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# Coherence in Three-Body Final States*

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The amplitude for a reaction leading to three free particles is studied in the context of nonrelativistic scattering theory with particular emphasis on the distribution of information about final-state interactions of a given pair. We show that contrary to the simple notions often read into the Faddeev equations, terms ending with the interaction of a given pair do not carry the major information about interactions of that pair. When a coherent combination of all terms contributing to the process is taken, a factor clearly carrying the features of a pair's final-state interactions appears multiplying the *entire* amplitude. This factor carries the elastic-scattering phase of the pair. It will dominate the variation of the amplitude with respect to final-state interactions of the pair if certain other terms vary slowly, and this seems plausible. Application of this result to triangle singularities shows that the coherent combination of terms with and without final rescattering changes the singularity qualitatively. The assignment of a final-state interaction factor onto the entire amplitude for each pair suggests a product-type final-state wave function for the process. The implications of this for overlapping resonances are discussed with particular reference to static models and the absence of the Peierls mechanism. The case of two particles interacting independently with a fixed center, for which the product wave function is exact, is discussed from the Faddeev viewpoint and shows that a multiple-scattering approach can be misleading.

## I. INTRODUCTION

In states involving more than two strongly interacting particles, the interactions between the various pairs are coherent. Thus for any process involving the particles, information about a given pair's interactions is distributed over the entire amplitude. Since one of the major motivations for the study of multiparticle states is to obtain this information about pair interactions, it is important to understand how it is distributed. In this paper we study this question for nonrelativistic three-body systems. It is usual in studying these to separate the amplitude into parts according to which pair interacts "last." Such a separation is particularly natural in the three-body theory of Faddeev where it is formally exact.[1] It is often further assumed on physical grounds that the major features of any pair's interactions are carried predominantly by that term in which the pair interacts last. We show that this is not so, and that when the coherence of terms is taken into account the entire amplitude carries the information. The way in which the term with final rescattering combines with those without this rescattering to distribute the information about the pair interaction over the entire amplitude is closely related to the way the integral term combines with the real Born term in the two-body Lippmann-Schwinger equation to give the complex two-body amplitude.

The impetus for examining this question came from our exact three-body calculation of the reaction $n+d \rightarrow n+n+p$.[2] This reaction is of particular interest for the information it can give on the strong low-energy neutron-neutron interaction. We found numerically that this $n$-$n$ interaction showed up more clearly in the *entire* reaction amplitude than it did in those terms involving only a final $n$-$n$ rescattering. In static models it is also known that the effects of each possible pairwise interaction on a three-body final state is carried in the full amplitude rather than being distributed among the various pieces.[3] In this paper we show that this effect, found numerically in the three-nucleon case and known in the static model, is a general feature of three-body states. In fact the result is easily generalized to final states of more than three particles.

Our criticism of the usual argument that in a reaction leading to three particles the terms ending with interactions between a given pair carry the primary information about that pair is based on showing that the part of the amplitude coming before the final interaction is not slowly varying in magnitude or phase and therefore the variation of a term ending with a given pair interaction as a function of that pair's energy does not have a simple interpretation in terms of the pair's two-body $t$ matrix. However, when the terms ending with interactions between the other pairs are added in to form the total reaction amplitude, a multiplicative factor emerges on the *entire* amplitude that does have a simple interpretation in terms of the pair's elastic-scattering amplitude and phase shift. Furthermore, it seems reasonable to assume that the terms coming before this factor are now slowly varying so that the primary information about a given pair's final-state interactions is carried in this final factor on the entire amplitude. One of the principal features of this factor is that, considered as a function of the center-of-mass energy of the pair in question, in a given partial wave for that pair, it has the elastic-scattering phase of the pair. If the phase of the rest of the amplitude varies slowly in this variable, the variation of phase of the entire amplitude as a function of the pair energy will be that of the elastic two-body scattering phase of the pair.

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[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. 39, 1459 (1960) [English transl.: Soviet Phys.—JETP 12, 1014 (1961)].

[2] R. Aaron and R. D. Amado, Phys. Rev. 150, 857 (1966).

[3] C. Goebel, Phys. Rev. Letters 13, 143 (1964).

If the entire amplitude for a process leading to three particles acquires a factor for each pair, we might expect it to carry a product of factors. In fact it is not possible in general to specify the angular momentum and energy of each pair in a three-body state, and hence it is not possible to assign a unique elastic-scattering state to each pair simultaneously. Nevertheless, there are special cases such as low-energy reactions or static models in which only one partial wave dominates for each pair and one can write the amplitude with a product of phase factors. This product form would emerge naturally if we wrote the three-body final state wave function as a product of two-body wave functions. The product wave function is a venerable approximation in the three-body problem, but does not seem to arise gracefully in the modern Faddeev formulation, which begins with a sum of terms partitioned according to sequential multiple scattering ideas. It has been found recently by Day in the study of three-body clusters in nuclear matters, for example, that a product type of wave function is far superior to the sum of correlated parts that seems to come naturally from the Faddeev approach.[4] That work and what we present here can be taken as a note of caution against reading simple, physical approximation schemes from the formally correct Faddeev equations. In view of this, it is instructive to study the Faddeev equation for the case of two particles interacting independently with a static potential and not with one another. This problem is exactly solved by a product wave function. In the Appendix we show how the Faddeev equations can also be made to yield this result, but it is also clear that it is not a very natural approach to the problem.

