Sources and Electrodynamics

JULIAN SCHWINGER*
Harvard University, Cambridge, Massachusetts
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A new kind of particle theory is being explored, one that is intermediate in concept between the extremes of S matrix and field theory. It employs the methods of neither approach. There are no operators, and there is no appeal to analyticity in momentum space. It is a phenomenological theory, and cognizant that measurements are operations in space and time. Particles are defined realistically by reference to their creation or annihilation in suitable collisions. The source is introduced as an abstraction of the role played by all the other particles involved in such acts. Through the use of sources the production and detection of particles, as well as their interaction, are incorporated into the theoretical description. There is a creative principle that replaces the devices of other formulations. It is an insistence upon the generality of the space-time description of the coupling among sources that is inferred from a specific spatio-temporal arrangement, in which various particles propagate between sources. Standard quantum-mechanical and relativistic requirements, imposed on the source description of noninteracting particles, imply the existence of the two statistics and the connection with spin. In this situation sources are only required to emit and absorb the mass of the corresponding particle. Particle dynamics is introduced by an extension of the source concept. It is considered meaningful for a source to emit several particles with the same total quantum numbers as a single particle, if sufficient mass is available. This is most familiar as the photon radiation that accompanies the emission of charged particles. The new types of sources introduced in this way imply new couplings among sources, which supply still further varieties of sources. This proliferation of interactions spans the full dynamical content of the initial primitive interaction. The ambition of the phenomenological source theory is to represent all dynamical aspects of particles, within a certain context, by a suitable primitive interaction. This paper is devoted to the reconstruction of electrodynamics.

INTRODUCTION

THE source concept has been proposed as a conceptually and practically useful way of characterizing particle phenomena.[1] The first applications were restricted to noninteracting particles. (The reader is reminded that this is a physical situation, not a hypothetical simplification of reality.) But the notion of source, as an abstraction of physical production and annihilation mechanisms, should also be well suited to describe particle interactions.

To represent the existence of a particle, the corresponding source function $S(p)$ need only be defined for the physical momentum values that are associated with the particle mass, $-p^2=m^2$. This is too restrictive for a theory of particle interactions, however. The much-used device of complex momenta seems inappropriate, at least in the initial stages of formulating a physical theory. Another direction is suggested by the most familiar of dynamical situations. The creation of a charged particle involves the transfer of charge to the particle from the source (which idealizes all the partners in a collision). An accelerated charge can radiate, and soft photons in particular are usually emitted with high probability. No sharp distinction can be made, in general, between creating a charged particle, and a charged particle that is accompanied by photons. It must, therefore, be physically meaningful, and useful, to define a charged-particle source for a range of momentum values such that $-p^2\geq m^2$. In the following pages we shall reconstruct some of the simpler aspects of electrodynamics on this basis. The well-known quantitative success of electrodynamics thus indicates the validity of the extended source description, at least for the range of mass values that contribute significantly to the various phenomena. There are some technical advantages in the phenomenological source method, which will be indicated in the text.

PHOTON AND ELECTRON SOURCES

The photon source is a real vectorial function $J^\mu(\xi)$. The vacuum amplitude that describes an arbitrary number of noninteracting photons is

$$\langle 0_+|0_-\rangle^J=\exp\left[i\frac{1}{2}\int (d\xi)(d\xi')J^\mu(\xi)D_+(\xi-\xi')J_\mu(\xi')\right],$$

where

$$D_+(\xi)=\Delta_+(\xi,m^2=0).$$

The necessity for a restriction on this source is made evident by the form of the vacuum-persistence probability,

$$|\langle 0_+|0_-\rangle^J|^2=\exp\left[-\int\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k^0}J^\mu(k)^*J_\mu(k)\right].$$

It is required that

$$J^\mu(k)^*J_\mu(k)=|\mathbf{J}(k)|^2-|J^0(k)|^2\geq 0,$$

and, in particular, $\mathbf{J}(k)=0$ must imply $J^0(k)=0$. The invariant restriction that accomplished this is

$$k^\mu J_\mu(k)=\mathbf{k}\cdot\mathbf{J}(k)-k^0J^0(k)=0$$

or

$$\partial_\mu J^\mu(\xi)=0.$$

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[1] J. Schwinger, Phys. Rev. 152, 1219 (1966).

The two real unit polarization vectors associated with each k^μ obey

$$e^\mu_{k\lambda}e_{k\lambda'\mu}=\delta_{\lambda\lambda'},\qquad \lambda,\lambda'=1,2$$
$$k_\mu e^\mu_{k\lambda}=0,\qquad \bar{k}_\mu e^\mu_{k\lambda}=0,$$

and $\bar{k}_\mu$ is obtained from k_μ by reversing the sign of the time component (or of the space components). We also have

$$g^{\mu\nu}=\sum_{\lambda=1,2} e^\mu_{k\lambda}e^\nu_{k\lambda}+(k^\mu\bar{k}^\nu+k^\nu\bar{k}^\mu)/(k\bar{k}),$$

which makes explicit that

$$\int\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k^0}J^\mu(k)^*J_\mu(k)=\sum_{k\lambda}|J_{k\lambda}|^2\geq 0,$$

where

$$J_{k\lambda}=\left[\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k^0}\right]^{1/2}e^\mu_{k\lambda}J_\mu(k).$$

The multiphoton states generated by the source, the conserved-current vector $J^\mu(\xi)$, are represented by

$$\langle\{n\}|0_-\rangle^J=\langle 0_+|0_-\rangle^J\prod_{k\lambda}\frac{(iJ_{k\lambda})^{n_{k\lambda}}}{[n_{k\lambda}!]^{1/2}}$$

and

$$\langle 0_+|\{n\}\rangle^J=\langle 0_+|0_-\rangle^J\prod_{k\lambda}\frac{(iJ^*_{k\lambda})^{n_{k\lambda}}}{[n_{k\lambda}!]^{1/2}}.$$

