrak{R}^4)=\{f(p):\|f\|_{n,m,A}<\infty \quad \text{for all } n, m, A\}. \tag{15}$$

When we consider all the various test-function spaces which meet these requirements, there is no one smallest space contained in all the others. *Hence there is no one test-function class suitable for all strictly localizable fields.* Each field will dictate which test-function space is appropriate for that field, and the relevant test functions will vary from problem to problem.

B. Test Functions over $\mathfrak{R}^l$

It is possible to define analogous test-function spaces over $\mathfrak{R}^l$, namely, $\mathfrak{C}(\mathfrak{R}^l)$, $\mathfrak{M}(\mathfrak{R}^l)$, or $\mathfrak{L}(\mathfrak{R}^l)$. Merely replace $\mathfrak{R}^4$ by $\mathfrak{R}^l$ in each definition. Note that the norms defined in (12) automatically entail T.F.3, the fact that the fields are Schwartz distributions in momentum space.

C. A High-Energy Bound Imposed by Strict Localizability

The property of strict localizability can be translated into a property of the growth-indicator function $g(t)$. In particular, strict localizability puts a high-energy bound on the growth of fields. It will be used in later works to give bounds on matrix elements.

Theorem 1. (High-Energy Bound) The space $\mathfrak{L}(\mathfrak{R}^l)$ is nontrivial (that is, there exists one local test function not identically zero), if and only if

$$\int_0^\infty \frac{\ln g(t^2)}{1+t^2}dt<\infty. \tag{16}$$

In terms of the power series coefficients of $g(t^2)$ defined in (14), the function $g(t^2)$ satisfies (15) if and only if

$$\sum_{r=0}^{\infty}\sup_{n\geq 0}\left[(c_{2r+2n})^{1/(2r+2n)}\right]<\infty. \tag{17}$$

We next see that whenever there exists *one* strictly local test function, a sufficiently large class must automatically exist.

[35] Nguyen Van Hieu, Ann. Phys. (N. Y.) 33, 428 (1965).

[36] W. Güttinger, Fortschr. Physik (to be published); Nuovo Cimento (to be published).

[37] Let T be a multilinear functional defined on functions in $\otimes_l \mathfrak{C}(\mathfrak{R}^1)$, and continuous in each variable, the other $(l-1)$ being held fixed. The nuclear theorem says that T has a unique extension to an element of $\mathfrak{C}'(\mathfrak{R}^l)$. This is a stronger requirement that the "abstract kernel theorem" proved in Ref. 33 for "nuclear spaces."

Theorem 2. If $\mathfrak{L}(\mathfrak{R}^l)$ is a nontrivial, then for any open region O in $\mathfrak{R}^l$, the space $\mathfrak{L}(O)=\mathfrak{C}(\mathfrak{R}^l)\cap\mathfrak{D}(O)$ is dense in the space $\mathfrak{D}(O)$.

Remarks

(1) If $g(t)$ is a polynomial, then $\mathfrak{C}(\mathfrak{R}^l)=\mathfrak{M}(\mathfrak{R}^l)=\mathfrak{S}(\mathfrak{R}^l)$, the Schwartz space, and $\mathfrak{L}(\mathfrak{R}^l)=\mathfrak{D}(\mathfrak{R}^l)$.

(2) Theorem 1 gives a high-energy bound on strictly localizable fields. For example, while growth of $g(\|p\|^2)$ as

$$\exp\{\|p\|/(\ln\|p\|)^{1+\epsilon}\} \tag{18}$$

or as

$$\exp\{\|p\|/(\ln\|p\|(\ln\ln\|p\|)^{1+\epsilon})\}$$

is acceptable, a growth as fast as

$$\exp\{\|p\|/\ln\|p\|\}$$

is not strictly localizable. In a later paper, we translate this bound into a bound on the growth of the momentum-space vacuum-expectation values.

(3) Theorem 1 provides the substance for the remarks made above that there is no one test-function class suitable for all strictly localizable fields. If we are given $g(t^2)$ for which (16), is finite, then there is a function $f(t^2)$ for which (16) is finite and such that for any A,

$$\lim_{t\to\infty} g(At^2)/f(t^2)=0. \tag{19}$$

(4) We postpone the proof of Theorems 1 and 2, and first define a strictly local field theory.

