this becomes

$$-t_1(E-h_2)\frac{1}{E-h_1-h_2}t_2(E-h_1)\times\left(\frac{1}{E-h_1-h_2}-\frac{1}{\epsilon_2-h_2}\right)t_2(\epsilon_2)\frac{1}{\epsilon_1-h_1}t_1(\epsilon_1). \quad \text{(A16)}$$

The following identity is easily verified for the Lippmann-Schwinger equation[19]:

$$t(a)[(a-h)^{-1}-(b-h)^{-1}]t(b)=t(a)-t(b). \quad \text{(A17)}$$

[19] The proof is

$$t(a)-t(b)=t(a)(a-h)^{-1}v-v(b-h)^{-1}t(b)=t(a)(a-h)^{-1}t(b)$$
$$-t(a)(a-h)^{-1}v(b-h)^{-1}t(b)-t(a)(b-h)^{-1}t(b)$$
$$+t(a)(a-h)^{-1}v(b-h)^{-1}t(b)=t(a)[(a-h)^{-1}-(b-h)^{-1}]t(b).$$

Therefore (A14) becomes

$$t_1(E-h_2)(E-h_1-h_2)^{-1}t_2(\epsilon_2)(E-h_1-h_2)^{-1}t_1(\epsilon_1)$$
$$-t_1(E-h_2)(E-h_1-h_2)^{-1}t_2(E-h_1)$$
$$\times(E-h_1-h_2)^{-1}t_1(\epsilon_1). \quad \text{(A18)}$$

The second term cancels the second term of (A8). The first term of (A18) and the first of (A8) combine again using the identity (A17) to give (A13). Hence the product form is the solution even though the equation naively does not seem to want it to be. This shows that in problems for which the product form is a good first approximation, simple interpretation of the Faddeev equations is particularly dangerous.

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Effect of the Quasipole on the π-N Forward Scattering Amplitudes

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It is shown that a quasipole plays an important role in low-energy meson-baryon scattering. The influence of this quasipole, which exists on the second Riemann sheet just below the elastic threshold and has a residue of the same order of magnitude as that of the nucleon pole, is analyzed by extrapolation of the data of the pion-nucleon scattering amplitude. Since the residue of the quasipole is related to the pion-nucleon coupling constant, this extrapolation gives an independent possibility for determining the coupling constant. The result is $g^2/4\pi=13.9$.

INTRODUCTION

JUST below the elastic threshold, the pion-nucleon scattering amplitude has a quasipole with the residue of $+0.77(g^2/4\pi)$ for $T=\frac{3}{2}$ on the second sheet.[1] Neglect of this fact has so far hindered a simple description of low-energy pion-nucleon scattering in terms of the nucleon poles and the 3-3 resonance pole. The purpose of this paper is to examine the influence of the quasipole on this scattering amplitude by analyzing the low-energy data of pion-nucleon scattering. We shall first repeat the definition of the quasipole and then compute its residue. It is to be noticed that this computation can be performed without making any approximation, solely by means of the unitarity condition and the spectral representation of the scattering amplitudes.

QUASIPOLE

Let us consider pion-nucleon scattering, with masses μ and m, respectively, first neglecting the spin of the nucleon; the results for actual physical case will be given later on. We will assume the following two conditions:

(i) Spectral representation.

For $l\geqq L$

$$a_l(s)=-\frac{g^2}{2q^2}Q_l\left(1+\frac{m^2+2\mu^2-s}{2q^2}\right)+\frac{g^2}{m^2-s}\delta_{l,0}+\frac{1}{2q^2}\frac{1}{\pi}\times\int_{4\mu^2}^{\infty}dt'A_t(t',s)Q_l\left(1+\frac{t'}{2q^2}\right)-\frac{1}{2q^2}\frac{1}{\pi}\times\int_{(m+\mu)^2}^{\infty}du'A_u(u',s)Q_l\left(1+\frac{2m^2+2\mu^2-u'-s}{2q^2}\right), \quad (1)$$

and for $0\leqq l<L$

$$a_l(s)=-\frac{g^2}{2q^2}Q_l\left(1+\frac{m^2+2\mu^2-s}{2q^2}\right)+f_l(s), \quad (2)$$

where $f_l(s)$ is an analytic function of s regular at $s=m^2+2\mu^2$, and L is a finite positive integer.

(ii) Unitarity condition.

$$a_l(s)-a_l^{\dagger}(s)=\frac{iq}{\sqrt{s}}a_l(s)a_l^{\dagger}(s) \quad (3)$$

[1] T. Sawada, Phys. Rev. Letters 15, 567 (1965).

for $(m+\mu)^2 \leq s < (m+2\mu)^2$. Since we are concerned with the neighborhood of a point $s=m^2+2\mu^2$, it is convenient to introduce a new infinitesimal variable η^2 defined by

$$\eta^2=\frac{m^2+2\mu^2-s}{2q^2}. \quad (4)$$

Then we can prove the following theorem concerning the second-sheet amplitude $A^{\mathrm{II}}(s,\cos\theta)$, which is defined by the analytic continuation through the elastic cut.