The electron-positron source $\eta(x)$ is a real totally anticommutative quantity with a fourfold multiplicity to represent $s=\frac{1}{2}$, and an additional twofold multiplicity to describe electric charge. The vacuum probability amplitude that gives an account of arbitrary numbers of noninteracting particles is

$$\langle 0_+|0_-\rangle^\eta=\exp\left[i\frac{1}{2}\int(dx)(dx')\eta(x)\gamma^0G_+(x-x')\eta(x')\right],$$

where

$$G_+(x-x')=\int\frac{(dp)}{(2\pi)^4}e^{ip(x-x')}\frac{1}{\gamma p+m-i\epsilon}$$
$$=i\int\frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}e^{ip(x-x')}(m-\gamma p),\quad x^0>x^{0\prime}.$$

The projection matrix that occurs in the latter form can be exhibited in a dyadic representation

$$(m-\gamma p)/2m=\sum_{\sigma l}u_{p\sigma l}u^*_{p\sigma l}\gamma^0.$$

Here,

$$(u^*_{p\sigma l}\gamma^0u_{p\sigma'l'})=\delta_{\sigma l,\sigma'l'},\quad \sigma,\sigma',l,l'=\pm 1$$

and σ, l are spin and charge values, respectively. Transposition or complex conjugation gives another version,

$$(-m-\gamma p)/2m=\sum_{\sigma l}u^*_{p\sigma l}u_{p\sigma l}\gamma^0.$$

The individual spinors obey

$$u^*_{p\sigma l}\gamma^0(m+\gamma p)=u_{p\sigma l}\gamma^0(-m+\gamma p)=0$$

and

$$(m+\gamma p)u_{p\sigma l}=(-m+\gamma p)u^*_{p\sigma l}=0.$$

Multiparticle states are represented by

$$\langle\{n\}|0_-\rangle^\eta=\langle 0_+|0_-\rangle^\eta\prod_{p\sigma l}(i\eta_{p\sigma l})^{n_{p\sigma l}}$$

and

$$\langle 0_+|\{n\}\rangle^\eta=\langle 0_+|0_-\rangle^\eta\prod_{p\sigma l}(i\eta^*_{p\sigma l})^{n_{p\sigma l}},$$

where

$$\eta_{p\sigma l}=\left[\frac{(d\mathbf{p})}{(2\pi)^3}\frac{m}{p^0}\right]^{1/2}[u^*_{p\sigma l}\gamma^0\eta(p)]$$

and each integer $n_{p\sigma l}$ can be either 0 or 1. Some standard multiplication order, and its inverse, is to be used in writing the products of the $\eta_{p\sigma l}$ and the $\eta^*_{p\sigma l}$, respectively.

The vacuum amplitude that describes both kinds of particles, under physical conditions of noninteraction, is

$$\langle 0_+|0_-\rangle^{\eta J}=\exp[iw_2(\eta J)],$$

with

$$w_2(\eta J)=w_{2,0}(\eta)+w_{0,2}(J)$$
$$=\frac{1}{2}\int(dx)(dx')\eta(x)\gamma^0G_+(x-x')\eta(x')$$
$$+\frac{1}{2}\int(d\xi)(d\xi')J^\mu(\xi)D_+(\xi-\xi')J_\mu(\xi').$$

There is another way of presenting w_2 in which the explicit integral construction is replaced by an implicit differential one. This is done with the aid of auxiliary quantities $\psi(x)$ and $A_\mu(\xi)$, which are numbers of the same type as $\eta(x)$ and $J_\mu(\xi)$, respectively. Consider

$$w_2(\eta J)=\int(dx)\eta(x)\gamma^0\psi(x)+\int(d\xi)J^\mu(\xi)A_\mu(\xi)$$
$$-\frac{1}{2}\int(dx)\psi(x)\gamma^0G_+^{-1}\psi(x)$$
$$-\frac{1}{2}\int(d\xi)A^\mu(\xi)D_+^{-1}A_\mu(\xi),$$

where

$$G_+^{-1}\psi(x)=[\gamma^\mu(1/i)\partial_\mu+m]\psi(x),$$
$$D_+^{-1}A_\mu(\xi)=-\partial^2A_\mu(\xi),$$

and it is understood that w_2 is stationary with respect to variations of the auxiliary quantities. That condition implies that

$$\eta(x)=G_+^{-1}\psi(x)\,,\quad J_\mu(\xi)=D_+^{-1}A_\mu(\xi)\,,$$

to which we add the boundary conditions necessary to give the explicit forms

$$\psi(x)=\int(dx')G_+(x-x')\eta(x')\,,$$

$$A_\mu(\xi)=\int(d\xi')D_+(\xi-\xi')J_\mu(\xi')\,.$$

The elimination of the auxiliary quantities, fields associated with the sources, recovers the integral expression for $w_2(\eta J)$.

PRIMITIVE PARTICLE INTERACTION

We now give quantitative form to the physical hypothesis that an electron source $\eta(p)$, with $-p^2>m^2$, can radiate an electron and a photon. This process must fit into the already established framework since the subsequent removal of the electron, or of the photon, gives a particular realization of a photon source, or of an electron source, respectively. Thus, a term quadratic in η, representing the electron photon-creation act and the subsequent annihilation of the electron, must play the role of photon source, and a term that is bilinear in η and J, representing the joint-creation process and the subsequent absorption of the photon, must act as an effective electron source. Accordingly, the function $w(\eta J)$ should be supplemented, at least, by a term of the general form

$$w_3(\eta J)=w_{2,1}(\eta J)=\frac{1}{2}\int(dx)\cdot\cdot(d\xi')\eta(x)\gamma^0G(x,x';\xi)^\mu \times\eta(x')D_+(\xi-\xi')J_\mu(\xi')\,.$$

The effective-photon-source description requires that

$$\partial^\mu{}_\xi G(x,x';\xi)_\mu=0\,.$$

The anticommutative nature of electron sources implies

$$[\gamma^0G(x',x;\xi)^\mu]^T=-\gamma^0G(x,x';\xi)^\mu\,,$$

and the circumstances that produce an effective electron source indicate that $G(x,x';\xi)^\mu$ contains a single-electron function G_+ referring to the point x, and one referring to the point x'. These considerations suggest the structure

$$G(x,x';\xi)^\mu=G_+(x-\xi)eq\gamma^\mu G_+(\xi-x') -ieq[f^\mu(x-\xi)-f^\mu(x'-\xi)]G_+(x-x')\,.$$

The antisymmetrical charge matrix q occurs here in order to assure the Fermi-Dirac (F.D.) antisymmetry of $\gamma^0G(x,x';\xi)^\mu$. The appearance of the term containing the numerical function f^μ is dictated by the fact that

$$\partial^\mu\,[G_+(x-\xi)eq\gamma_\mu G_+(\xi-x')] =-ieq[\delta(x-\xi)-\delta(x'-\xi)]G_+(x-x')\neq 0\,.$$

This is compensated by choosing

$$-\partial^\mu{}_\xi f_\mu(x-\xi)=\delta(x-\xi)\,,$$

which evidently describes the exchange of charge between the source and the charged particle. The two terms of $G(x,x';\xi)^\mu$ make explicit that both the charged particle and the source are involved in the mechanism of photon radiation.