(5) In Ref. 38 we apply the bound of Theorem 1 to derive a bound on the decay of form factors at large momentum transfer.

IV. A STRICTLY LOCAL FIELD THEORY

We define an SLFT as a local field theory of an SLF. We adopt the usual Wightman assumptions listed in the introduction[20–22] and we now give them in a form applicable to our fields.

A. A Hilbert Space of States

The state space is a (separable) Hilbert space H. There is a unitary representation of the Lorentz transformations on H. More precisely, there is a strongly continuous unitary representation $U(a,M)$ of the covering group of the Poincaré group, namely, the inhomogeneous $SL(2;C)$ group.

B. Fields as Operator-Valued Generalized Functions

To each test function $f(x)\in\mathfrak{C}(\mathfrak{R}^4)$, a field A assigns an operator $A(f)$. All such field operators are defined on a common, dense, invariant domain $D\subset H$. The domain D is invariant under Lorentz transformations, under space-time translations, and under application of the field operators

$$U(a,M)D\subset D,$$
$$A(f)D\subset D, \tag{20}$$

and

$$A^*(f)D\subset D.$$

For each ψ, Φ, in D, the form

$$(\psi,A(f)\Phi)$$

is continuous in f in the topology of $\mathfrak{C}(\mathfrak{R}^4)$. That is, $(\psi,A\Phi)$ is a generalized function in $\mathfrak{C}'(\mathfrak{R}^4)$.

C. Covariance of the Fields

The field A with components A_j transforms under the Poincaré group as

$$U(a,M)A_j(f)U(a,M)^{-1}\psi = \sum_{k=1}^{N} S_{jk}(M^{-1})A_k(f_{\{a,M\}})\psi, \tag{21}$$

where ψ is any vector in D,

$$(f_{\{a,M\}})(x)=f(\Lambda(M^{-1})(x-a)), \tag{22}$$

and $S_{jk}(M^{-1})$ is a finite-dimensional representation of $SL(2;C)$, the covering group of the Lorentz group.

D. Positive Energy

By Stone's theorem,

$$U(a,1)=\exp(iP^\mu a_\mu), \tag{23}$$

where P^μ is interpreted as the energy-momentum operator. The spectrum of the energy-momentum is assumed to lie in the closure of the forward light cone. In other words, for any vector ψ in the domain of P^μ, the numbers

$$k^\mu=(\psi,P^\mu\psi)$$

form a vector in $\bar{V}^+$. We assume that there exists a unique vector ψ_0 in H, invariant under Poincaré transformations, and denote ψ_0 the physical vacuum.

$$U(a,M)\psi_0=\psi_0,$$
$$P^\mu\psi_0=0. \tag{24}$$

The vacuum ψ_0 is assumed to be cyclic for the smeared fields.

E. Strict Localizability and Locality

We assume that the field A is strictly localizable; in other words, $\mathfrak{L}(\mathfrak{R}^4)$ is assumed nontrivial. Then A is

[38] A. M. Jaffe, Phys. Rev. Letters **17**, 661 (1966).

local if whenever f and g in $\mathfrak{L}(\mathfrak{R}^4)$ have spacelike separated supports,

$$A_j(t)A_k(g)\psi=\pm A_k(g)A_j(f)\psi. \qquad (25)$$

Here ψ is any vector in D.

F. Particle Interpretation

We wish to ensure a particle interpretation and a connection with an S matrix. This will be discussed in a later work.

V. QUASIANALYTIC CLASSES AND THE PROOF OF THEOREMS 1 AND 2

In this section we shall prove Theorems 1 and 2. Since we shall use the theory of quasianalytic classes of functions,[39–42] we review some definitions.

A. Quasianalytic Classes

An important property of analytic functions is the fact that they are uniquely determined by their derivatives at a point. Taking this property as basic, a class of functions is called *quasianalytic* if any function in the class in uniquely determined by giving all its derivatives at a point. Thus analytic functions form a quasianalytic class, but there may be other quasianalytic classes which contain functions that do not have everywhere convergent power series.

