Mass Formula and "Broken $SU(3)$" in an Associative Algebra

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An associative algebra, which contains the enveloping algebra of the Poincaré group and the enveloping algebra of $SL(3,c)$, is considered. Its reduction with respect to irreducible representations of $SU(3)$ is studied and a mass formula is derived, which contains the mass formula for mesons of "broken" $SU(3)$ as a special case. The results are compared with experimental data.

I. INTRODUCTION

ATTEMPTS to understand the breaking of the $SU(3)$ symmetry as exhibited by the mass formula have mainly been undertaken in the framework of a large group G, containing the Poincaré group $\mathcal{P}$ and the intrinsic "broken symmetry" group. The program of the dynamical group approach[1] was to find a large group G such that an irreducible component of its enveloping algebra $\mathcal{E}(G)$ describes various particles and resonances as different states of one physical system represented by an irreducible representation space of $\mathcal{E}(G)$, and that the mass spectrum is a consequence of the property of $\mathcal{E}(G)$. The existence of such a group has been disproved by the O'Raifeartaigh theorem[2] and Werle[3] was the first to suggest combining the algebras of $SU(3)$ and the Poincaré group $\mathcal{P}$ to an algebra A, which was not a Lie algebra. The structure that "remained of A in the rest frame" was however again a Lie algebra, as a consequence of which he obtained a mass formula, which was linear in the mass and the intrinsic quantum numbers, whereas experimental data favor quadratic mass formulas.[1a]

Generalizing the original idea of a dynamical group, we have studied in a previous work a simple associative algebra of continuous operators in a rigged Hilbert space, which contains $\mathcal{E}(\mathcal{P})$ and $\mathcal{E}(SL(2,c))$, and gives rise to a quadratic mass formula. This model was simple enough to learn about the mathematical problems connected with such an approach. In the present paper we shall investigate a more complicated and more realistic model in the same mathematical frame as in I.[4]

II. DEFINING RELATIONS OF THE ALGEBRA

The algebra $\mathcal{B}$ of our model is a combination of the enveloping algebra of the Poincaré group $\mathcal{E}(\mathcal{P})$ with the enveloping algebra of the intrinsic spectrum generating (or noninvariance) group $SL(3,c)$. Thus $\mathcal{B}$ is generated by

$$\begin{array}{llll} P_\mu, & L_{\mu\nu}, & M=(P_\mu P^\mu)^{1/2}, & \mu,\nu=0,1,2,3, \\ H_i, & E_{\pm\alpha}, & i=1,2, & \\ G_i, & F_{\pm\alpha}, & \alpha=1,2,3, & \end{array} \tag{1}$$

where the multiplication is defined by the relations[5]:

$$[P_\mu,P_\nu]=0,\quad [L_{\mu\nu},P_\rho]=i(g_{\nu\rho}P_\mu-g_{\mu\rho}P_\nu)\,, \tag{2}$$

$$[L_{\mu\nu},L_{\rho\sigma}]=-i(g_{\mu\rho}L_{\nu\sigma}+g_{\nu\sigma}L_{\mu\rho}-g_{\mu\sigma}L_{\nu\rho}-g_{\nu\rho}L_{\mu\sigma})\,,$$

$$[H_i,H_j]=0\,, \tag{3a}$$

$$[E_\alpha,E_\beta]=N_{\alpha\beta}E_\gamma\,, \tag{3b}$$

$$[H_j,E_\alpha]=r_j(\alpha)E_\alpha\,, \tag{3c}$$

$$[E_\alpha,E_{-\alpha}]=r^i(\alpha)H_i\,, \tag{3d}$$

$$[G_i,G_j]=0\,, \tag{4a}$$

$$[G_j,F_\alpha]=-r_j(\alpha)E_\alpha\,, \tag{4b}$$

$$[F_\alpha,F_\beta]=-N_{\alpha\beta}E_\gamma\,, \tag{4c}$$

$$[F_\alpha,F_{-\alpha}]=-r^i(\alpha)H_i\,, \tag{4d}$$

$$[H_i,G_j]=0\,, \tag{4e}$$

$$[H_j,F_\alpha]=r_j(\alpha)F_\alpha\,, \tag{4f}$$

$$[E_\alpha,F_\beta]=N_{\alpha\beta}F_\gamma\,, \tag{4g}$$

$$[E_\alpha,F_{-\alpha}]=r^i(\alpha)G_i\,, \tag{4h}$$

$$[G_j,E_\alpha]=r_j(\alpha)F_\alpha\,. \tag{4i}$$

$$[L_{\mu\nu},H_i]=0\,, \tag{5a}$$

$$[L_{\mu\nu},E_\alpha]=0\,, \tag{5b}$$

$$[L_{\mu\nu},G_i]=0\,, \tag{5c}$$

$$[L_{\mu\nu},F_\alpha]=0\,, \tag{5d}$$

$$[P_\mu,H_i]=0\,, \tag{5e}$$

$$[P_\mu,E_{\pm1}]=0\,, \tag{5f}$$

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[1] (a) A. O. Barut, in *Proceedings of the Seminar on High Energy Physics, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965), and references therein; (b) A. Böhm, Nuovo Cimento **43**, 665 (1966); (c) N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan, Phys. Letters **19**, 322 (1965); (d) Y. Dothan, M. Gell-Mann, and Y. Ne'eman, *ibid.* **17**, 143 (1965); Y. Ne'eman, in *Proceedings of the 1965 Tokyo International Seminar in Theoretical Physics* (W. A. Benjamin, Inc., New York, 1966).