A specific choice for $f^\mu(x-\xi)$ is part of the characterization of a charged-particle source. We try to give covariant expression to the picture of radiation emanating from the source, with its implication of a degree of temporal localizability. An example of an *unacceptable* covariant function is

$$f^\mu(x-\xi)=-(\partial^\mu{}_\xi/\partial^2{}_\xi)\delta(x-\xi) =\partial^\mu{}_\xi\frac{i}{4\pi^2}\frac{1}{(x-\xi)^2}\,,$$

which has no intrinsic temporal scale. We propose, instead, to have the source function $\eta(x)$ supply a timelike vector, as represented by $-i\partial^\mu{}_\eta$, and then construct the spacelike vector

$$\nabla^\mu=\partial^\mu{}_\xi-\partial^\mu{}_\eta(\partial_\xi\partial_\eta)/\partial^2{}_\eta\,,$$
$$\partial^\mu{}_\eta\nabla_\mu=0\,.$$

This gives the function

$$f^\mu(x-\xi)=-(\nabla^\mu/\nabla^2)\delta(x-\xi)\,,$$

where

$$\nabla^2=\partial^2{}_\xi-(\partial_\xi\partial_\eta)^2/\partial^2{}_\eta$$

and

$$\partial^\mu{}_\xi\nabla_\mu=\nabla^2\,.$$

To verify time locality, use a coordinate system for which a particular numerical assignment of $-i\partial^\mu{}_\eta$ has only a time component. Then ∇^μ has only spatial components, and

$$f^k(x-\xi)=\partial^k{}_\xi\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\boldsymbol{\xi}|}\delta(x^0-\xi^0)\,.$$

An important aspect of this f^μ is best described with the aid of the explicit probability amplitude for electron and photon emission. The treatment of the quadratic η combination as a weak effective photon source gives

$$\langle 1_{k\lambda}|0_-\rangle^\eta=i\left[\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k^0}\right]^{1/2}e^\mu{}_{k\lambda}\int(d\xi)\cdot\cdot(dx') \times e^{-ik\xi}\tfrac{1}{2}\eta(x)\gamma^0G(x,x';\xi)_\mu\eta(x')\,.$$

Part of this structure is

$$e^{\mu}{}_{k\lambda}\int(d\xi)e^{-ik\xi}i[f_\mu(x-\xi)-f_\mu(x'-\xi)]$$
$$=-\frac{e_{k\lambda}\partial_\eta}{k\partial_\eta}(e^{-ikx}-e^{-ikx'}),$$

where it has been recognized that $\eta(x)$ and $\eta(x')$ give equivalent results, since $ek=0$, $k^2=0$. Now let η be the superposition of an extended source and of an electron-detection source, the latter containing only momenta that obey $-p^2=m^2$. The result is

$$\langle 1_{k\lambda}1_{p\sigma l}|0_-\rangle^\eta=iel\left[\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k^0}\frac{(d\mathbf{p})}{(2\pi)^3}\frac{m}{p^0}\right]^{1/2}$$
$$\times e^{\mu}{}_{k\lambda}u^*{}_{p\sigma l}\gamma^0\left[\gamma_\mu\frac{1}{\gamma P+m}-2P_\mu\frac{1}{P^2+m^2}\right]\eta(P),$$

where

$$P^\mu=k^\mu+p^\mu$$

and

$$P^2+m^2=2kp.$$

The two terms can be combined:

$$\gamma_\mu\frac{1}{\gamma P+m}-2P_\mu\frac{1}{P^2+m^2}=(\gamma P+m)\gamma_\mu\frac{1}{P^2+m^2}.$$

The factor $\gamma p+m$ vanishes when applied to $u^*{}_{p\sigma l}\gamma^0$, and

$$\langle 1_{k\lambda}1_{p\sigma l}|0_-\rangle^\eta=-el\left[\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k^0}\frac{(d\mathbf{p})}{(2\pi)^3}\frac{m}{p^0}\right]^{1/2}$$
$$\times u^*{}_{p\sigma l}\gamma^0\sigma_{\mu\nu}e^{\mu}{}_{k\lambda}k^\nu\eta(P)\frac{1}{P^2+m^2},$$

which ascribes the radiation to the electron magnetic dipole moment.

Let us compare this result with one obtained by an alternative source definition, in which the timelike vector $-i\partial^\mu{}_\eta$ is replaced by a constant vector parallel to the time axis. With the latter choice, the source term gives no contribution, since $e^0{}_{k\lambda}=0$. This version is characterized by

$$\gamma_\mu\frac{1}{\gamma P+m}\to(2p_\mu+i\sigma_{\mu\nu}k^\nu)\frac{1}{P^2+m^2}.$$

The additional effect vanishes if the charged particle is created at rest, but otherwise implies the strong radiation of soft photons that is associated with accelerated charges. The definition we have adopted avoids charge acceleration in the creation process and minimizes the accompanying radiation.

One may raise the question whether some modification of the primitive interaction could remove the radiation that accompanies charged-particle creation by a source. An obvious addition would be

$$\delta G(x,x';\xi)_\mu=\partial^\eta{}_\xi[G_+(x-\xi)\mu q\sigma_{\mu\nu}G_+(\xi-x')],$$

which maintains the general properties of $G(x,x';\xi)_\mu$. The effect of this alteration is described by the substitution

$$ei\sigma_{\mu\nu}k^\nu\to(e+2m\mu)i\sigma_{\mu\nu}k^\nu+\mu[2p_\mu\gamma k-\gamma_\mu(P^2+m^2)].$$

If

$$(e/2m)+\mu=0,$$

strong suppression of the radiation does occur, for charged-particle creation at rest. But with very energetic charged particles the situation is completely reversed, owing to the p_μ term.