Theorem: In a small neighborhood of $s=s_+\equiv m^2+2\mu^2$,

$$A^{\mathrm{II}}(s,\cos\theta)=-\frac{g^2}{2q^2}\sum_{l=0}^{\infty}(2l+1)\frac{Q_l(1+\eta^2)}{1+a^{-1}Q_l(1+\eta^2)}\times P_l(\cos\theta)+O\left(\frac{1}{|\ln\eta|^2}\right), \quad (5)$$

where

$$\frac{1}{a}=\frac{g^2}{2iq\sqrt{s}}. \quad (6)$$

The proof is given in Ref. 1. From this theorem we see that the information about the spectral functions $A_t(t',s)$ and $A_u(u',s)$ is *not* needed in order to obtain the value of $A^{\mathrm{II}}(s,\cos\theta)$ in the small neighborhood of $s=m^2+2\mu^2$. If we use the formula relating Q_l to the Bessel function K_n,

$$Q_l(1+\eta^2)=K_0(\xi)+\frac{\eta^2}{12}\left(-\frac{K_1(\xi)}{\xi}+\xi K_1(\xi)-2K_0(\xi)\right)+O(l^{-4}), \quad (7)$$

where

$$\xi=\sqrt{2}(l+\tfrac{1}{2})\eta, \quad (8)$$

and change the summation of Eq. (5) into integration over ξ, then we have

$$\sum_{l=0}^{\infty}(2l+1)\frac{Q_l(1+\eta^2)}{1+a^{-1}Q_l(1+\eta^2)}=\frac{1}{\eta^2}R_+ + C_+ + O\left(\frac{1}{|\ln\eta|^2}\right), \quad (9)$$

with

$$R_+=\int_0^{\infty}\frac{\xi K_0(\xi)}{1+a^{-1}K_0(\xi)}d\xi, \quad (10)$$

and a similar expression for C_+. By combining Eqs. (4), (5), and (9), the forward scattering amplitude becomes

$$A^{\mathrm{II}}(s,1)=\frac{g^2(R_+-1)}{s-s_+}+\frac{g^2}{s-s_+}+C'+O(|\ln\eta|^{-2}). \quad (11)$$

The second term of the right-hand side of Eq. (11) is the nucleon u pole $g^2/(m^2-u)$, while the first term is called the quasipole which is characteristic to the second-sheet amplitude.[1]

TABLE I. $[\mathrm{Re}B^{3/2}(s,1)/4\pi m]$ minus the nucleon-pole contribution.

$\omega_{\mathrm{lab}}-\mu$ (MeV)	$-I/\pi$ (in units of pion mass; $\mu=1$)
15	0.183±0.003
25	0.198±0.003
35	0.213±0.003
40	0.221±0.003
58	0.252±0.003
80	0.293±0.004
100	0.314±0.004
120	0.293±0.003
143	0.219±0.004
150	0.179±0.004
170	0.040±0.003

When the nucleon spin is taken into account, the pion-nucleon scattering can be described in terms of two independent amplitudes, A and B. As usual, let us introduce two other independent functions, f_1 and f_2, which are more convenient for the partial-wave expansion[2]:

$$f_1=[(E+m)/8\pi W][A+(W-m)B],$$
$$f_2=[(E-m)/8\pi W][-A+(W+m)B]. \quad (12)$$

If we fix $\cos\theta=1$, the pole and the quasipole of the second-sheet function in the small neighborhood of $s=s_+$ are given by

$$(f_1+f_2)^{\mathrm{II}}=-\mu\left(\frac{1-\mu^2/4m^2}{1+2\mu^2/m^2}\right)^{1/2}\times\frac{1}{a}\left[\frac{1}{s-s_+}+\frac{r(a)}{s-s_+}\right]+O(1), \quad (13)$$

$$f_2^{\mathrm{II}}=\frac{\mu}{2}\left(\frac{3m}{(m^2+2\mu^2)^{1/2}}-1\right)\left(\frac{1-\mu^2/4m^2}{1+2\mu^2/m^2}\right)^{1/2}\times\frac{1}{a}\left[\frac{1}{s-s_+}+\frac{r(a)}{s-s_+}\right]+O(1), \quad (14)$$

where

$$r(a)=-\frac{1}{a}\int_0^{\infty}\frac{\xi(K_0(\xi))^2}{1+a^{-1}K_0(\xi)}d\xi, \quad (15)$$

and

$$a^{-1}=+\frac{2g^2}{4\pi}\frac{\mu}{2m}\left(1-\frac{\mu^2}{4m^2}\right)^{-1/2} \quad \text{for } T=\tfrac{3}{2},$$
$$a^{-1}=-\frac{g^2}{4\pi}\frac{\mu}{2m}\left(1-\frac{\mu^2}{4m^2}\right)^{-1/2} \quad \text{for } T=\tfrac{1}{2}. \quad (16)$$