[2] L. O'Raifeartaigh, Phys. Rev. Letters **14**, 332 (1965).

[3] J. Werle, Nuovo Cimento (to be published).

[4] A. Böhm, J. Math. Phys. (to be published); hereafter referred to as I.

[5] Notation $\{A,B\}=AB+BA$; $[A,B]=AB-BA$.

$$[MP_\mu,E_\alpha M]=g(r_2(\alpha)\{H_2,E_\alpha\}-3r_1(\alpha)\{H_1,E_\alpha\} -3N_{-1\alpha}\{E_{+1},E_{\alpha-1}\} -3N_{1\alpha}\{E_{-1},E_{\alpha+1}\})P_\mu, \quad (5g)$$

$$[MP_\mu,G_iM]=3g\sum_{\alpha=\pm2,\pm3} r_1(\alpha)\{E_\alpha,F_{-\alpha}\}P_\mu, \quad (5h)$$

$$[MP_\mu,F_\alpha M]=4gr_2(\alpha)\{H_2,F_\alpha\}P_\mu +3g\sum_{\beta=\pm2,\pm3} N_{-\beta\alpha}\{E_\beta,E_{\alpha-\beta}\}P_\mu; \quad (5i)$$

$$(P_\mu XM-MX_iP_\mu)=0 \text{ for every } X_i\in\{F_\alpha,G_i,E_\alpha\}. \quad (6)$$

Here g is a constant of the dimension $(\mathrm{MeV})^2$ (in the units we use, $\hbar=c=1$), the value of which we determine later from experimental data.

The root vectors are (in the normalization we use[6])

$$r_i(1)=(1/\sqrt{3})(1,0),\quad r_i(2)=(1/2\sqrt{3})(1,\sqrt{3}), \quad r_i(3)=(1/2\sqrt{3})(-1,\sqrt{3}), \quad (7a)$$

$$r_i(-\alpha)=-r_i(\alpha), \quad (7b)$$

and

$$N_{\alpha\beta}=\pm\sqrt{\tfrac{1}{6}} \text{ if } r(\alpha)+r(\beta)=r(\gamma) \text{ is also a nonvanishing root vector} \quad (7c)$$
$$=0 \text{ otherwise};$$

in particular,

$$N_{1,3}=-N_{3,1}=N_{-3,-1}=-N_{-1,-3}=N_{3,-2} =-N_{-2,3}=N_{-2,1}=-N_{1,-2}=N_{2,-3} =-N_{-3,2}=N_{-1,2}=-N_{2,-1}=\sqrt{\tfrac{1}{6}}. \quad (7d)$$

In the normalization we have used here,[6] the hypercharge is

$$Y=2H_2 \quad (8)$$

and the isospin is

$$I_3=\sqrt{3}H_1,\quad (I_1\pm iI_2)=(\sqrt{\tfrac{1}{6}})E_{\pm1}, \quad (9a)$$

so that the isospin operator [Casimir operator of $SU^I(2)$] is

$$\mathbf{I}^2=3(H_1^2+E_{+1}E_{-1}+E_{-1}E_{+1}). \quad (9b)$$

The second-order Casimir operator of $SU(3)$ is

$$\mathcal{C}=3(H_1^2+H_2^2+\textstyle\sum_\alpha E_\alpha E_{-\alpha}). \quad (10)$$

On the irreducible representation $D(\lambda_1,\lambda_2)$ of $SU(3)$, $\mathcal{C}$ has the value

$$\mathcal{C}(\lambda_1,\lambda_2)=\tfrac{1}{3}(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)+(\lambda_1+\lambda_2); \quad (11a)$$

in particular for those representations where $\lambda_1=\lambda_2=\lambda$ (which we shall be mainly interested in)

$$\mathcal{C}(\lambda)=\lambda^2+2\lambda. \quad (11b)$$

As in I, we restrict ourselves here to those irreducible components of $\mathcal{B}$ for which $(f,M^2f)\geq m_0^2\|f\|^2$, $m_0>0$.

[6] R. E. Behrends, J. Dreitlein, C. Fronsdal, and W. Lee, Rev. Mod. Phys. 34, 1 (1962).