Let $\{M_n\}$ be a sequence of non-negative numbers, and consider the class of infinitely differentiable functions $C\{M_n\}$ defined by the following: A function $f(x)$ of one real variable belongs to $C\{M_n\}$ if and only if there exist constants A_1 and A_2 such that the derivatives of $f(x)$ satisfy

$$\sup_{x\in\mathfrak{R}^1} |D^n f(x)| \le A_1(A_2)^n M_n. \qquad (26)$$

The class $C\{M_n\}$ is a *quasianalytic class*, if and only if any function $f(x)\in C\{M_n\}$ which vanishes along with all its derivatives at one point,

$$(D^n f)(x_0)=0,$$

must vanish identically:

$$f(x)\equiv 0, \quad \text{for all } x.$$

Thus no quasianalytic class of functions will contain a nontrivial function with compact support. Conversely, the following is known (see Mandelbrojt[40,41]):

Theorem 3. Every class $C\{M_n\}$ which is *not* quasianalytic, contains a nontrivial, positive function with compact support.

[39] T. Carleman, *Les Fonctions Quasianalytiques* (Gauthier-Villars, Paris 1926).

[40] S. Mandelbrojt, Rice Inst. Pam. **29**, No. 1 (1942).

[41] S. Mandelbrojt, *Séries Adhérantes, Régularisation des Suites, Applications* (Gauthier-Villars, Paris, 1952).

[42] P. J. Cohen, Stanford Report, 1966 (unpublished).

The classical theorem of Denjoy and Carleman gives the condition on the coefficients M_n which is necessary and sufficient for the class $C\{M_n\}$ to be quasianalytic. A related condition was given by Ostrowski.[39–42]

Theorem 4. (Denjoy-Carleman) The class $C\{M_n\}$ is quasianalytic if and only if

$$\sum_{r=0}^{\infty} \sup_{n\ge 0} [(M_{r+n})^{-1/(r+n)}] = \infty. \qquad (27)$$

Theorem 5. (Ostrowski) Let

$$H(t)=\sup_{r\ge 0} [t^r M_r^{-1}].$$

Then the condition (27) is valid if and only if

$$\int_1^\infty \frac{\ln H(t)}{t^2} dt = \infty. \qquad (28)$$

B. Some Useful Results

Recall that $\mathfrak{L}(\mathfrak{R}^1)=\mathfrak{F}\mathfrak{M}(\mathfrak{R}^1)\cap\mathfrak{D}(\mathfrak{R}^1)$, where $\mathfrak{F}$ stands for Fourier transformation. We start with

Theorem 6. The space $\mathfrak{L}(\mathfrak{R}^1)$ is nontrivial if and only if

$$\sum_{r=0}^{\infty} \sup_{n\ge 0} [(c_{2r+2n})^{1/(2r+2n)}] < \infty, \qquad (29)$$

where c_{2r} is defined in (14).

Proof. Suppose that $f(x)$ is a nontrivial element of $\mathfrak{L}(\mathfrak{R}^1)$, with Fourier transform $\tilde{f}(p)\in\mathfrak{M}(\mathfrak{R}^1)$, normalized so that $\int g(p^2)|\tilde{f}(p)|dp=1$. Then

$$\sum_{r=0}^{N} c_{2r} \sup_{x\in\mathfrak{R}^1} |D^{2r} f(x)| \le \int \sum_{r=0}^{N} c_{2r} p^{2r} |\tilde{f}(p)| dp. \qquad (30)$$

By the monotone convergence theorem, (30) remains bounded as $N\to\infty$ and therefore

$$c_{2r} \sup_{x\in\mathfrak{R}^1} |D^2 f(x)| \le 1,$$

or

$$\sup_{n\ge 0} [(c_{2r+2n})^{1/(2r+2n)}] \le \sup_{m\ge 0} [(\sup_{x\in\mathfrak{R}^1} |D^{2r+2m} f(x)|)^{-1/(2r+2m)}]. \qquad (31)$$