In Eqs. (13) and (14) the first term in the square brackets is the nucleon u pole which also exists in the first sheet of the scattering amplitude, while the second term is the quasipole characteristic to the second-sheet

[2] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

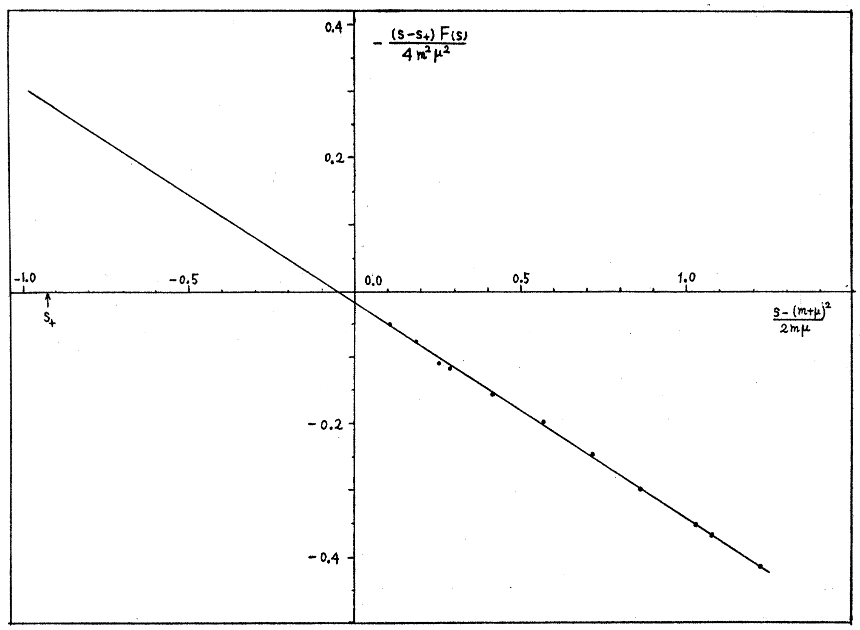

FIG. 1. The extrapolation of the low-energy pion-nucleon scattering amplitude for $T=\frac{3}{2}$ versus laboratory energy. $F(s)$ is related to $\mathrm{Re}B(s,1)$ by Eqs. (19) and (18). $s_+=m^2+2\mu^2$ is the position of the quasipole.

singularity. In this paper the low-energy data of the pion-nucleon scattering will be compared with the predicted values of Eqs. (13) and (14) for the case of isospin $\frac{3}{2}$. For the case of isospin $\frac{1}{2}$, a little more elaborate analysis is needed, since the denominator of the integrand of Eq. (15) vanishes, which corresponds to the appearance of the series of poles. This case will be treated elsewhere.

COMPARISON WITH EXPERIMENTS

The pion-nucleon scattering data are cited from the article by Hamilton and Woolcock.[3] Let us consider an analytic function of s,

$$\hat{B}(s,1)=\tfrac{1}{2}[B^{\mathrm{I}}(s,1)+B^{\mathrm{II}}(s,1)], \tag{17}$$

which is regular at the threshold $s=(m+\mu)^2$ and coincides with $\mathrm{Re}B^{\mathrm{I}}(s,1)$ when s takes a physical value. The quantity defined by

$$\frac{I(s)}{\pi}=\left\{\frac{\mathrm{Re}B(s,1)}{4\pi m}+\frac{1}{m}\frac{2g^2}{4\pi}\frac{1}{s-s_+}\right\} \tag{18}$$

is given in Table I.[3] In order to eliminate the singularity

TABLE II. Predicted values of the residue of the quasipole $\hat{R}$ for various values of the pion-nucleon coupling constant $g^2/4\pi$.