III. INVARIANT OPERATORS

We see that relations (5) and (6) are built in analogy to relations (7) and (8) of I. A consequence of relations (6) and (5a), (5b), (5c) is that

$$P_\mu XM-MXP_\mu=0 \quad (12)$$

for every $X\in\mathcal{E}(SL(3,c))$. In the same way as in I, one finds Werle's equation

$$[P_\mu/M,X]=0 \quad (13)$$

for every $X\in\mathcal{E}(SL(3,c))$ and therefore for the spin operator

$$\Gamma=-(P_\rho P^\rho)^{-1}\Gamma_\mu\Gamma^\mu,\quad \Gamma_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu L^{\rho\sigma}, \quad [\Gamma,X]=0 \quad (14)$$

for every $X\in\mathcal{E}(SL(3,c))$.

From (14) it follows that the spin operator Γ is an invariant operator of the whole algebra $\mathcal{B}$. So an irreducible component of $\mathcal{B}$—and therewith the physical system described by it—is characterized by one spin s.

From comparison with I we presume that the 4 generators of the center of $\mathcal{E}(SL(3,c))$, $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$, are also invariants of $\mathcal{B}$. That $\mathbf{C}_i$ commutes with $L_{\mu\nu}$ follows from the relations (5a), (5b), so that it remains to show that $[\mathbf{C}_i,P_\mu]=0$. The easiest way of seeing this is with the help of the other central element of $\mathcal{B}$:

$$Z=M^2-g(4H_2^2+3\sum_{\alpha=2,3}\{E_\alpha,E_{-\alpha}\}). \quad (15)$$

As Z is a central element it is in particular $[\mathbf{C}_i,Z]=0$ and therefore follows from (15):

$$[\mathbf{C}_i,M^2]=0, \quad (16a)$$

and therefore

$$[\mathbf{C}_i,M]=0. \quad (16b)$$

From Eq. (12) it follows immediately that

$$[P_\mu,X]M=[M,X]P_\mu \quad (17a)$$

for every $X\in\mathcal{E}(SL(3,c))$, so that we obtain from (16b):

$$[\mathbf{C}_i,P_\mu]=[\mathbf{C}_i,M]P_\mu/M=0. \quad (18)$$

So it remains to show that Z is an invariant operator of $\mathcal{B}$.

$$[Z,L_{\mu\nu}]=0 \quad (19a)$$

follows from relation (5a), (5b);

$$[Z,H_i]=0 \quad (19b)$$

follows from relation (5c), from (3a), and from

$$\sum_{\alpha=\pm2\pm3}[H_i,E_\alpha E_{-\alpha}]=-[H_i,\{E_{+1},E_{-1}\}] =\{E_{+1},[E_{-1},H_i]\}+\{E_{-1},[E_{+1},H_i]\}=0,$$

where the first equality is a consequence of (10) and the last a consequence of (3b) and (7b).

$$[Z,E_{\pm 1}]=0 \tag{19c}$$

follows from (5f) and $[H_2,E_{\pm 1}]=0$ and

$$3\sum_{\pm 2,\pm 3}[E_\alpha E_{-\alpha},E_{\pm 1}]=-3[(H_1^2+H_2^2+\{E_{+1},E_{-1}\}),E_{\pm 1}]$$
$$=-[\mathbf{I}^2,E_{\pm 1}]=0,$$

which is a consequence of (9) and (10). To show that

$$[Z,E_\alpha]=0,\quad \alpha=\pm 2,\pm 3 \tag{19d}$$

we first calculate $[M^2,E]$. From (17a) we obtain

$$[MP_\mu,XM]=[M^2,X]P_\mu \tag{17b}$$

for every $X\in\mathcal{E}(SL(3,c))$, so that we obtain with (5g):

$$[M^2,E_\alpha]=g(r_2(\alpha)\{H_2,E_\alpha\}-3r_1(\alpha)\{H_1,E_\alpha\}$$
$$-3N_{-1\alpha}\{E_{+1},E_{\alpha-1}\}-3N_{1\alpha}\{E_{-1},E_{\alpha+1}\}). \tag{20}$$

With (10), (3b), and (3d) we have for

$$3\sum_{\beta=\pm 2,\pm 3}[E_\beta E_{-\beta},E_\alpha]$$
$$=-3\{H_1,E_\alpha\}r_1(\alpha)-3r_2(\alpha)\{H_2,E_\alpha\}-3[\{E_{+1},E_{-1}\},E_\alpha]$$
$$=3(-\{H_1,E_\alpha\}r_1(\alpha)-\{H_2,E_\alpha\}r_2(\alpha)$$
$$-\{E_{+1},E_{\alpha-1}\}N_{-1\alpha}-\{E_{-1},E_{\alpha+1}\}N_{1\alpha}),$$

so that we obtain

$$[(4H_2^2+3\sum_{\alpha=2,3}\{E_\alpha,E_{-\alpha}\}),E_\alpha]$$
$$=r_2(\alpha)\{H_2,E_\alpha\}-3r_1(\alpha)\{H_1,E_\alpha\}$$
$$-3N_{-1\alpha}\{E_{+1},E_{\alpha-1}\}-3N_{1\alpha}\{E_{-1},E_{\alpha+1}\}. \tag{21}$$