Use

$$\sup_{m\ge 0}(\sup_{x\in\mathfrak{R}^1} |D^{2r+2m} f(x)|)^{-1/(2r+2m)} \le \sup_{m\ge 0}(\sup_{x\in\mathfrak{R}^1} |D^{2r+m} f(x)|)^{-1/(2r+m)},$$

sum (31) over r, and add the odd terms to the right-hand side in order to get

$$\sum_{r=0}^{\infty} \sup_{n\geq 0} \left[(c_{2r+2n})^{1/(2r+2n)}\right] \leq \sum_{r=0}^{\infty} \left[\sup_{n\geq 0}\left(\sup_{x\in\mathfrak{R}^1} |D^{r+n}f(x)|\right)^{-1/(r+n)}\right]. \quad (32)$$

Since $f(x)\not\equiv 0$, and f has compact support, it does not belong to any quasianalytic class. Therefore, by defining

$$M_n = \sup_{x\in\mathfrak{R}^1} |D^n f(x)|,$$

we infer from Theorem 4 that the sum on the right side of (32) is finite, which is the desired result.

Conversely, let us suppose that the sum (29) is finite; we then construct a nontrivial function in $\mathfrak{L}(\mathfrak{R}^1)$. The first step is to note that any infinitely differentiable function $f(x)$ is an element of $\mathfrak{C}(\mathfrak{R}^1)$ if it has the following property: For each n, m, and B, there exists a constant $M(n,m,B)$ such that

$$\sup_{x\in\mathfrak{R}^1} |(1+x^2)D^{2r+n}\{x^m f(x)\}| \leq \frac{M(n,m,B)}{d_{2r}B^r}, \quad (33)$$

where

$$d_{2r} = [\sup_{n\geq 0}\{c_{2r+2n}{}^{1/(2r+2n)}\}]^{2r}. \quad (34)$$

We now verify that $\tilde{f}(p)$, the Fourier transform of $f(x)$, is an element of $\mathfrak{M}(\mathfrak{R}^1)$, which means that $\|\tilde{f}\|_{n,m,A} < \infty$ for all the norms defined in (12). Clearly,

$$\|\tilde{f}\|_{n,m,A} \leq 2^n \sum_{r=0}^{\infty} c_{2r}A^r\{|\tilde{f}|_{2r,m} + |\tilde{f}|_{2r+2n,m}\}, \quad (35)$$

where

$$|\tilde{f}|_{n,m} = \sup_{p\in\mathfrak{R}^1} |p^n D^m \tilde{f}(p)|. \quad (36)$$

However,

$$|\tilde{f}|_{n,m} \leq \alpha \sup_{x\in\mathfrak{R}^1} |(1+x^2)D^n\{x^m f(x)\}|, \quad (37)$$

where

$$\alpha = \int dx(1+x^2)^{-1}.$$

Combining (37) with the assumption (33) leads to

$$\|\tilde{f}\|_{n,m,A} \leq 2^n\{M(0,m,B)+M(2n,m,B)\} \times \sum_{r=0}^{\infty}\left(\frac{A}{B}\right)^r\left(\frac{c_{2r}}{d_{2r}}\right). \quad (38)$$

Since by definition $c_{2r}\leq d_{2r}$, the series on the right-hand side of (38) converges whenever we choose the arbitrary constant B greater than A. Thus $\tilde{f}(p)$ is an element of $\mathfrak{M}(\mathfrak{R}^1)$, and $f(x)$ an element of $\mathfrak{C}(\mathfrak{R}^1)$. We now need to show it possible to construct such an $f(x)$ with compact support. If $g(t^2)$ is a polynomial, $\mathfrak{C}(\mathfrak{R}^1) = \mathfrak{S}(\mathfrak{R}^1) \supset \mathfrak{D}(\mathfrak{R}^1)$, so we can assume that not to be the case.