The study of coherence in three-body final states can also be made to shed light on the question of triangle singularities.[5] These occur when in a three-body final state the particles can propagate on the energy shell for large distances before the final rescattering. If this final rescattering is large, this should give an anomalously large (singular) amplitude. This process is coherent with the same process without the final rescattering and the combination of the two terms changes qualitatively the nature of the process.

Another question of considerable interest in multiparticle states is overlapping resonances. The existence of a product form for the three-body amplitude would seem to reinforce the case for enhancement of the three-body amplitude when two two-body resonances overlap. In numerical investigations of exactly soluble static models no such enhancement is found.[6] We show how this can be in spite of the product form which is exactly applicable to these models.

[4] Ben Day, Phys. Rev. **151**, 826 (1966).

[5] I. J. Aitchison and C. Kacser, Phys. Rev. **142**, 1104 (1966). This detailed paper contains many references to earlier work.

[6] F. S. Chen-Cheung and C. M. Sommerfield, Phys. Rev. **152**, 1401 (1966).

In Sec. II we show how the terms in the amplitudes leading to three free particles combine coherently so that the information about a given pair's interactions seems to reside most naturally in the entire amplitude for the process rather than in the terms ending with interactions of that pair only. Section III discusses triangle singularities and shows in particular that the coherent addition of the rescattering term to the term without final rescattering gives a vanishing triangle singularity for resonant rescattering. In Sec. IV some of the consequences of a product form are discussed with particular reference to overlapping resonances. An exact solution of the Faddeev equation for a product wave function is given in the Appendix.

## II. COHERENCE IN THE THREE-BODY AMPLITUDE

We now show that in a three-body final state, the terms involving final interactions between the various pairs combine coherently or interfere so that the information about a particular pair's interaction is not carried by the terms that end with rescatterings of that pair but by the entire amplitude.

For concreteness we will illustrate this result in the context of a reaction in which two particles interact to give three, all different. Labeling the final particles 1, 2, and 3, the reaction is $1+(23)\to 1+2+3$, where (23) stands for a bound state of 2 and 3 or an elementary particle coupled to 2 and 3. In either case, we assume (23) is stable against spontaneous decay into $2+3$. We shall use a Faddeev formalism to describe the reaction.[1] In such a formulation the complete amplitude $(1,(23)|T|1,2,3)$ is written as the sum of three parts

$$T=T_1+T_2+T_3. \tag{1}$$

These satisfy the set of coupled equations

$$\begin{aligned} T_1&=(T_2+T_3)G_0t_{23}, \\ T_2&=H'G_0t_{13}+(T_1+T_3)F_0t_{13}, \\ T_3&=H'G_0t_{12}+(T_1+T_2)G_0t_{12}, \end{aligned} \tag{2}$$

where $t_{ij}$ is the two-body scattering amplitude in the three-body space, $G_0$ is the three-body free particle Green's function, and $H'$ is the interaction mechanism for the coupling $(23)\rightleftarrows 2+3$. These equations are represented graphically in Fig. 1. The two-body amplitudes which appear are in general off the energy shell, a typical term being in momentum space

$$(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3|t_{12}|\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3') = (\mathbf{p}_1,\mathbf{p}_2|\hat{t}_{12}(E-\epsilon_{p_3})|\mathbf{p}_1',\mathbf{p}_2')\delta(\mathbf{p}_3-\mathbf{p}_3'), \tag{3}$$

where $\hat{t}$ is a solution of the ordinary two-body Lippmann-Schwinger equation with energy variable $E-\epsilon_{p_3}$. $E$ is the total three-body energy variable and $\epsilon_{p_3}$ the energy appropriate to momentum $p_3$.

From (2) we can read off the significance of $T_1$, $T_2$, $T_3$. $T_i$ is the sum of all terms contributing to $T$ which

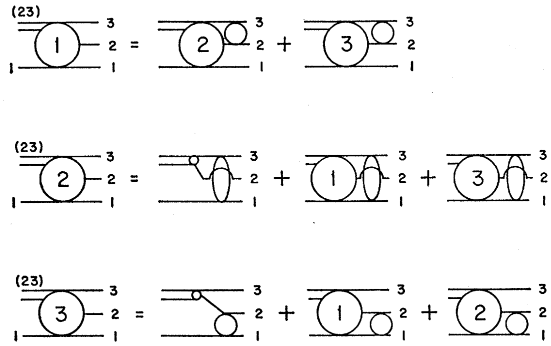

FIG. 1. Graphical representation of the coupled Faddeev equations [Eq. (2)] for the reaction $1+(23)\rightarrow 1+2+3$.