Comparing (20) and (21), we obtain (19d). The relation

$$[Z,G_i]=0 \tag{19e}$$

is proven in the same way: From (5h) we obtain with (17a)

$$[M^2,G_i]=3g\sum_{\alpha=\pm 2,\pm 3}r_i(\alpha)\{E_\alpha,F_{-\alpha}\}. \tag{22}$$

From (4e) and (4i) it follows[7] that

$$[(4H_2+3\sum_{\alpha=2,3}\{E_\alpha,E_{-\alpha}\}),G_i]$$
$$=3\sum_{\alpha=2,3}[\{E_\alpha,E_{-\alpha}\},G_i] \tag{23}$$
$$=3\sum_{\alpha=2,3}-\{E_\alpha,F_{-\alpha}\}r_i(-\alpha)-\{E_{-\alpha},F_\alpha\}r_i(\alpha).$$

Comparing (22) and (23) we obtain (19e). In the same way it is seen that

$$[Z,F_\alpha]=0. \tag{19f}$$

To prove the last equation:

$$[Z,P_\mu]=0, \tag{19g}$$

it is sufficient, because of (5e), to show that

$$\sum_{\alpha=2,3}[\{E_\alpha,E_{-\alpha}\},P_\mu]=0. \tag{24}$$

Because of (17a) this follows from

$$\sum_{\alpha=2,3}[\{E_\alpha,E_{-\alpha}\},M^2]=0, \tag{25}$$

which we now show. With (20) we obtain[7]

$$\sum_{\alpha=2,3}[\{E_\alpha,E_{-\alpha}\},M^2]$$
$$=g\sum_{\alpha=\pm 2,\pm 3}(-r_2(-\alpha)\{E_\alpha,\{H_2,E_{-\alpha}\}\}$$
$$+3r_1(-\alpha)\{E_\alpha,\{H_1,E_{-\alpha}\}\})$$
$$=+g3\sum_{\alpha=\pm 2,\pm 3}(N_{-1,-\alpha}\{E_\alpha,\{E_{+1},E_{-\alpha-1}\}\}$$
$$+N_{1,-\alpha}\{E_\alpha,\{E_{-1},E_{-\alpha+1}\}\}).$$

The first sum on the right-hand side is zero, which follows with (3b) from the property (7b). So we obtain using (7d) and (7c):

$$\sum_{\alpha=2,3}[\{E_\alpha,E_{-\alpha}\},M^2]$$
$$=3g(\sqrt{\tfrac{1}{6}})(\{E_{-2},\{E_{+1},E_{+3}\}\}-\{E_2,\{E_{-1},E_{-3}\}\}$$
$$-\{E_3,\{E_{+1},E_{-2}\}\}+\{E_{-3},\{E_{-1},E_{+2}\}\})=0$$

because of (3c) and (7d). Thus (19g) has been proven.

Therewith it has been proved that Z is an invariant operator.[8]

IV. COMMUTING SYSTEM

The internal invariance group or the degeneracy group of M^2 is the direct product of the isospin and hypercharge group $SU(2)^I\times U(1)^Y$, which follows from relations (5a)–(5f). Therefore

$$P_i,\quad S_3,\quad \mathbf{I}^2,\quad I_3,\quad Y \tag{26}$$

together with the generators of the center $\mathbf{C}_i, \Gamma, Z$ belong to a system of commuting operators.

From (5a)–(5f), together with (24), it follows that also the Casimir operator $\mathfrak{C}$ of $SU(3)$ commutes with the elements of (26). In the same way one might be able to show that the third-order Casimir operator $\mathfrak{C}_3$ of $SU(3)$ also commutes with (26); we shall not show it here, because in those irreducible components of $\mathfrak{G}$ which we are interested in [which contain the degenerate series representation $(m=0,\rho)$ of $SL(3,c)$], $\mathfrak{C}_3$ will always be zero. So we have found in

$$\mathbf{C}_i,\ \ \Gamma,\ \ Z,\ \ P_i,\ \ S_3,\ \ \mathbf{I}^2,\ \ I_3,\ \ Y,\ \ \mathfrak{C},\ \ \mathfrak{C}_3=0 \tag{27}$$

a commuting system.

[7] $[\{A,B\},C]=\{A,[B,C]\}+\{B,[A,C]\}$.

[8] The relations (5) have been set up such that this is fulfilled.