Let

$$\alpha_{2r} = (d_{2r})^{1/2r}, \quad (39)$$

where d_{2r} is defined by (34). Also, let

$$\eta_{2r} = \sum_{m=r}^{\infty} \alpha_{2m},$$
$$\beta_{2r} = \alpha_{2r}/(\eta_{2r})^{1/2}, \quad (40)$$

and

$$\gamma_{2r} = \inf_{m\leq r} \beta_{2m}.$$

By hypothesis (29), we have $\eta_0 < \infty$. It is then easy to demonstrate that

$$\sum_{r=0}^{\infty} \beta_{2r} < \infty, \quad (41)$$

and hence that

$$\sum_{r=0}^{\infty} \gamma_{2r} < \infty. \quad (42)$$

It is no loss of generality to assume $\gamma_{2r}\leq 1$. Note that α_{2r}, γ_{2r}, and $\gamma_{2r}{}^{2r}$ all decrease monotonically. Thus

$$(\alpha_{2r}/\gamma_{2r}) \leq \sqrt{\eta_{2s}} \quad (43)$$

for some $s(r)\leq r$. Furthermore, by (42) we see that $\gamma_{2r}\to 0$ as $r\to\infty$, which implies that $s(r)\to\infty$ as $r\to\infty$. Thus $\eta_{2s(r)}\to 0$ as $r\to\infty$, and

$$\lim_{r\to\infty} (\alpha_{2r}/\gamma_{2r}) = 0. \quad (44)$$

Define

$$M_{2r} = (\gamma_{2r})^{-2r},$$
$$M_{2r+1} = (\gamma_{2r})^{-(2r+1)}, \quad (45)$$

and consider the class of infinitely differentiable functions $C\{M_r\}$. From (42) we see that $C\{M_r\}$ is not a quasi-analytic class. Hence it contains a positive function $h(x)$ with compact support. Since $C\{M_r\}$ is invariant under translations and dilations, we can assume that $h(x)$ vanishes outside the interval $I=[-\frac{1}{2}, \frac{1}{2}]$.

We now show that

$$f(x) = (h*h)(x) = \int h(x-y)h(y)dy \quad (46)$$

is an element of $\mathfrak{L}(\mathfrak{R}^1)$. Since $f(x)$ is infinitely differentiable and vanishes outside the interval $2I$, it is sufficient to prove that $f\in\mathfrak{C}(\mathfrak{R}^1)$. Keeping the support of

$f(x)$ in mind, we have

$$\sup_{x\in\Re^1} |(1+x^2)D^{2r+n}\{x^m f(x)\}|$$
$$\leq 2 \sup_{x\in\Re^1} |D^{2r+n}\{x^m f(x)\}|$$
$$\leq 2 \sum_{\alpha=0}^{m} \sup_{x\in\Re^1} (|D^\alpha x^m||D^{2r+n-\alpha}f(x)|)\binom{2r+n}{\alpha}$$
$$\leq \left[2^{2r+n+1}(m+1)!\int |D^n h(y)|dy\right] \times \sup_{x\in\Re^1, 0\leq\alpha\leq \min(m,2r)} [D^{2r-\alpha}h(x)].$$

Recall that $h(x)\in C\{M_r\}$, and that M_r defined in (45) increases monotonically. Thus

$$\sup_{x\in\Re^1} |(1+x^2)D^{2r+n}\{x^m f(x)\}|$$
$$\leq 2^{2r+n+1}(m+1)!\int |D^n h(y)|dy A_1(A_2)^{2r}M_{2r}$$
$$= C(n,m)\left(\frac{2B^{1/2}A_2\alpha_{2r}}{\gamma_{2r}}\right)^{2r}\frac{1}{d_{2r}B^r}, \quad (47)$$

where $C(n,m)$ is a constant independent of r. By relation (44), we infer that

$$\sup_r C(n,m)\left(\frac{2B^{1/2}A_2\alpha_{2r}}{\gamma_{2r}}\right)^{2r} \equiv M(n,m,B)<\infty. \quad (48)$$

Therefore we conclude that

$$\sup_{x\in\Re^1} |(1+x^2)D^{2r+n}\{x^m f(x)\}| \leq \frac{M(n,m,B)}{d_{2r}B^r},$$

which is precisely relation (33). The above argument then shows that $f(x)\in\mathfrak{S}(\Re^1)$, which completes the proof of Theorem 6.