a^{-1}	$g^2/4\pi$	$\hat{R}$
2.0	13.44	−0.272
2.1	14.10	−0.291
2.2	14.76	−0.311

[3] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

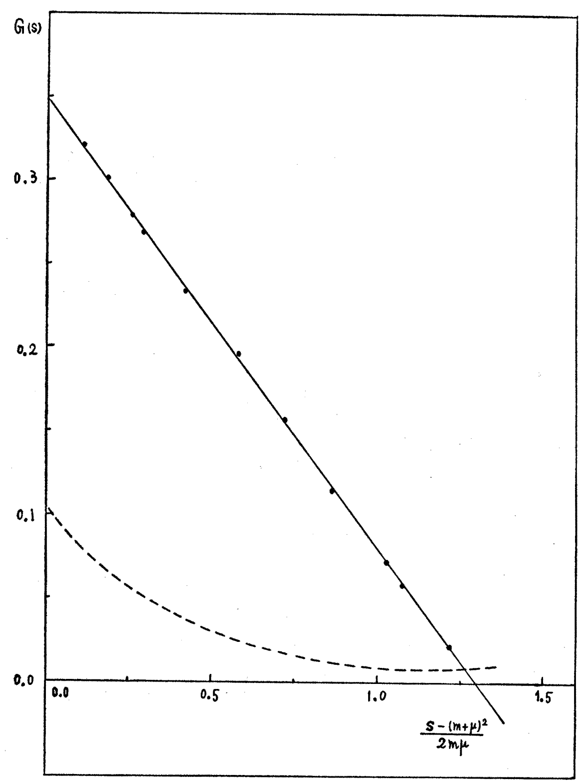

FIG. 2. Plot versus laboratory energy of the low-energy data on pion-nucleon scattering for $T=\frac{3}{2}$ with the nucleon pole and the quasipole subtracted. $G(s)$ is related to $\mathrm{Re}B(s,1)$ by Eq. (21). The dashed line is the subtracted quasipole.

of N^*, a function defined by

$$F(s)=\frac{1}{s-(m+\mu)^2}\left\{(s-s_R)(s-s_R{}^*)\frac{I(s)}{\pi}-\left[(s-s_R)(s-s_R{}^*)\frac{I(s)}{\pi}\right]_{s=(m+\mu)^2}\right\} \quad (19)$$

will be analyzed, where S_R is the position of the 3-3 resonance, $\sqrt{S_R}=8.66+i0.41$ in units of pion mass. S_R is determined by the 3-3 partial-wave fit done by Noyes and Edwards.[4] The residue of the quasipole is obtained, just as in the usual extrapolation, by plotting the function $(s-s_+)F(s)$, which is shown in Fig. 1. From this extrapolation we have as the residue

$$\hat{R}=\lim_{s\to s_+}\frac{(s-s_+)F(s)}{4m^2\mu^2}=-0.285 \quad \text{in units of pion mass,} \quad (20)$$

while predicted values are given in Table II for various values of the coupling constant. These two values become identical, if $g^2/4\pi=13.9$. This reasonable agreement definitely shows the existence of the quasipole at $s=s_+$. In Fig. 2 is plotted the B function with the nucleon u pole and the quasipole subtracted, i.e., the function

$$G(s)=\frac{(s-s_R)(s-s_R{}^*)}{4m^2\mu^2}\left\{\frac{\mathrm{Re}B(s,1)}{4\pi m}+\frac{1}{m}\frac{2g^2}{4\pi}\frac{1}{s-s_+}-\frac{1}{2m}\frac{g^2}{4\pi}\frac{0.77}{s-s_+}\right\}. \quad (21)$$

For comparison, the contribution of the quasipole to $G(s)$ is indicated by the dashed line. Notice that the plot of Fig. 2 becomes a straight line, which corresponds to the vanishing of the background term. This shows that the low-energy value of $B^{(3/2)}(s,1)$ comes solely from the nucleon pole, the 3-3 resonance pole, and the quasipole. The reason that the existence of the quasipole has been overlooked for such a long time is that this singularity can be found neither by the perturbation expansion of finite order nor by the summation of the partial-wave functions up to a finite l; furthermore it exists on the second sheet of the scattering amplitude although its position is very close to the physical region.

[4] H. P. Noyes and D. N. Edwards, Phys. Rev. **118**, 1409 (1960).

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Theory of Stability of Tensor Operators under Perturbations and its Application to Particle Physics*

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We investigate the perturbations of tensor operators due to symmetry breaking and consequent representation mixing. A group-theoretical stability principle, valid for an arbitrary (simple, compact) group is formulated, which in many cases assures the vanishing of the first-order perturbation when it is constrained to leave a certain component unaltered. The physically interesting case of unitary symmetry is discussed in detail. All previously known results are recovered and several new results are deduced. As an application we discuss the conditions under which the universality of the Cabibbo angles for leptonic decays is valid.

INTRODUCTION

HIGHER symmetries of strong interactions that have been proposed in recent years have been remarkably successful in the organization of the data on particles and resonances. But they all share the property of being broken appreciably, either by virute of interactions of lesser strength which violate these symmetries or by virtue of some other mechanism. In the case of unitary symmetry and the spin-dependent symmetries these symmetry violations as manifested by the observed mass differences are appreciable. Nevertheless, it is remarkable that a considerable remnant of the symmetry survives in the observable features like supermultiplets and mass and coupling constant sum rules. Thus, for example, the identification of the sources of electromagnetic and weak interactions to be octet currents seems to be in quantitative agreement with experiment in spite of the large representation mixing expected in view of the departures from unitary symmetry. We should therefore search for a

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