end with $t_{jk}$ ($i\neq j\neq k$). Furthermore, on-shell for the three-body state a given final $t_{jk}$ will be half-on-shell. That is, the final two-body energy will be equal to the energy variable in the amplitude. An important property of such two-body-half-on-shell amplitudes in a given partial wave is that they have the elastic-scattering phase. That is, in the two-body center of mass the scattering amplitude from relative momentum $p$ to $q$ in the $l$th partial wave with energy $\epsilon_q$ appropriate to momentum $q$ can be written

$$(p|t_l(t_q)|q)=A_l(p,q)t_l(q)\,, \tag{4}$$

where $t_l(q)=(q|t_l(\epsilon_q)|q)$ is the on-shell amplitude and $A$ is *real*. This theorem seems not to be universally known although it is in fact implied by the off-shell equation of Kowalski[7] and Noyes.[8] A simple proof of it can be given from off-shell unitarity alone without recourse to Lippmann-Schwinger–type equations. The half-off-shell unitarity relation is

$$\mathrm{Im}(p|t_l(\epsilon_q)|q)=(p|t_l(\epsilon_q)|q)t_l^*(q)\rho(q)\,, \tag{5}$$

where $\rho(q)$ is some real density-of-states factor. One can always write $(p|t_l(\epsilon_q)|q)$ in the form (4) with $A$ arbitrary and hence (5) becomes

$$\mathrm{Im}A_l(p,q)t_l(q)=A_l(p,q)|t_l(q)|^2\rho(q)\,; \tag{6}$$

since the left-hand side is real so $A$ must be and the theorem is proved.

In view of this result and of the interpretation of $T_i$ given above, it seems tempting to assert that if in a three-body final state a given pair $jk$ has a strong final-state interaction then that interaction will manifest itself most clearly in $T_i$ and not particularly strongly in $T_j$ or $T_k$. We would expect this dependence to show up most clearly in the dependence of the amplitudes on the $j$-$k$ relative energy, and of the three parts only $T_i$ seems to have a simple dependence on this variable. We might expect, for example, that the dominant variation of the phase of $T_i$ as a function of the $j$-$k$ relative energy is given by the rapidly varying phase of the final half-on-shell $t_{jk}$, which phase is the elastic $j$-$k$ phase. For this to be so what comes before this last $t_{jk}$, in particular $G_0$, must have a slowly varying phase. We shall show that this is not so. To see this we note that using (2) the entire amplitude can be written in any of three equivalent ways.

$$T=(T_2+T_3)(1+G_0t_{23})\,, \tag{7a}$$

$$T=H'G_0t_{13}+(T_1+T_3)(1+G_0t_{13})\,, \tag{7b}$$

$$T=H'G_0t_{12}+(T_1+T_2)(1+G_0t_{12})\,. \tag{7c}$$

In each of these there appears a factor $1+G_0t_{ij}$ with $t_{ij}$ half-on-shell. $1+G_0t_{ij}$ has nontrival dependence on the relative momentum of $i$ and $j$ and is proportional to delta functions in all the other variables. Its dependence on the relative $i$-$j$ momentum in a given $i$-$j$ partial-wave $l$ can be studied from the Lippmann-Schwinger equation for $t_{ij}$ in that partial wave in the two-body center of mass

$$t_l=v_l+v_lG_0t_l\,. \tag{8}$$

Dividing by $v_l$, we get

$$v_l^{-1}t_l=1_l+G_0t_l\,, \tag{9}$$

where $1_l$ is the partial-wave projection of "one." Since $t_l$ has the phase $\delta_l$ half-on-shell and since $v_l$ is real, $v_l^{-1}t_l$ has the phase $\delta_l$ and therefore so does $1_l+G_0t_l$. It is clear from (8) or (9) how this comes about. The $t_l$ on the right has the phase $\delta_l$, but $G_0$ has an imaginary part. Hence the phase of say $v_lG_0t_l$ is not $\delta_l$ but is cleverly organized to cancel the real $v_l$ term so that the sum has the scattering phase. It is even clearer in the form $1_l+G_0t_l$ that it is the imaginary or on-shell part of $G_0$ that will combine coherently with $1_l$ to give the entire term the scattering phase. Put in terms of magnitudes rather than phases we can say that if $v_l$ is slowly varying but $t_l$ is rapidly varying in some range, $v_lG_0t_l$ must have a slowly varying part organized so as to cancel the slow variation of $v_l$. In the three-body problem with Faddeev amplitudes, we do not expect a $j$-$k$ final-state interaction to manifest itself strongly in $T_j$ or $T_k$. These then play the role of the potential, but $T_j$ is analogous to $v_lG_0t_l$ and has a part organized to cancel the variation of $T_j+T_k$ and put a factor containing the variation of $t_{jk}$ on to the entire amplitude.

This factor $1+G_0t$ occurs in each of the forms of $T$ in (7); in addition (7b) and (7c) contain $H'G_0t_{13}$ and $H'G_0t_{12}$. In this case, however, no additional phase arises from the $G_0$ since (23) is stable against spontaneous decay into 2 and 3 and that assures that the intermediate state in $H'G_0t_{ij}$ cannot propagate on the energy shell and therefore that $G_0$ is always real and slowly varying. Hence a term like $H'G_0t_{12}$ has the phase and variation in the 1-2 center of mass in a given 1-2 partial wave of 1-2 elastic scattering in that wave from the half-on-shell factor $t_{12}$ and that is all. Therefore each of the forms for $T$ in (7) shows that as a function of the $i$-$j$ center of mass energy in an $i$-$j$ partial wave, the

[7] K. L. Kowalski, Phys. Rev. Letters **15**, 798 (1965).
[8] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965).

*entire* $T$ acquires a factor that gives it the variation and phase of $i$-$j$ elastic scattering. This is so for each pair.