V. LIMITING CASE OF ZERO COUPLING

In the limit of coupling constant $g \to 0$, $\mathcal{B}$ goes into the enveloping algebra of the direct product $\mathcal{P} \times SL(3,c)$. Because then we obtain from (5g), (5h), (5i):

$$[MP_\mu, X_i M] = 0 \tag{28a}$$

for $X_i \in \{E_{\pm 2, \pm 3}, F_\alpha, G_i\}$ and therefore with (17b)

$$[M^2, X_i] = 0 \tag{28b}$$

and with (17a)

$$[P_\mu, X_i] = 0. \tag{28c}$$

To construct the representation space of $\mathcal{B}$ we start with the representation space of $\mathcal{P}$ and $SL(3,c)$.

VI. RIGGED HILBERT SPACE OF $SL(3,c)$

The rigged Hilbert space of $SL(3,c)$ is constructed in complete analogy to the $SL(2,c)$ case described in Andersen *et al.*,[9] Sec. II C. To obtain the space Ψ of analytic vectors we use the prescription given in the second part of Sec. II F of Ref. 9. For this purpose we need to know the content of representations of the maximal compact subgroup $SU(3)$ in a global representation of $SL(3,c)$. This we obtain from the Gelfand and Neumark theory of the classical groups.[10]

The unitary irreducible representations of the principal and degenerate series of $SL(3,c)$ are characterized by the 4 numbers $(m_2, m_3, \rho_2, \rho_3)$ and the 2 numbers (m_3, ρ_3), respectively, where m_k are integers and ρ_k are arbitrary pure imaginary numbers. Thus the Casimir operators $\mathbf{C}_i$ are on an irreducible representation functions of these numbers:

$$\mathbf{C}_i = f_i(m_2, m_3, \rho_2, \rho_3) \tag{29a}$$

and

$$\mathbf{C}_i = f_i(m_3, \rho_3), \tag{29b}$$

respectively; in the last case only two of the four Casimir operators are independent.

We consider first the principal series representation characterized by $(m_2, m_3, \rho_2, \rho_3)$. From theorem 5 and theorem 6 of Sec. 10, part I of Ref. 10 follows the theorem:

Let $g \to T_g$ be an irreducible representation of $SL(3,c)$ characterized by $(m_2, m_3, \rho_2, \rho_3)$; let T_u be the representation of its simple unitary subgroup $SU(3)$ when $SL(3,c)$ is restricted to $SU(3)$ in the representation $g \to T_g$. T_u contains an irreducible representation $D(\lambda_1, \lambda_2)$ of $SU(3)$ if the representation space $\mathcal{H}(\lambda_1, \lambda_2)$ contains a weight vector of the weight[11] $((1/2\sqrt{3})m_2, \frac{1}{6}(2m_3 - m_2))$.

The irreducible representation $D(\lambda_1, \lambda_2)$ of $SU(3)$ is contained in the representation T_u characterized by $(m_2, m_3, \rho_2, \rho_3)$ as many times, as the space $\mathcal{H}(\lambda_1, \lambda_2)$ contains linear independent weight vectors of the weight $((1/2\sqrt{3})m_2, \frac{1}{6}(2m_3 - m_2))$.

From this theorem we see that in any irreducible representation of the principal series the irreducible representation $D(\lambda_1, \lambda_2)$ of $SU(3)$ occur in general more than once, so that the eigenvalues of (27) will not be sufficient to characterize the states of an irreducible representation space, and we need additional quantum numbers, labeling the different irreducible representations of type $D(\lambda_1, \lambda_2)$ of $SU(3)$. Though the introduction of new quantum numbers might be an interesting aspect, we want to restrict ourselves here to the case where the eigenvalues of (27) are sufficient to characterize the states, i.e., where (27) is a maximal commuting system. Therefore we are interested in irreducible representations of $SL(3,c)$ which contain an irreducible representation of $SU(3)$ at most once. These representations we find in the degenerate series.

In accordance with Sec. 15 of Ref. 10 we call a vector ψ of the irreducible representation space $\mathcal{H}(\lambda_1, \lambda_2)$ of $SU(3)$ a weight vector with respect to the subgroup $SU(2)^I$ if the subspace spanned by ψ is invariant with respect to all transformations of $SU(2)^I$, i.e., if ψ is an eigenvector with eigenvalue zero of $\mathbf{I}^2$.

For the degenerate series representation (m_3, ρ_3) of $SL(3,c)$ the above-mentioned theorem is valid, if one replaces "weight vector" by "weight vector with respect to $SU(2)^I$" and the weight "$((1/2\sqrt{3})m_2, \frac{1}{6}(2m_3 - m_2))$" by "$(0, \frac{1}{3}m_3)$."