Theorem 7. $\mathfrak{L}(\Re^l)$ is nontrivial if and only if $\mathfrak{L}(\Re^1)$ is nontrivial.

Proof. If $f(x)\in\mathfrak{L}(\Re^1)$, then $\prod_{j=1}^l f(x_j)\in\mathfrak{L}(\Re^l)$. This follows from the fact that $g(t_1^2+\cdots+t_l^2)\leq g(lt_1^2)+\cdots+g(lt_l^2)$. Conversely, if $f(x_1,\cdots,x_l)\in\mathfrak{L}(\Re^l)$, then fixing $x_2, x_3,\cdots, x_l$ yields a function in $\mathfrak{L}(\Re^1)$.

Theorem 8. $\mathfrak{L}(\Re^l)$ is nontrivial if and only if

$$\int_0^\infty \frac{\ln g(t^2)}{1+t^2}dt<\infty. \quad (49)$$

Proof. By Theorem 7, it is sufficient to prove the case $l=1$. Suppose first that (49) holds, and consider the class $C\{M_r\}$, where we define

$$M_{2r-1}=M_{2r}=(c_{2r})^{-1}. \quad (50)$$

Then for $t\geq 1$, $g(t^2)>H(t)$, where $H(t)$ is defined in Theorem 5. Hence (49) leads to

$$\int_0^\infty \frac{\ln H(t)}{t^2}dt<\infty,$$

which by Theorem 5 assures us that $C\{M_r\}$ is not quasi-analytic. Therefore Theorem 4 gives that

$$\sum_{r=0}^\infty \sup_{n\geq 0} [(c_{2r+2n})^{1/(2r+2n)}]<\infty,$$

which by Theorem 6 is equivalent to a nontrivial space $\mathfrak{L}(\Re^1)$.

Conversely, suppose that $f(x)$ is a nontrivial element of $\mathfrak{L}(\Re^1)$, whose transform $\tilde{f}(p)$ is normalized so that $\int g(p^2)|\tilde{f}(p)|dp=1$. Then by (31),

$$|D^{2r}f(x)|\leq 1/c_{2r}.$$

Consider the class of functions $C\{M_r\}$, defined by

$$M_r=A^r \sup_{x\in\Re^1} |D^r f(x)|, \quad (51)$$

where A is a given constant. Since $M_{2r}\leq A^{2r}c_{2r}^{-1}$, the function $H(t)$ defined in Theorem 5 satisfies

$$A^{2r}H(t)\geq c_{2r}t^{2r}, \quad \text{for all } r. \quad (52)$$

Choosing $A<1$ and summing over r yields

$$H(t)\geq(1-A)g(t^2). \quad (53)$$

Since $f(x)\in C\{M_r\}$, the class $C\{M_r\}$ is not quasi-analytic. By Theorem 5,

$$\int_1^\infty \frac{\ln H(t)}{t^2}dt<\infty,$$

which combined with (53) yields

$$\int_0^\infty \frac{\ln g(t^2)}{1+t^2}dt<\infty.$$

This completes the proof of Theorem 8.

C. Proof of Theorems 1 and 2

Theorem 1 is a combination of Theorems 6–8, and hence has already been proved. We now proceed to Theorem 2. First note that it is sufficient to prove that if $\mathfrak{L}(\Re^l)$ is nontrivial, then it is dense in $\mathfrak{D}(\Re^l)$. Secondly, convolution by $\mathfrak{D}$ maps $\mathfrak{L}$ into $\mathfrak{L}$, $\mathfrak{D}*\mathfrak{L}\subset\mathfrak{L}$.

Let us suppose that $\mathfrak{L}$ is nontrivial, but not dense in $\mathfrak{D}$. Then there exists a nonzero Schwartz distribution $\chi\in\mathfrak{D}'$ which annihilates $\mathfrak{L}$, $\chi(\mathfrak{L})=0$. In other words, for any $f\in\mathfrak{L}(\Re^l)$, $h\in\mathfrak{D}(\Re^l)$,

$$\chi(f*h)=0=(\hat{h}*\chi)(f), \quad (54)$$

where $(\hat{h})(x)=h(-x)$. Here $\hat{h}*\chi$ is a regularized distribution, and hence an infinitely differentiable function.