In multiple-scattering language one can see why this arises by noting that each term contributing to the amplitude that ends with a $G_0t_{ij}$ is fed by a term which also appears without the $G_0t_{ij}$ and this is just the combination needed to get a factor for the total amplitude with the $i$-$j$ scattering variation and phase. Of course this feeding term is also complex and a function of the relative pair energy under study. Hence this result by no means establishes precisely the phase or variation of the entire amplitude in any variable. It is possible to take the three relative pair energies as independent variables and therefore to discuss the variation of the amplitude in one of these while the other two are kept fixed. (To stay on the energy shell the total energy must vary as one pair energy does.) What is hoped is that when the other variables are kept fixed, the variation in phase of the amplitude due to variation of the relative $i$-$j$ center-of-mass energy in a fixed $l$ wave is dominated by the last factor and this carries the $i$-$j$ elastic-scattering variation and phase. Although we cannot prove this in general, we see from (7) that now the piece before the phase-bearing factors has a chance to be slowly varying since it is made up of terms ending with an interaction between a non-dominant pair. It is also clear that in a reaction leading to more than three particles one can again divide the amplitude so that each term ending in $G_0t_{ij}$ is "fed" by a term that appears without $G_0t_{ij}$, so that again the *entire* amplitude has a factor with the phase and variation of $t_{ij}$ in the appropriate variable.

These results imply that in the three-body case the Faddeev separation of the amplitude into terms according to which pair interacts last is misleading in the way in which it distributes information about the final-state interactions among the pairs. The term $1+G_0t_{ij}$ which occurs in the regrouping of the Faddeev amplitudes is just the two-body scattering wave function. Since this occurs for all pairs, the results arrived at above indicate that in some sense a product of two-body wave functions is a better approximation to the three-body final-state wave function than the sum of terms, each dominated by a given interaction as one might normally expect from the Faddeev equation. Unfortunately, it does not seem possible to specify the relative pair energy and the partial wave for each pair independently, and therefore one cannot in general use a product form directly. However, in some sense a product is better. This has been noticed by Day in the study of three-body correlations in nuclear matter where a product type of approximation seems to do a much better job than a *sum* of correlated two-body terms.[4] In Sec. IV we conjecture a product form motivated by the results of this section which may be valid in a broad class of cases, and in the Appendix we discuss the case in which the product is exact (two independent systems), and show how the Faddeev equations, although formally correct and capable of giving the answer, are misleading.

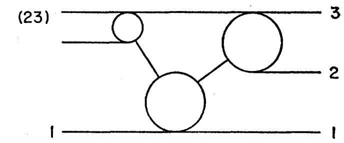


FIG. 2. Typical triangle graph for the reaction $1+(23)\rightarrow 1+2+3$.

## III. TRIANGLE SINGULARITIES

Much attention has focused in recent years on a particular subclass of three-body final states, those having triangle singularities.[5] In this section we show that the coherent combination of the triangle graph with other lower-order graphs acts to change qualitatively the nature of the singular contribution and in fact to remove it in many cases.

We study the situation in the example of the previous section, the reaction $1+(23)\rightarrow 1+2+3$ in which particle 1 is incident on a bound state of 2 and 3, (23), to give three particles 1, 2, and 3. A typical triangle graph comes from a term in the total amplitude of the form

$$H'G_0t_{12}G_0t_{23}, \tag{10}$$

where $H'$ is the mechanism for $(23)\rightarrow 2+3$, and $t_{12}$ and $t_{23}$ are two-particle $t$ matrices in the three-body space. They are appropriately off-shell. The graph itself appears in Fig. 2. The point of the triangle singularity is that although the energy denominator in the first $G_0$ of (10) cannot vanish because (23) is stable, it can in the second. Hence that intermediate state can propagate on the energy shell. If the masses of the particles are right, this state will propagate a long way before the final 2-3 collision. If further $t_{23}$ is big at that energy, the contribution from this graph will be anomalously large in the physical region. This is the triangle singularity. Experimentally such singularities have proved very elusive.[9]

From the previous section it is clear that the consideration of (10) above is not enough. If (10) contributes to the reaction, so must the term without the final $t_{23}$,

$$H'G_0t_{12}. \tag{11}$$

We also saw in the last section that the argument that this piece knows nothing of $t_{23}$ and therefore cannot interfere with (10) is wrong. In fact the sum of the two terms is

$$H'G_0t_{12}(1+G_0t_{23}). \tag{12}$$

It has the familiar $1+G_0t_{23}$ factor. We saw in the previous section that this factor has the phase of $t_{23}$ in a given particle wave, and in particular that the 1 is

[9] The effect of such a singularity may have been observed by J. Lang, R. Muller, W. Wolfli, R. Bosch, and P. Marmier [Phys. Letters 15, 248 (1965)], but in their case the final rescattering is low-energy $n$-$p$ scattering which has a large scattering length, but is not resonant.

coherent with the energy-conserving part of $G_0$. Since it is just that part that plays the key role in the triangle singularity, we must be particularly suspicious of leaving the 1 out.