We mention this ambiguity of the electron-photon primitive interaction, not to invoke a speculative dynamical principle that would eliminate it, but to emphasize that some arbitrary elements must appear at the phenomenological level. They can be removed only by introducing experimental characteristics of the specific particles that are being described. It is the ambition of the phenomenological source theory that all direct appeals to experiment be limited to determining the form of the primitive interaction, which will then imply a complete dynamical theory, within a framework of suitably limited objectives. The accurate predictions of quantum electrodynamics are reproduced with $\mu=0$, which we accept as a basic phenomenological assertion about photons and electrons (or muons).

A photon source $J^\mu(\xi)$ and the effective source given by

$$J^\mu{}_{\text{eff}}(\xi)=\frac{1}{2}\int(dx)(dx')\eta(x)\gamma^0G(x,x';\xi)^\mu\eta(x')$$

are not equivalent in one respect. The former is real. The lack of reality displayed by the effective source arises from the internal emission and absorption of an electron, a degree of dynamical detail that is not contemplated in J^μ. The distinction should disappear, then, if we can extrapolate $J^\mu{}_{\text{eff}}$ to physical circumstances in which the inner propagation of a particle does not occur. This will be the situation if the source concept is also extended to include sources that contain only momenta such that $-p^2<m^2$. Then $J^\mu{}_{\text{eff}}$ is real, if the phenomenological parameter e has that property. As a related requirement, let no source be capable of emitting a particle. The probability of maintaining the vacuum must then be unity, and w real, which asserts that e is real.

INTERACTION SKELETON

The source is defined through an abstraction of physical acts of particle production and annihilation. The primitive interaction initiates the introduction of realistic mechanisms and implies certain types of effective sources. The particle-mediated couplings between these sources give new interaction mechanisms and additional effective sources. This proliferation should ultimately span the full dynamical content of the primitive interaction for the specified set of particles. The resulting account of particle dynamics incorporates a symbolic transcription of the experimenter's art. Particles are prepared by creation in an initial configuration and detected by annihilation from their final states. These aspects of sources refer to the known properties of noninteracting particles and the desired information about particle interactions can be isolated.

In this section we consider a first stage of the development. Only single-particle exchanges between sources are considered. That already implies an infinite variety of interactions, which appear in skeletal form, however, since the complete physical structure of these interactions, can emerge only by considering multiparticle exchanges.

The effective photon source supplied by the quadratic combination of electron sources gives the simplest illustration. The interaction of two such sources through the exchange of a photon is described by

$$w_{4,0}(\eta)=\frac{1}{2}\int(dx)\cdots(d\xi')\tfrac{1}{2}\eta(x)\gamma^0G(x,x';\xi)^\mu\eta(x')\\ \times D_+(\xi-\xi')\tfrac{1}{2}\eta(x'')\gamma^0G(x'',x''';\xi')_\mu\eta(x''').$$

This form can also be applied in circumstances for which the photon momentum is spacelike, as realized in the scattering of two charged particles, and gives a skeletal picture of that process. One should notice that $\eta(x)\eta(x')\eta(x'')\eta(x''')$, a totally antisymmetrical function of its arguments (including matrix indices), selects a coefficient of corresponding symmetry. Permutation symmetries associated with statistics, which include so-called crossing symmetry, are guaranteed automatically by the algebraic properties of the sources. In the scattering application the sources emit and absorb charged particles, without accompanying photons. The interaction term can then be simplified to

$$w_{4,0}(\eta)=\frac{1}{2}\int(d\xi)(d\xi')\tfrac{1}{2}\psi(\xi)\gamma^0\gamma^\mu eq\psi(\xi)D_+(\xi-\xi')\\ \times\tfrac{1}{2}\psi(\xi')\gamma^0\gamma_\mu eq\psi(\xi'),$$

where

$$\psi(x)=\int(dx')G_+(x-x')\eta(x'),$$

which implies the known single-photon-exchange matrix elements for electron-electron and positron-positron scattering.[2]

[2] It does not seem to have been noticed that the high-energy limit of the center-of-mass angular distribution for unpolarized electron-electron scattering is a perfect square,

$$\left(\frac{1}{\sin^2(\frac{1}{2}\theta)}+\frac{1}{\cos^2(\frac{1}{2}\theta)}-1\right)^2,$$

or that the corresponding electron-positron angular distribution is obtained by multiplication with $\cos^4(\frac{1}{2}\theta)$.

The effective electron source, bilinear in η and J, that is implied by the primitive interaction can be written

$$\eta'_{\rm eff}(x)=eq\gamma^\mu A_\mu(x)\int(dx')G_+(x-x')\eta(x')\\ +ieq\Lambda(x,A)\eta(x),$$

where

$$A_\mu(\xi)=\int(d\xi')D_+(\xi-\xi')J_\mu(\xi')$$

and

$$\Lambda(x,A)=\int(d\xi)f^\mu(x-\xi)A_\mu(\xi).$$

The interaction of two such sources through the intermediary of a single electron is represented by

$$w_{2,2}(\eta,J)=\frac{1}{4}\int(dx)\cdot\cdot(d\xi')\eta(x)\gamma^0\\ \times G(x,x';\xi\xi')^{\mu\nu}\eta(x')A_\mu(\xi)A_\nu(\xi').$$

The following symmetries are required:

$$[\gamma^0G(x',x;\xi\xi')^{\mu\nu}]^T=-\gamma^0G(x,x';\xi\xi')^{\mu\nu},\\ G(x,x';\xi'\xi)^{\nu\mu}=G(x,x';\xi\xi')^{\mu\nu},$$

and the consideration of effective photon sources demands that

$$\partial^\mu{}_\xi G(x,x';\xi\xi')_{\mu\nu}=0.$$

The latter condition is certainly satisfied under the circumstances for which the formula is derived—each photon is associated with a different extended electron source. It will fail, however, if applied to more general spatio-temporal arrangements. The situation is rectified by including the possibility that both photons may be radiated from the same source, in successive emission acts, as described by the additional effective electron source

$$\eta''_{\rm eff}(x)=\tfrac{1}{2}[ieq\Lambda(x,A)]^2\eta(x).$$

The result so obtained is

$$G(x,x';\xi,\xi')^{\mu\nu}=G_+(x-\xi)eq\gamma^\mu G_+(\xi-\xi')eq\gamma^\nu G_+(\xi'-x')\\ +G_+(x-\xi')eq\gamma^\nu G_+(\xi'-\xi)eq\gamma^\mu G_+(\xi-x')\\ -ieq[f^\mu(x-\xi)-f^\mu(x'-\xi)]G_+(x-\xi')eq\gamma^\nu G_+(\xi'-x')\\ -ieq[f^\nu(x-\xi')-f^\nu(x'-\xi')]G_+(x-\xi)eq\gamma^\mu G_+(\xi-x')\\ +ieq[f^\mu(x-\xi)-f^\mu(x'-\xi)]\\ \times ieq[f^\nu(x-\xi')-f^\nu(x'-\xi')]G_+(x-x'),$$

which exhibits all the necessary properties.