Thus the problem of finding which representation $D(\lambda_1, \lambda_2)$ of $SU(3)$ is contained in an irreducible representation of the degenerate series of $SL(3,c)$ (m_3, ρ_3), is reduced to the problem of finding in which representation $D(\lambda_1, \lambda_2)$ an isospin singlet with the hypercharge $Y = \frac{2}{3}m_3$ is contained and how often it is contained

This is easiest found with the help of the Gelfand pattern[12] of a basis vector of $\mathcal{H}(\lambda_1 \lambda_2)$. From the Gelfand pattern one sees by inspection that in the representation $D(\lambda_1, \lambda_2)$ only one isospin singlet $I = 0$ is contained, which has the hypercharge $Y = \frac{2}{3}(\lambda_2 - \lambda_1)$. Thus in the representation (m_3, ρ_3) of $SL(3,c)$ those representations $D(\lambda_1, \lambda_2)$ of $SU(3)$ are contained for which $m_3 = (\lambda_2 - \lambda_1)$ and they are contained exactly once. In particular in the representation $(m_3 = 0, \rho_3 = \rho)$ of $SL(3,c)$ the representations $D(\lambda) = D(\lambda_1 = \lambda, \lambda_2 = \lambda)$ of $SU(3)$ occur exactly once; these are the 1-, 8-, 27-, and 64-dimensional representations. Only in this representation of the degenerate series the octet occurs. We want to restrict ourselves therefore to this representation $(0, \rho)$.

We can now construct the rigged Hilbert space of the representation $(m_3 = 0, \rho)$ of $SL(3,c)$ in complete analogy to the $SL(2,c)$ case described in Sec. II C of Ref. 9.

[9] C. M. Andersen, A. Böhm, and A. M. Bouncristiani, Boulder Lectures, 1966 (unpublished).

[10] I. M. Gelfand and M. A. Neumark, Unitäre Darstellungen der Klassischen Gruppen, Berlin, 1957 (unpublished).

[11] Weight is the eigenvalue of the two-component "vector" (H_1, H_2); weight vector is its eigenvector.

[12] L. C. Biedenharn, Lecture Notes CERN 65-41 (unpublished); G. E. Baird and L. C. Biedenharn, J. Math. Phys. **4**, 825 (1963); J. D. Louck, *ibid.* **6**, 1786 (1965); I. M. Gelfand and M. L. Zetlin, Dokl. Akad. Nauk. SSSR **71**, 825 (1950).

The space of analytic vectors Ψ is

$$\Psi=\sum_{\lambda=0}^{\infty}\mathcal{H}(\lambda)\,,\quad \text{(algebraic sum)} \tag{30}$$

where $\mathcal{H}(\lambda)$ is the representation space of $D(\lambda)$ in which the basis

$$|I,I_3,Y;\lambda) \tag{31}$$

of eigenvectors of

$$\mathbf{I}^2,\quad I_3,\quad Y,\quad \mathcal{C} \tag{32}$$

has been chosen. [In all representation spaces $\mathcal{H}(\lambda)$ the third-order Casimir operator $\mathcal{C}_3$ is zero.[13]]

In Ψ we introduce, besides the Hilbert-space topology given by the scalar product $(\varphi|\psi)$, $\varphi,\psi\in\Psi$, with respect to which the elements of $\mathcal{L}(SL(3,c))$ are symmetric, the nuclear topology by the countable number of scalar products

$$(\varphi|\psi)_P=(\varphi|(\mathcal{C}+1)^P|\psi). \tag{33}$$

The completion of Ψ with respect to this topology gives ϕ_2. This topology is obviously equivalent to the topology given by the scalar products (AA1) in the Appendix of I [because $\Delta_{SL(3,c)}=2\mathcal{C}+\mathbf{C}_2(0,\rho)$]. Thus by the results of Appendix AA of I, we have in $\phi_2\subset\mathcal{H}_2\subset\phi_2^\times$ the rigged Hilbert space in which $\mathcal{E}(SL(3,c))$ is an algebra of continuous operators.

VII. REPRESENTATION SPACE OF THE ALGEBRA AND MASS FORMULA

To obtain the rigged Hilbert space for $\mathcal{B}$ we proceed in the same way as in I. Let $\tilde{\phi}_1\subset\mathcal{H}_1\subset\tilde{\phi}_1^\times$ again denote the rigged Hilbert space for (a reducible component of) $\mathcal{E}(\mathcal{P})$. Then we form the direct product of linear spaces $\tilde{\phi}_1\otimes\phi_2$ and equip it with the countable Hilbert-space topology given by the scalar products

$$(\varphi|\psi)_P=(\varphi,\theta^P\psi);\quad \varphi,\psi\in\tilde{\phi}_1\otimes\phi_2, \tag{34a}$$

where

$$\theta=\mathcal{C}+\Delta_{\mathcal{P}}=3(H_1^2+H_2^2+\sum_{\alpha=\pm1,\pm2,\pm3}E_\alpha E_{-\alpha})+(1/g)P_0^2+(1/g)\mathbf{P}^2+\mathbf{N}^2+\mathbf{M}^2. \tag{34b}$$