In order to see explicitly how this coherence arises and how the factor $1+G_0t_{23}$ acquires the 2-3 elastic-scattering phase we pass to the 2-3 center-of-mass system, and factor out the delta function on the momentum of particle 1 and the center-of-mass momentum of 2 and 3. Thus the factor $1+G_0t_{23}$ in the 2-3 $l$th partial wave for going from relative 2-3 momenta $k$ to $p$ is

$$\frac{\delta(k-p)}{4\pi p^2}-\frac{(k|t_l(p^2)|p)}{2\pi^2(p^2-k^2+i\epsilon)}, \tag{13}$$

where we have taken $\hbar=1$ and the reduced 2-3 mass$=\frac{1}{2}$, and used the fact that the center-of-mass energy is $p^2$. The normalization and phase conventions are those of Merzbacher.[10] We now separate the Green's function into principal part and delta function to give for (13)

$$\frac{\delta(k-p)}{4\pi p^2}(1+ipt_l(p^2))-\frac{P}{2\pi^2}\frac{(k|t_l(p^2)|p)}{p^2-k^2}, \tag{14}$$

where $t_l(p^2)$ is the on-shell amplitude and in this normalization is

$$t_l(p^2)=\frac{\sin\delta_l(p^2)}{p}\exp[i\delta_l(p^2)], \tag{15}$$

and therefore (14) becomes

$$\frac{\delta(k-p)}{4\pi p^2}\cos\delta_l(p^2)e^{i\delta_l(p^2)}-\frac{P}{2\pi^2}\frac{(k|t_l(p^2)|p)}{p^2-k^2}. \tag{16}$$

Since the half-on-shell amplitude has the phase $\delta$ and since the principal part factor is real, the form (16) shows explicitly that $1_l+G_0t_l$ has the scattering phase. Furthermore, the terms are now incoherent in the sense that there is no cross-term since $\delta(x)P(1/x)=0$.

We see from (16) that the effect of combining the terms is to make the coefficient of the on-shell part $\cos\delta$ rather than the $\sin\delta$ one would expect from (10) alone. This qualitative change in the nature of the contribution of the final 2-3 scattering to the triangle singularity is particularly important if the final rescattering is resonant, since in that case $\cos\delta=0$, and the amplitude of the singular term is zero. This may account for the difficulty in observing triangle singularities since examples with resonant final scatterings have usually been chosen in hopes of making the effect big.[11]

*Note added in proof.* The argument of the preceding paragraph is incorrect. The principal-part term is also singular in the case of a triangle singularity. As Schmid shows,[11] the leading singularity then is proportional to $\exp(2i\delta)$ rather than $\exp(i\delta)\cos\delta$ as in (16). Thus the cross section as a function of $p^2$ only will show *no effect* of the singularity for *any* $\delta$. In a Dalitz plot there may be effects of the triangle graph, but a correct account of these still requires a study of the triangle graph *and* the coherent nonrescattering graph.

## IV. DISCUSSION—OVERLAPPING RESONANCES

We have seen in Sec. II that coherent combination of the terms contributing to a three-body amplitude will give the *entire* amplitude as a function of the center-of-mass energy of a given pair in a given two-body partial wave, a factor with the phase appropriate to two-body elastic scattering of that pair in that partial wave. Since the three-body amplitude gets a phase factor from each pair, it is tempting to assume that the entire amplitude carries a product of three-phase factors $\exp(i\delta)$, one from each pair. This is generally not the case since we cannot independently specify the energy and angular momentum of each pair in its own center of mass. It is true, however, that for a given three-body momentum and angular momentum, the relative energies of the pairs in their own centers of mass form a set of independent variables. This follows from the work of Omnes,[12] who showed that in the three-body center-of-mass system, with the three-body angular momentum and its components fixed, the state of the system is specified by giving the energy of each of the particles. These three energies are linearly related to the three-pair energies, and therefore these form a perfectly good set of independent variables. It is not possible in general also to specify the angular momentum of each pair. However, there are many interesting cases in which the angular momentum is fixed for dynamical reasons. For example, in three-body reactions or decays at low energies each pair is predominantly in a relative $s$ state, or sometimes because of selection rules a $p$ state. In these cases in which each pair angular momentum is dominated by one partial wave, the conclusions of Sec. II would lead us to try to write the full amplitude with a product of factors carrying the two-body variation and phase of each pair. We could then assume that the remaining variation of the three-body amplitude depends only on the total energy.

[10] E. Merzbacher, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1961), Chap. 12.

[11] The fact that terms with and without final rescattering combine coherently in this way, and that in particular the on-shell part is zero for resonant rescattering seems to be an old but not generally known result. [Cf. G. Chew, M. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957); C. Goebel and H. Schnitzer, *ibid.* **123**, 1021 (1961).] So far as I know it has not been previously applied to triangle singularities. However, criticism of the usual triangle-singularity discussion has been given recently from a different point of view by C. Schmid [*ibid.* **154**, 1363 (1967)]. I would like to thank F. Low for bringing an unpublished report of this work to my attention.

[12] R. L. Omnes, Phys. Rev. **134**, B1358 (1964).