The interaction term $w_{2,2}(\eta,J)$ contains a description, in skeleton form, of electron-photon scattering. For this application, in which accompanying photons are not involved in the action of the η sources, one can use the simpler version

$$w_{2,2}(\eta,J)=\frac{1}{2}\int(dx)(dx')\psi(x)\gamma^0 eq\gamma^\mu A_\mu(x)\times G_+(x-x')eq\gamma^\nu A_\nu(x')\psi(x'),$$

with the anticipated result.

In order to provide a systematic survey of other interaction skeletons, let us omit temporarily the direct photon radiation from charged sources. We consider the dependence of w on η in relation to a total effective source, as expressed by the differential form

$$\delta_\eta w=\int(dx)(dx')\delta\eta(x)\gamma^0 G_+(x-x')\eta_{\rm eff}(x').$$

The consideration of w_2+w_3 gives

$$\eta_{\rm eff}(x)=\eta(x)+eq\gamma^\mu A_\mu(x)\int(dx')G_+(x-x')\eta_{\rm eff}(x').$$

But, merely by writing $\eta_{\rm eff}$ rather than η in the second dynamical term, we have automatically included the processes that are obtained by repetition of the primitive interaction. Another form of this integral equation is produced by the definition

$$\int(dx')G_+(x-x')\eta_{\rm eff}(x')=\int(dx')G(x,x';A)\eta(x'),$$

namely,

$$G(x,x';A)=G_+(x-x')+\int(d\xi)G_+(x-\xi)eq\gamma^\mu A_\mu(\xi)\times G(\xi,x';A),$$

which also implies the differential equation

$$[\gamma^\mu(-i\partial_\mu-eqA_\mu(x))+m]G(x,x';A)=\delta(x-x').$$

The integral equation, if considered to uniquely determine $G(x,x';A)$, admits the alternative form

$$G(x,x';A)=G_+(x-x')+\int(d\xi)\times G(x,\xi;A)eq\gamma^\mu A_\mu(\xi)G_+(\xi-x'),$$

which also asserts that

$$[\gamma^0 G(x',x;A)]^T=-\gamma^0 G(x,x';A).$$

The latter property is required in order to state the integrated version,

$$w(\eta,J)=\frac{1}{2}\int(dx)(dx')\eta(x,A)\gamma^0 G(x,x';A)\eta(x',A)+\cdots,$$

where we have anticipated that the charge properties of the source are reinstated by writing

$$\eta(x,A)=\exp[ieq\Lambda(x,A)]\eta(x).$$

To verify the last statement we consider the dependence of the quadratic η term on the photon source $J^\mu(\xi)$, as represented by the differential form

$$\int(d\xi)(d\xi')\delta J^\mu(\xi)D_+(\xi-\xi')\frac{\delta}{\delta A^\mu(\xi')}\int(dx)(dx')\times\eta(x,A)\gamma^0 G(x,x';A)\eta(x',A).$$

The effective-photon-source interpretation requires that

$$\partial^\mu{}_\xi\frac{\delta}{\delta A^\mu(\xi)}\{\exp[-ieq\Lambda(x,A)]G(x,x';A)\times\exp[ieq\Lambda(x,A)]\}=0.$$

This is equivalent to the demand that the quantity in brackets be left unchanged by the substitution

$$A^\mu(\xi)\to A^\mu(\xi)+\partial^\mu{}_\xi\lambda(\xi),$$

where $\lambda(\xi)$ is an arbitrary gauge function. It is known that

$$G(x,x';A+\partial\lambda)=\exp[ieq\lambda(x)]G(x,x';A)\times\exp[-ieq\lambda(x')],$$

and indeed,

$$\Lambda(x,A+\partial\lambda)=\Lambda(x,A)+\lambda(x)$$

does supply the compensating terms.

These results are embodied in the following implicit differential construction:

$$w(\eta J)=\int(dx)\eta(x,A)\gamma^0\psi(x)+\int(d\xi)J^\mu(\xi)A_\mu(\xi)-\frac{1}{2}\int(dx)\psi(x)\gamma^0[\gamma^\mu(-i\partial_\mu-eqA_\mu(x))+m]\psi(x)-\frac{1}{2}\int(d\xi)A^\mu(\xi)(-\partial^2)A_\mu(\xi).$$

The stationary requirement on $\psi(x)$ gives the differential equation

$$[\gamma^\mu(-i\partial_\mu-eqA_\mu(x))+m]\psi(x)=\eta(x,A)$$

and the elimination of ψ recovers the quadratic η expression. But this version of w gives an account of the mutual action between photon and charged particle. The stationary property with respect to $A^\mu(\xi)$ asserts that

$$-\partial^2 A^\mu(\xi)=J^\mu(\xi)+j^\mu(\xi),$$

where

$$j^\mu(\xi)=\tfrac{1}{2}\psi(\xi)\gamma^0 eq\gamma^\mu\psi(\xi)+\int(dx)\psi(x)\gamma^0 ieq\eta(x,A)f^\mu(x-\xi)$$

and

$$\partial^\mu{}_\xi j_\mu(\xi)=0.$$

It is clear that $j^\mu(\xi)$ is the effective photon source produced by charged particles and their sources, including effects induced by the presence of other photons.