The completion of $\tilde{\phi}_1\otimes\phi_2$ with respect to this topology gives the space $\tilde{\phi}$ in which $\mathcal{B}$ is an algebra of continuous operators.[14] Thus we have in

$$\tilde{\phi}\subset\tilde{\mathcal{H}}_1\otimes\mathcal{H}_2\subset\tilde{\phi}^\times \tag{35a}$$

the canonical triplet of spaces[15] for the algebra $\mathcal{B}$. Again $\mathcal{B}$ is not irreducible in (35a). However, it contains one irreducible representation of $\mathcal{E}(SL(3,c))$ characterized by $(m_3=0,\rho)$ and therefore the direct sum of the representation spaces $\mathcal{H}(\lambda)$, $\lambda=0,1,2\cdots$, of $SU(3)$. Also it contains one spin s. The irreducible component of $\mathcal{B}$, $\mathcal{B}_{z,s,(m_3=0,\rho)}$—which describes one physical system—we obtain if we choose out of the space (35a) the subspace with eigenvalue z of the operator Z:

$$\phi\subset\mathcal{H}(z,s,(m_3=0,\rho))\subset\phi^\times. \tag{35b}$$

One concludes the nuclearity of ϕ in the same way as in I, so (35b) is the rigged Hilbert space of $\mathcal{B}_{z,s,(0,\rho)}$. In ϕ we obtain a basis by the direct product of the basis vectors (31) and (28) of I. However, as we have seen in I, the result does not depend upon the choice of this basis or the basis of generalized eigenvectors of the maximal commuting system (27). Because of the nuclearity of ϕ and the assumption that (27) is a complete system of commuting operators, we apply the Dirac spectral theorem to obtain the basis of generalized eigenvectors

$$|I,I_3,Y,\lambda,p_i,s_3;\,(0,\rho),s,z\rangle\,, \tag{36}$$

where $(m_3=0,\rho)$, s, and z are fixed.

We obtain the mass formula and therewith the reduction of $\mathcal{B}_{z,s,(0,\rho)}$ with respect to irreducible representations of the Poincaré group, if we take the expectation value (eigenvalue) of (15) in the states (36):

$$m^2=\langle|M^2|\rangle=z+g\langle|4H_2^2+3\sum_{\alpha=2,3}\{E_\alpha,E_{-\alpha}\}|\rangle;$$

if we use (8), (9), and (10) we can write this

$$m^2=z+g\langle\cdots|\tfrac{1}{4}Y^2-\mathbf{I}^2+\mathcal{C}|\cdots\rangle$$

and obtain with (11) the mass formula

$$m^2=z+g[\tfrac{1}{4}Y^2-I(I+1)+\lambda^2+2\lambda]. \tag{37}$$

Using the reduction of $\mathcal{H}(\lambda)$ with respect to $U^Y(1)\times SU^I(2)$,

$$\mathcal{H}(\lambda)=\sum_{Y,I}^{(\lambda)}(\mathcal{H}^Y\otimes\mathcal{H}^I)_\lambda=\sum_{\lambda\geq Y\geq-\lambda}\ \sum_{\lambda-\frac{1}{2}Y\geq I\geq\frac{1}{2}Y}(\mathcal{H}^Y\otimes\mathcal{H}^I)_\lambda\,, \tag{38}$$

which one obtains immediately from the Gelfand pattern we can write $\mathcal{H}(z,s,(0,\rho))$:

$$\begin{aligned}\mathcal{H}(z,s,(0,\rho))&=\sum_{\lambda=0}^{\infty}\oplus(\mathcal{H}(m,s)\overline{\otimes}\mathcal{H}(\lambda))\\&=\sum_{\lambda=0}^{\infty}\sum_{Y,I}^{(\lambda)}\oplus(\mathcal{H}(m_{I,Y,\lambda},s)\overline{\otimes}(\mathcal{H}^Y\otimes\mathcal{H}^I)_\lambda).\end{aligned} \tag{39}$$

Thus, we have obtained in (35b) the irreducible representation space of $\mathcal{B}$, describing our physical system, which contains according to (39) a discrete number of irreducible representation spaces of the Poincaré group $\mathcal{H}(m_{I,Y,\lambda},s)$, where $m_{I,Y,\lambda}$ is given by the mass formula (37).

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[13] $\mathcal{C}_3(\lambda_1,\lambda_2)=(\lambda_1-\lambda_2)[(2/9)(\lambda_1+\lambda_2)^2+(1/9)\lambda_1\lambda_2+\lambda_1+\lambda_2+1]$.

[14] The proof of this statement carries through in complete analogy to the proof of the continuity of the algebra $\mathcal{A}$ in Appendix A of I.

[15] $\mathcal{H}_1\otimes\mathcal{H}_2$ is the completion of the direct product of the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with respect to the Hilbert space topology.

VIII. COMPARISON WITH EXPERIMENTAL DATA

The result of our model $\mathcal{B}$ is worth comparing with the experimental data.