We are, of course, more interested in the dependence of the magnitude of the amplitude on the pair energy than the phase. The most likely candidate for carrying information on both the magnitude and the phase of the two-body scattering is the Fredholm denominator function or the $D$ function of the two-body $t$ matrix. It is $D^{-1}$ that has the scattering phase in the two-body case. Hence, in the reaction leading to three particles with total three-body angular momentum $J$ where each pair is mostly in one partial wave, we conjecture an amplitude of the form

$$\frac{N_J(\epsilon_{12},\epsilon_{13},\epsilon_{23};E)}{\Delta_J(E)D_{12}(\epsilon_{12})D_{13}(\epsilon_{13})D_{23}(\epsilon_{23})}, \tag{17}$$

where $D_{ij}(\epsilon_{ij})$ are the two-body $D$ functions, each a function of the relative pair energy $\epsilon_{ij}$. $\Delta_J(E)$ is the irreducible three-body part of the denominator. It contains the three-body bound state poles of angular momentum $J$ and any genuine three-body resonances. $E$ is the total three-body energy. It is not linearly independent of the $\epsilon$'s, but we write the amplitude this way to indicate which combination of variables plays a role where. $N_J$ is a real function and is slowly varying if the form (17) is to be useful. An amplitude with the product of $D$ functions like this was previously obtained by Blankenbecler on different grounds.[13]

Forms such as (17) also arise naturally in the three-body sector of static models, where the restriction of pair angular momentum is a natural one.[3] In particular these forms have been used to discuss overlapping resonances in these static models. In these it is usually assumed that mesons, free to propagate, interact with a static source in some partial wave, but that there is no meson-meson interaction. Then the form (17) emerges exactly, but with only two final $D$ functions. If each of these two-body $D$ functions has a resonance at some particular energy, then at the three-body energy appropriate to both on resonance, the three-body amplitude given by (17) should be enhanced and perhaps resonant. This is the Peierls mechanism in this context.[14] This mechanism has been investigated in a solvable version of the three-body sector of the Lee model. The model is complicated so as to introduce a resonance in the $N$-$\theta$ sector[6,15] and the three-body ($V$-$\theta$) sector solved numerically.[6] In fact, no such complication is necessary to discuss the problem since as Fonda, Ghirardi, and Rimini[16] have shown the ordinary Lee model admits a stable V particle and an $N$-$\theta$ resonance by a choice of the source function. In the numerical calculation no enhancement emerges in the three-body system at the double resonance energy. This can also be seen to be the case analytically in the simpler Lee model[17] with resonances produced by the sources. This absence of Peierls mechanism occurs even though the form (17) is exact in these cases. How this comes about can be seen by considering the unitarity relation for elastic scattering in the three-body sector ($V$-$\theta$ scattering in the Lee model case). This relation is

$$\mathrm{Im}T_{22}=|T_{22}|^2\rho_2+\int|T_{23}|^2\rho_3\delta(E), \tag{18}$$

where $T_{22}$ is the elastic-scattering amplitude and $T_{23}$ is the breakup amplitude, which is asserted to have the form (17). $\rho_2$ and $\rho_3$ are phase-space factors and $\delta(E)$ is an energy-conserving delta function. Inserting (17) for $T_{23}$ in (18), we get

$$\mathrm{Im}T_{22}(E)=|T_{22}(E)|^2\rho_2(E)$$
$$+\frac{1}{|\Delta(E)|^2}\int_0^E\left|\frac{N(\epsilon,E-\epsilon;E)}{D(\epsilon)D(E-\epsilon)}\right|^2\rho(\epsilon,E-\epsilon)d\epsilon, \tag{19}$$

where $\epsilon$ is the energy of one of the mesons produced and we have assumed they have the same $D$ function for simplicity. If $N$ is slowly varying and $D(E)$ has a resonance at $\epsilon_r$, it is easy to see that the integral on the right of (19) will resonate at $E=2\epsilon_r$. Hence if $1/|\Delta(E)|^2$ is smooth, $\mathrm{Im}T_{22}$ will resonate at $E=2\epsilon_r$ and so will $T_{22}$. But the usual arguments would lead us to hope that a resonance in $T_{22}$ should show up in $1/\Delta(E)$, since that is the three-body part of $T_{23}$. We would expect the three-body $D$ functions of $T_{22}$ and of $T_{33}$ to have the same resonances and bound states. But if $1/\Delta$ resonates at $E=2\epsilon_r$ and the integral does as well, the contribution of $T_{23}$ to (18) will be a double resonance, contrary to assumption. The way out is that $\Delta$ has a term in it proportional to the integral so that at $E=2\epsilon_r$, $\Delta$ grows by just the amount needed to cancel the enhancement from the integral and the entire term does not resonate.[17] This is precisely how the Peierls enhancement is cancelled in the Lee model. Since this is not a resonance in the three-body part of the production-amplitude $D$ function, we need not expect it to show up in the elastic scattering. It seems likely that all this happens in general.

If such an interplay of two- and three-body parts occurs in the general case, the situation with overlapping resonances is quite complicated. The discussion of Sec. II shows that we cannot write the amplitude as a simple sum of parts each with one of the resonances, but this discussion indicates that the straightforward interpretation of the product form is also too simple.

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[13] R. Blankenbecler, Phys. Rev. **122**, 983 (1961).
[14] R. F. Peierls, Phys. Rev. Letters **6**, 641 (1961).
[15] T. M. Luke, Phys. Rev. **141**, 1373 (1966).
[16] L. Fonda, G. C. Ghirardi, and A. Rimini, Phys. Rev. **133**, B196 (1964).
[17] R. D. Amado, Phys. Rev. **122**, 696 (1961).

ing my ideas on this work. I would particularly like to thank J. Noble for discussions leading to the proof given in Sec. II that the half-off-shell two-body $t$ matrix has the elastic-scattering phase.