In the proof of Theorem 6, it was shown that if $\mathfrak{L}(\mathfrak{R}^1)$ is nontrivial, then it contains a nontrivial, non-negative function. In the proof of Theorem 7 this positive function yields a nontrivial, non-negative function $f(x)$ in $\mathfrak{L}(\mathfrak{R}^l)$. Furthermore, since $\mathfrak{L}(\mathfrak{R}^l)$ is translation and dilation-invariant, it is possible to choose the support of $f(x)$ in an arbitrarily small neighborhood of any given point.

We now use this fact to show that χ must vanish. Suppose not; then for some $h\in\mathfrak{D}$, the regularization $(\hat{h}*\chi)(x)$ is not identically zero. Choose a point x_0 where $(\hat{h}*\chi)(x_0)\neq 0$, and chooses a sufficiently small neighborhood N of x_0, so that the real or imaginary part of the infinitely differentiable function $(\hat{h}*\chi)(x)$ has a constant sign. Choose the support of $f(x)$, a positive element of $\mathfrak{L}(\mathfrak{R}^1)$ to lie in N. This contradicts (54), unless $\chi=0$, and therefore it completes the proof of Theorem 2.

VI. EXAMPLES

In this section we discuss some simple examples of SLF's which are not operator-valued distributions. While the examples given have trivial scattering, they give a concrete illustration of how to deal with singular high-energy behavior. The most straightforward example of an SLF is obtained by exponentiating a free scalar field $\varphi(x)$. It was explained in Sec. II that

$$A(x)=:\exp\lambda\varphi:(x) \tag{55}$$

cannot be an operator-valued distribution. Nevertheless, if we choose the indicator function g for $\mathfrak{C}(\mathfrak{R}^4)$,

$$g(t^2)=\sum_{r=0}^{\infty} c_{2r}t^{2r},$$

to have exponential order in t greater than $\frac{2}{3}$, then $A(x)$ is an SLF in $\mathfrak{C}'(\mathfrak{R}^4)$. For instance, given any $0<\epsilon<1$, an acceptable choice for g would be given by

$$c_{2r}=1/(3r-\epsilon r)!. \tag{56}$$

In this case, the two-point function of the exponential can be written

$$\begin{aligned}(\psi_0,A^*(x)A(y)\psi_0)&=\exp\{|\lambda|^2(\psi_0,\varphi(x)\varphi(y)\psi_0)\}\\&=\exp\{|\lambda|^2(m^2/8\pi i)[m^2(x-y)^2]^{-1/2}\\&\qquad\times H_1^{(1)}((m^2(x-y)^2)^{1/2})\}\\&=\int_0^\infty \rho(M^2)\frac{1}{i}\Delta^{(+)}(M^2;x-y)dM^2,\end{aligned} \tag{57}$$

where $\rho(M^2)$ is a positive measure, and

$$\int_0^\infty \frac{\rho(M^2)}{g(M^2)}dM^2<\infty. \tag{58}$$

Here g is an acceptable indicator function described above. Thus at large values of the invariant mass M^2, the spectral weight $\rho(M^2)$ grows slower than the indicator function $g(M^2)$.

More generally, if one were interested in exponentiating any free-field component defined over l-dimensional space-time, this can be done to give an SLF in $\mathfrak{C}'(\mathfrak{R}^l)$. The strict localizability of such functions of free fields was discussed in Ref. 43. In addition, any entire function of a free field[44] can be realized in four-dimensional space-time as an SLF. It is possible even to include a wider class of functions.

In all these cases, the discussion of convergence of infinite series of free fields can be dealt with by using techniques similar to those in Ref. 44. The required technical tools will be developed in later works. In particular, there is a limit theorem[45] associated with $\mathfrak{C}'(\mathfrak{R}^l)$, and this allows a discussion of convergence of fields in terms of their analytically continued, vacuum-expectation values.

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