Faced with a structure as familiar as that for $w(\eta J)$, the reader may need to be reminded that the fields $\psi(x)$ and $A^\mu(\xi)$ are numerical quantities (of a nature appropriate to the statistics). Furthermore, the infinite series in η and J that results from the complete elimination of ψ and A is a sequence of increasingly elaborate interaction skeletons. Later terms in this series do not contain modifications of earlier ones, including the examples already discussed, which are the leading terms of the infinite series.

TWO-PARTICLE EXCHANGE

The picture of particles and their skeletal interactions becomes more substantial when two-particle exchanges between sources are included. The first objects to be modified are the propagation functions $G_+(x-x')$ and $D_+(\xi-\xi')$. The introduction of the primitive interaction implies that extended η sources can interact by exchanging an electron, or by the exchange of an electron and a photon. The primitive interaction may also be interpreted by extending the concept of photon source, particularly to such momenta that $-k^2>(2m)^2$, when electron-positron emission from the extended photon source takes place. Two such sources interact by exchanging a photon, or an electron-positron pair. In taking this step we have rejected the possibility that the validity of the extended-source concept be contingent upon the absence of a gap between the one-particle and two-particle mass spectra. We regard it as meaningful and useful that a source can be extended to include any combination of particles with the same properties, apart from mass, as the specified particle.

The probability amplitude for emission of an electron and a photon from a weak source has already been stated in a form equivalent to

$$\langle 1_{k\lambda}1_{p\sigma l}|0_-\rangle^\eta = iel[d\omega_k d\omega_p 2m]^{1/2} \times u^*_{p\sigma l}\gamma^0\gamma k\gamma e_{k\lambda}\eta(P)\frac{1}{P^2+m^2},$$

where

$$d\omega_k=\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k^0},\quad d\omega_p=\frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}.$$

Its counterpart in absorption is

$$\langle 0_+|1_{k\lambda}1_{p\sigma l}\rangle^\eta = iel\frac{1}{P^2+m^2} \times\eta(-P)\gamma^0\gamma e_{k\lambda}\gamma k u_{p\sigma l}[d\omega_k d\omega_p 2m]^{1/2}.$$

The contribution to the vacuum amplitude associated with electron and photon exchange between two sources, η and η_2, is

$$\sum_{k\lambda,p\sigma l}\langle 0_+|1_{k\lambda}1_{p\sigma l}\rangle^{\eta_1}\langle 1_{k\lambda}1_{p\sigma l}|0_-\rangle^{\eta_2} = -2e^2\int d\omega_k d\omega_p\frac{1}{P^2+m^2}\eta_1(-P)\gamma^0\gamma k\eta_2(P),$$

in which the elementary summations over σ, l, and $\lambda=1$, 2 have already been performed. We concentrate our attention on the total momentum P by introducing a unit factor in the form

$$(2\pi)^3\int dM^2\frac{(d\mathbf{P})}{(2\pi)^3}\frac{1}{2P^0}\delta(p+k-P),$$

where

$$-P^2=M^2>m^2,$$

and then integrating over k and p. A calculation in the rest frame of the timelike vector P quickly verifies that

$$(2\pi)^3\int d\omega_k d\omega_p\delta(p+k-P)\gamma k=\frac{1}{32\pi^2}\left(\frac{M^2-m^2}{M^2}\right)^2\gamma P,$$

and the two-particle summation becomes $(\alpha=e^2/4\pi)$

$$\frac{\alpha}{2\pi}\int_m^\infty\frac{dM}{M}\frac{(d\mathbf{P})}{(2\pi)^3}\frac{1}{2P^0}\frac{M^2-m^2}{M^2}\eta_1(-P)\gamma^0\gamma P\eta_2(P)$$

or

$$-i\frac{\alpha}{2\pi}\int_m^\infty\frac{dM}{M}\left(1-\frac{m^2}{M^2}\right)\int(dx)(dx')\eta_1(x)\gamma^0\gamma^\mu\frac{1}{i}\partial_\mu \times\Delta_+(x-x',M^2)\eta_2(x').$$

We recognize in the latter a modification of the propagation function $G_+(x-x')$, which we now designate as

$$\bar{G}_+(x-x')=\int\frac{(dp)}{(2\pi)^4}e^{ip(x-x')}\bar{G}_+(p).$$

Our result is expressed by

$$\bar{G}_+(p)=\frac{1}{\gamma p+m-i\epsilon}+\frac{\alpha}{4\pi}\int_m^\infty\frac{dM}{M}\left(1-\frac{m^2}{M^2}\right) \times\left[\frac{1}{\gamma p+M-i\epsilon}+\frac{1}{\gamma p-M+i\epsilon}\right].$$

Two features of this entirely finite spectral structure should be noted. The positive coefficients of $1/(\gamma p\pm M)$ assure the validity of the positiveness property (η_1 is a commuting spinor):

$$-\mathrm{Re}\,i\int(dx)(dx')\epsilon(x-x')\eta_1(x)\gamma^0\bar{G}_+(x-x')\eta_1(x')\geq 0;$$

the discrete mass value at m is effectively isolated from the continuous spectrum by the vanishing of the spectral-weight factor at $M=m$.

Comparison is inevitable with the textbook version of the renormalized electron Green's function, in a related approximation.[3] This uses a photon Green's function with the covariant factor $g_{\mu\nu}$ rather than the divergenceless structure of the Lorentz gauge. A spectral presentation of the result is not easily located. In some old notes (Harvard lectures, 1952, unpublished) I find that

$$G_+(p)]_{\text{renorm.}}{}^{\text{cov. gauge}} = \frac{1}{\gamma p+m-i\epsilon}+\frac{\alpha}{4\pi}\int_m^\infty \frac{dM}{M}\left(1-\frac{m^2}{M^2}\right) \times\left\{\frac{1-[2mM/(M-m)^2]}{\gamma p+M-i\epsilon}+\frac{1+[2mM/(M+m)^2]}{\gamma p-M+i\epsilon}\right\}.$$

This structure lacks both of the properties we have commented on, and it is infrared-divergent. It can be produced, in the present context, by omitting the dynamical effect of the source, and summing over four photon polarizations. Much closer to our new result is the renormalization calculation using the radiation gauge. Indeed, the two are identical in the rest frame of the momentum, but the radiation-gauge result is not covariant in form. The particle-source formulation has enabled us to resolve the ancient gauge dilemma between physical-positiveness requirements and covariance.