## APPENDIX

In this Appendix we study the Faddeev equation for three-body problems in which two particles interact independently with a fixed scattering center and not with one another. This problem is trivially solvable in the Schrödinger-equation form since the Hamiltonian is a sum of two commuting parts and therefore the wave function is a product. Since, however, the Faddeev approach insists on starting by writing the solution as a sum, the problem is not trivial in that formalism.

Since the idea of a scattering amplitude is not very natural for this problem, let us study the wave function. The scattering wave function for particle 1 on the fixed center may be written

$$\phi_1 = 1_1 + g_1(\epsilon_1)t_1(\epsilon_1)\,, \tag{A1}$$

where $1_1$ is the plane wave and $g_1$ is the free-particle Green's function for particle 1, $1/(\epsilon_1 - h_1)$ where $h_1$ is the free-particle Hamiltonian for 1. $t_1$ is the scattering amplitude and $\epsilon_1$ is the energy eigenvalue. Correspondingly for particle 2

$$\phi_2 = 1_2 + g_2(\epsilon_2)t_2(\epsilon_2)\,. \tag{A2}$$

Then the full solution of

$$(h_1 + h_2 + v_1 + v_2)\Psi = E\Psi \tag{A3}$$

may be written

$$\Psi = \phi_1\phi_1\,, \quad E = \epsilon_1 + \epsilon_2\,, \tag{A4}$$

or

$$\Psi = 1_1 1_2 + 1_1 g_2(\epsilon_2)t_2(\epsilon_2) + 1_2 g_1(\epsilon_1)t_1(\epsilon_1) + g_1(\epsilon_1)t_1(\epsilon_1)g_2(\epsilon_2)t_2(\epsilon_2)\,. \tag{A5}$$

In the Faddeev formalism we write

$$\Psi = \Psi_1 + \Psi_2 - 1_1 1_2\,, \tag{A6}$$

and the Faddeev equation becomes

$$\begin{aligned}\Psi_1 &= 1_1 1_2 + G_0(E)t_1(E-h_2)\Psi_2\,,\\ \Psi_2 &= 1_1 1_2 + G_0(E)t_2(E-h_1)\Psi_1\,,\end{aligned} \tag{A7}$$

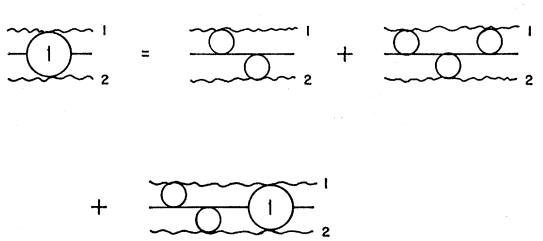

FIG. 3. Graphical representation of Eq. (A8) for a Faddeev amplitude in the case of the independent scattering of two particles (represented by wavy lines) from a fixed center (solid line).

where $G_0 = 1/(E - h_1 - h_2)$ is the full free Green's function and the $t$ matrices now appear off-shell. $\Psi_2$ can be eliminated to give an equation for $\Psi_1$. If we define $T_1$ by $T_1 = t_1(E-h_2)G_0 t_2(E-h_1)\Psi_1$, then $T_1$ satisfies

$$\begin{aligned}T_1 = t_1(E-h_2)G_0t_2(E-h_1) &+ t_1(E-h_2)G_0t_2(E-h_1)G_0t_1(E-h_2)\\ &+ t_1(E-h_2)G_0t_2(E-h_1)G_0T_1\,.\end{aligned} \tag{A8}$$

If we call

$$T = T_1 + T_2\,, \tag{A9}$$

where $T_2$ is defined like $T_1$, then we see that $T$ is the connected part of the scattering amplitude for this problem, and (A8) is the Faddeev equation for that amplitude. We did not write it down first since it is not the natural thing to study in this problem. Equation (A8) is represented diagrammatically in Fig. 3. We note that each term in $T_1$ begins with $t_1(E-h_2)$ so we might expect $T_1$ to begin that way. On the other hand, looking at (A9) we see that

$$T = T_1 + T_2 = G_0^{-1}g_1(\epsilon_1)g_2(\epsilon_2)t_1(\epsilon_1)t_2(\epsilon_2)\,. \tag{A10}$$

The problem is how to split this into $T_1$ and $T_2$. This can be done by noting that $\Psi$ satisfies[18]

$$\Psi = 1_1 1_2 + G_0(v_1 + v_2)\Psi\,, \tag{A11}$$

so that

$$\Psi_1 = 1_1 1_2 + G_0 v_1 \Psi\,, \tag{A12}$$

and since we have $\Psi$, $\Psi_1$ can be constructed. Alternatively, one can guess at $T_1$ and $T_2$ from (A10). The correct ansatz is

$$T_1 = t_1(\epsilon_1)G_0t_2(\epsilon_2)\,. \tag{A13}$$

In (A13) if $E = \epsilon_1 + \epsilon_2$, the final $t$ will always be on the energy shell. Then the difference between the answer (A13) and the first term of (A8) is only that the first $t_1$ is put on the final energy shell. There does not seem to be any way to represent this term graphically. Presumably it represents the fact that both particles are interacting *at the same time* with the scattering center, whereas our graphical language is sequential. That (A13) is the solution is not obvious from the form of the equation. To show that it does satisfy (A8) we study the last term of (A8) with (A13) inserted for $T_1$.