According to the reduction of the representation space of our physical system $\mathcal{H}(z,s,(0,\rho))$ with respect to irreducible representations of $SU(3)$:

$$\mathcal{H}(z,s,(0,\rho))=\sum\oplus\mathcal{H}(\lambda)=\mathcal{H}_1\oplus\mathcal{H}_8\oplus\mathcal{H}_{27}\oplus\cdots, \quad (40)$$

we should have a singlet, octet, 27-plet, etc., of the same spin s in which the mass increases according to the mass formula (37).

In Ref. 16 the problem of the mass spectrum was reduced to the problem of the simpler spectrum of $\alpha^2=\alpha^2(I,Y\cdots)$ resolving it with respect to the spin (cf. figure in Ref. 16). There we saw that the same regularity of the masses appeared for every spin $s=0$, 1, 2, i.e., the dependence of α^2 and herewith of m^2 on the internal quantum numbers I, Y should be the same, a fact reflected by (37) if g is—as it should be according to this model—independent of s. To check (37) we might therefore choose any of the spin values, e.g., $s=2$, where the regularity (40) shows up best, for the other spin the situation should be roughly the same according to Ref. 16.

For the singlet we choose f_0 with the mass approximately[17] $m_f^2=1.6$ (BeV)^2, so that z in (37) will be $z=1.6$ (BeV)^2. For g we choose $g=1/2R=0.142$ (BeV)^2, where R is the radius of the miniature de Sitter world of the model in Ref. 9. We have no other reason for this choice than that this gives good agreement with the experimental data.[18]

With these values of z and g, (37) predicts masses for the members of the octet, which are in complete agreement with the experimental masses of A_2, $K^*(1400)$, and E. That the spin of A_2 and $K^*(1400)$ is 2^+ has been recently confirmed; there are no experimental data against the conjecture that the spin of $E(Y=0,I=0)$ is also 2.

For the neutral member of the 27-plet ($I=0$, $Y=0$), (37) predicts a mass of $m^2\approx2.78$ BeV^2; recently there has been found a new $I=0$, $Y=0$ resonance $g_0(1670)$ with this mass. Our model predicts further the following $S=2$ 27-plet resonances:

$$\begin{aligned}
&Y=0,\quad I=1,\quad m^2\approx2.46\ \text{BeV}^2;\\
&Y=0,\quad I=2,\quad m^2\approx1.8\ \text{BeV}^2;\\
&Y=1,\quad I=\tfrac{1}{2},\quad m^2\approx2.7\ \text{BeV}^2;\\
&Y=1,\quad I=\tfrac{3}{2},\quad m^2\approx2.22\ \text{BeV}^2;\\
&Y=2,\quad I=1,\quad m^2\approx2.62\ \text{BeV}^2.
\end{aligned}$$

None of these masses has been observed.[19] The recently discovered $K^*(1800)$ resonance has a too high mass compared to our predicted value $m^2=2.7$ BeV^2 and seems to fit the scheme of Ref. 16 reasonably well with a spin predicted to $s=3$. The new $Y=0$, $I=1$ resonances have all too high masses compared to our prediction $m^2=2.5$ BeV^2. The other members of the 27-plet are difficult to observe.

IX. BROKEN $SU(3)$ AS SPECIAL CASE

We conclude with the same remark as in I. Instead of the irreducible representation space $\mathcal{H}(m_2=0,\rho)$ with ρ imaginary, of a unitary representation of $SL(3,c)$, we could have chosen an irreducible representation space $\mathcal{H}(m_2=0,\rho=r=\text{real})$ of a nonunitary representation of $SL(3,c)$. In that case the representation of the Lie algebra of $SU(3)$ would still be symmetric and there would be no reason why this kind of representations should be unphysical. The foregoing consideration would remain the same with the only difference that instead of (39) we would obtain a finite sum over λ;

$$\mathcal{H}(z,s,(0,r))=\sum_{\lambda=0}^{n(r)}\oplus(\mathcal{H}(m_{I,Y,\lambda},s)\otimes\mathcal{H}(\lambda)), \quad (39')$$

where n is determined by the choice of r. In that case the irreducible component of $\mathcal{B}$ would contain only a finite number of $SU(3)$ multiplets, i.e., we would have a finite mass spectrum. In particular one can choose a nonunitary representation (of the degenerate or principal series) of $SL(3,c)$ such that it contains only one $SU(3)$ multiplet, so that "broken" $SU(3)$ is contained in this model as a special case.

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[16] A. Böhm, in Proceedings of the Seminar on Elementary Particle Physics, Boulder, 1966 (to be published).

[17] The experimental data have been chosen from A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz, and M. Roos, Rev. Mod. Phys. **37**, 633 (1965); G. Goldhaber, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, California, 1967).

[18] It should however be interesting to investigate, whether there is a deeper reason behind this experimental connection of the constants g and $1/R^2$ by perhaps combining the model of Ref. 16 with the present model.

[19] *Note added in proof.* Recently there has been observed an $I=2$ resonance with the above predicted mass [R. Vanderhagen *et al.*, Phys. Letters **24B**, 493 (1967)].