The simple calculation we have performed might have been presented in a different way. The term in the vacuum amplitude that describes one-electron exchange between sources η_1, η_2 and one-photon exchange between sources J_1, J_2, with no interaction between the particles, is

$$\int (dx)\cdots(d\xi')iJ_1{}^\mu(\xi)\eta_1(x)\gamma^0G_+(x-x') \times D_+(\xi-\xi')i\eta_2(x')J_{2\mu}(\xi').$$

There is a related contribution of the primitive interaction in which a noninteracting electron and photon are detected by sources η_1 and J_1. It is

$$\int (dx)\cdots(d\xi')iJ^\mu{}_1(\xi)\eta_1(x)\gamma^0G_+(x-x')D_+(\xi-\xi') \times[eq\gamma_\mu\delta(x'-\xi')G_+(x'-\bar{x}) +ieq\delta(x'-\bar{x})f_\mu(\bar{x}-\xi')]\eta(\bar{x}),$$

which exhibits the effective electron-photon source realized by an extended electron source. Electron-photon exchange between two such sources is then described by the vacuum-amplitude term

$$\int (dx_1)\cdots(d\xi')\eta_1(x_1)\gamma^0 \times[G_1(x_1-x)\delta(x-\xi)eq\gamma^\mu-ieq\delta(x-x_1)f^\mu(x_1-\xi)] \times G_+(x-x')D_+(\xi-\xi')[eq\gamma_\mu\delta(x'-\xi')G_+(x'-x_2) +ieq\delta(x'-x_2)f_\mu(x_2-\xi')]\eta_2(x_2),$$

which is identical with the previous version, particularly if the implicit tensor $g_{\mu\nu}$ is replaced by a polarization vector summation. When used as it stands, the equivalence emerges from the relation

$$\gamma^\mu\gamma k\gamma_\mu=2\gamma k.$$

There is a close resemblance here to a field-theoretic perturbative calculation of the electron Green's function. But there are fundamental differences in meaning. The particle-source formulation describes the physical presence of an electron and a photon in relation to an idealized mechanism of emission and absorption. The field approach gives an approximation to the self-action of a field source and derives a corresponding approximate particle interpretation by the process of renormalization. It is unfortunate, incidentally, that the renormalization concept is usually so tied to the vagaries of the perturbation method that its general significance in the transformation from a field to a particle description tends to be misunderstood. There is no conceptual improvement in the absence of renormalization from a phenomenological particle theory—that concept is simply foreign to the latter's more limited objectives.

The term in the vacuum amplitude that describes the noninteracting propagation of two charged particles can be written

$$-\frac{1}{2}\int (dx)\cdots(dx''')\eta_1(x)\gamma^0G_+(x-x')\eta_2(x') \times\eta_1(x'')\gamma^0G_+(x''-x''')\eta_2(x''') =-\frac{1}{2}\int (dx)\cdots(dx''')(\eta_1(x'')\gamma^0)_c(\eta_1(x)\gamma^0)_a \times G_+(x-x')_{ab}G_+(x''-x''')_{cd}\eta_2(x')_b\eta_2(x''')_d,$$

where η_1 and η_2 are detection and production sources, respectively. The subscripts $a, \cdots, d$ combine Dirac spin indices and charge labels. An analogous display of the primitive interaction, specialized to simple electron-positron sources,

$$-\frac{1}{2}\int (dx)\cdots(dx'')(\eta(x'')\gamma^0)_c(\eta(x)\gamma^0)_aG_+(x-x')_{ab} \times G_+(x''-x')_{cd}(-i)(eq\gamma^\mu\gamma^0A_\mu(x'))_{bd},$$

identifies the effective electron-positron source generated by an extended photon source. The interaction of two such currents, J_1 and J_2, gives the following contribu-

[3] See, for example, J. Jauch and F. Rorhlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1955).

tion to the vacuum amplitude:

$$-\frac{1}{2}\int (d\xi)\cdot\cdot(d\xi')J^{\mu}{}_1(\xi)D_+(\xi-\xi_1)\times\mathrm{tr}[eq\gamma_\mu G_+(\xi_1-\xi_1')eq\gamma_\nu G_+(\xi_1'-\xi_1)]\times D_+(\xi'_1-\xi')J_2{}^{\nu}(\xi').$$

This is equivalent to

$$-\tfrac{1}{2}e^2\int d\omega_p d\omega_{p'}\frac{1}{(k^2)^2}J^{\mu}{}_1(-k)\times\mathrm{tr}[q\gamma_\mu(m-\gamma p)q\gamma_\nu(-m-\gamma p')]J^{\nu}{}_2(k),$$

where

$$k=p+p'$$

is characterized by

$$-k^2=M^2>(2m)^2.$$

We now introduce the unit factor

$$(2\pi)^3\int dM^2\frac{(d\mathbf{k})}{(2\pi^3)}\frac{1}{2k^0}\delta(p+p'-k),$$

and use the integral

$$(2\pi)^3\int d\omega_p d\omega_{p'}\delta(p+p'-k)\times\mathrm{tr}[q\gamma_\mu(m-\gamma p)q\gamma_\nu(-m-\gamma p')]=\frac{1}{6\pi^2}M^2\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2}\left(1+\frac{2m^2}{M^2}\right)\left(g_{\mu\nu}+\frac{1}{M^2}k_\mu k_\nu\right),$$

which is also easily derived in the rest frame of the timelike vector k. This gives the vacuum-amplitude term

$$i\frac{2\alpha}{3\pi}\int_{2m}^{\infty}\frac{dM}{M}\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2}\left(1+\frac{2m^2}{M^2}\right)\int(d\xi)(d\xi')\times J_1{}^{\mu}(\xi)\Delta_+(\xi-\xi',M^2)J_{2\mu}(\xi'),$$

which is conveyed by the modified photon-propagation function

$$\bar{D}_+(k)=\frac{1}{k^2-i\epsilon}+\frac{2\alpha}{3\pi}\int_{2m}^{\infty}\frac{dM}{M}\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2}\times\left(1+\frac{2m^2}{M^2}\right)\frac{1}{k^2+M^2-i\epsilon}.$$

The result is well known in renormalized perturbative field theory, and the resemblance of the two types of calculations will be evident. But we emphasize again the basic conceptual difference between the explicit consideration of particles propagating between sources, and the examination of the self-action of a field source, interpreted in particle language by means of renormalization. This distinction is closely connected with a conceivable mathematical ambiguity in our procedure, which we now point out in order to stress the consideration that has been implicitly used to resolve it.