$$t_1(E-h_2)(E-h_1-h_2)^{-1}t_2(E-h_1)(E-h_1-h_2)^{-1} \times t_1(\epsilon_1)(\epsilon_2-h_2)^{-1}t_2(\epsilon_2)\,. \tag{A14}$$

We have written out the $G_0$ terms and used the fact that $E = \epsilon_1 + \epsilon_2$ so that particle 1 is on the energy shell in the next to last intermediate state. Now since we have

$$[h_2, t_1] = [h_1, h_2] = [t_1(\epsilon_1), t_2(\epsilon_2)] = 0\,, \tag{A15}$$

[18] This method of constructing $\psi_1$ and $\psi_2$ is due to I. Alexandrov. I am grateful to him for explaining it to me.

this becomes

$$-t_1(E-h_2)\frac{1}{E-h_1-h_2}t_2(E-h_1)\times\left(\frac{1}{E-h_1-h_2}-\frac{1}{\epsilon_2-h_2}\right)t_2(\epsilon_2)\frac{1}{\epsilon_1-h_1}t_1(\epsilon_1). \quad \text{(A16)}$$

The following identity is easily verified for the Lippmann-Schwinger equation[19]:

$$t(a)[(a-h)^{-1}-(b-h)^{-1}]t(b)=t(a)-t(b). \quad \text{(A17)}$$

[19] The proof is
$$t(a)-t(b)=t(a)(a-h)^{-1}v-v(b-h)^{-1}t(b)=t(a)(a-h)^{-1}t(b)-t(a)(a-h)^{-1}v(b-h)^{-1}t(b)-t(a)(b-h)^{-1}t(b)+t(a)(a-h)^{-1}v(b-h)^{-1}t(b)=t(a)[(a-h)^{-1}-(b-h)^{-1}]t(b).$$

Therefore (A14) becomes

$$t_1(E-h_2)(E-h_1-h_2)^{-1}t_2(\epsilon_2)(E-h_1-h_2)^{-1}t_1(\epsilon_1)-t_1(E-h_2)(E-h_1-h_2)^{-1}t_2(E-h_1)\times(E-h_1-h_2)^{-1}t_1(\epsilon_1). \quad \text{(A18)}$$

The second term cancels the second term of (A8). The first term of (A18) and the first of (A8) combine again using the identity (A17) to give (A13). Hence the product form is the solution even though the equation naively does not seem to want it to be. This shows that in problems for which the product form is a good first approximation, simple interpretation of the Faddeev equations is particularly dangerous.

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# Effect of the Quasipole on the $\pi$-$N$ Forward Scattering Amplitudes

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It is shown that a quasipole plays an important role in low-energy meson-baryon scattering. The influence of this quasipole, which exists on the second Riemann sheet just below the elastic threshold and has a residue of the same order of magnitude as that of the nucleon pole, is analyzed by extrapolation of the data of the pion-nucleon scattering amplitude. Since the residue of the quasipole is related to the pion-nucleon coupling constant, this extrapolation gives an independent possibility for determining the coupling constant. The result is $g^2/4\pi=13.9$.

## INTRODUCTION

JUST below the elastic threshold, the pion-nucleon scattering amplitude has a quasipole with the residue of $+0.77(g^2/4\pi)$ for $T=\frac{3}{2}$ on the second sheet.[1] Neglect of this fact has so far hindered a simple description of low-energy pion-nucleon scattering in terms of the nucleon poles and the 3-3 resonance pole. The purpose of this paper is to examine the influence of the quasipole on this scattering amplitude by analyzing the low-energy data of pion-nucleon scattering. We shall first repeat the definition of the quasipole and then compute its residue. It is to be noticed that this computation can be performed without making any approximation, solely by means of the unitarity condition and the spectral representation of the scattering amplitudes.

## QUASIPOLE

Let us consider pion-nucleon scattering, with masses $\mu$ and $m$, respectively, first neglecting the spin of the nucleon; the results for actual physical case will be given later on. We will assume the following two conditions:

[1] T. Sawada, Phys. Rev. Letters **15**, 567 (1965).

(i) Spectral representation.

For $l\geqq L$

$$a_l(s)=-\frac{g^2}{2q^2}Q_l\left(1+\frac{m^2+2\mu^2-s}{2q^2}\right)+\frac{g^2}{m^2-s}\delta_{l,0}+\frac{1}{2q^2}\frac{1}{\pi}\times\int_{4\mu^2}^{\infty}dt'A_t(t',s)Q_l\left(1+\frac{t'}{2q^2}\right)-\frac{1}{2q^2}\frac{1}{\pi}\times\int_{(m+\mu)^2}^{\infty}du'A_u(u',s)Q_l\left(1+\frac{2m^2+2\mu^2-u'-s}{2q^2}\right), \quad (1)$$

and for $0\leqq l<L$

$$a_l(s)=-\frac{g^2}{2q^2}Q_l\left(1+\frac{m^2+2\mu^2-s}{2q^2}\right)+f_l(s), \quad (2)$$

where $f_l(s)$ is an analytic function of $s$ regular at $s=m^2+2\mu^2$, and $L$ is a finite positive integer.

(ii) Unitarity condition.

$$a_l(s)-a_l^\dagger(s)=\frac{iq}{\sqrt{s}}a_l(s)a_l^\dagger(s) \quad (3)$$