Our discussion concerns two extended sources, $J^{\mu}{}_1(x)$ and $J^{\mu}{}_2(x)$, which are so arranged in space–time that particles emitted from J_2 are detected by J_1. The description of this situation is then presented in a way that is applicable to any spatio-temporal arrangement of the sources. As a mathematical procedure, we take a structure involving $J(k)$, $-k^2>(2m)^2$, and rewrite it in coordinate space. This is certainly not unique, in the absence of other information about $J(k)$. Thus, the factor $1/(k^2+M^2-i\epsilon)$ could be replaced by

$$-\frac{1}{k^2+M^2-i\epsilon}\frac{M^2}{k^2-i\epsilon}=\frac{1}{k^2+M^2-i\epsilon}-\frac{1}{k^2-i\epsilon},$$

without altering its behavior in the neighborhood of $-k^2=M^2>(2m)^2$. The outcome of that replacement, incidentally, would be the unrenormalized Green's function of perturbative field theory. But we do possess additional information about $J(k)$. It is embodied in the initial specification of the source in relation to the one-particle spectrum, which should not be altered in the process of extending the source to permit two-particle radiation. This spectral *normalization* emphasizes that sources are introduced in relation to specific physical circumstances and do not change their significance in those connections when the class of phenomena under examination is enlarged. The consistency of this physical attitude is emphasized by a mathematical fact. The possible alternative given by the factor $-M^2/(k^2-i\epsilon)$ is an unacceptable spatio-temporal generalization of the two-particle exchange calculation, since that $\bar{D}_+(k)$ does not exist. There is another class of alternative representations, the simplest example of which is produced by the additional factor $-k^2/M^2$. This $\bar{D}_+(k)$ differs from our actual choice only by a finite additive constant, which changes $\bar{D}_+(x-x')$ by an additive four-dimensional delta function. The vacuum amplitude is thereby modified by a phase factor, which has no physical consequences, since it contributes neither to the vacuum-persistance probability nor the particle-mediated coupling of different sources.

The introduction of the modified propagation function $\bar{D}_+$ and $\bar{G}_+$ must occur in all interaction aspects of the theory that can be analyzed into effective photon or electron sources. That does not exhaust the implications of two-particle exchanges, however. Physical mechanisms are not represented completely by the idealized sources $J^{\mu}(\xi)$, $\eta(x)$, and additional effects that are characteristic of the specific interaction will appear. The simplest illustration is supplied by the primitive interaction itself. The latter describes an extended η as the source of an electron and a photon, emitted under circumstances in which the particles have no opportunity to interact, and an extended J as the source of an analogously noninteracting electron and positron. (The long range of the Coulomb interaction may prevent the complete realization of this situation, however.) In the next stage we enlarge the physical circumstances to permit such interactions to occur. The electron-photon interaction can be visualized, in part, as the fusion and subsequent reseparation of the particles. Here is just the mechanism that substitutes $\bar{G}_+$ for G_+. It is not a complete description of the interaction effect, however, since the latter has another contribution demanded by the Bose statistics of the photons. The electron-positron interaction also contains two parts, which are connected by the statistics of the particles. One is the normal interaction of charged particles. It is the other, visualized as the annihilation and subsequent re-creation of the particles, that replaces D_+ with $\bar{D}_+$.

The skeletal description of electron-positron scattering is contained in the following contribution to the vacuum amplitude:

$$i\int(dx)(dx')\tfrac{1}{2}\psi_1(x)\gamma^0\gamma^\mu eq\psi_1(x)D_+(x-x')\times\tfrac{1}{2}\psi_2(x')\gamma^0\gamma_\mu eq\psi_2(x')+i\frac{1}{2}\int(dx)(dx')\psi_1(x)\gamma^0\gamma^\mu eq\times\psi_2(x)D_+(x-x')\psi_2(x')\gamma^0\gamma_\mu eq\psi_1(x'),$$

where the implicit sources η_1 and η_2 are single-particle detection and production sources, respectively. The primitive interaction describes an extended photon source as the effective electron-positron source given by

$$\psi_{2a}(x)\psi_{2b}(x')]_{\text{eff}}=-i\left[\int(d\xi)G_+(x-\xi)eq\gamma A(\xi)G_+(\xi-x')\gamma^0\right]_{ab}.$$

The vacuum-amplitude term that states the effect of interactions on the charged-particle emission from an extended photon source is thus

$$i\int(dx)\tfrac{1}{2}\psi_1(x)\gamma^0\gamma_\mu eq\psi_1(x)[\bar{A}_\mu(x)-A_\mu(x)]+\int(dx)\cdots(d\xi)\tfrac{1}{2}\psi_1(x)\gamma^0\gamma^\mu eqG_+(x-\xi)eq\gamma A(\xi)\times G_+(\xi-x')\gamma_\mu eq\psi_1(x')D_+(x-x'),$$

where $\bar{A}$ is the potential constructed from the photon source J with the aid of the modified propagation function $\bar{D}_+$. The physical context of the second term is made explicit by writing it as

$$\alpha\int\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k^0}dM^2d\omega_{p_1}d\omega_{p_1'}(2\pi)^4\delta(p_1+p_1'-k)A_\mu(k)\times\eta_1(-p_1)\gamma^0(m-\gamma p_1)eqI^\mu(-m-\gamma p_1')\eta(-p_1'),$$

with

$$I^\mu=\int d\omega_p d\omega_{p'}(2\pi)^4\delta(p+p'-k)\gamma^\nu(m-\gamma p)\times\gamma^\mu(-m-\gamma p')\gamma_\nu\frac{1}{(p-p_1)^2}.$$

The evaluation of I^μ is performed without difficulty by introducing the simplifications permitted by the projection factors $m-\gamma p_1$, $-m-\gamma p_1'$, and by using the rest frame of the timelike vector k^μ. One thereby encounters the elastic Coulomb scattering amplitude $1/(\mathbf{p}-\mathbf{p}_1)^2$, integrated over all solid angles. The logarithmic divergence of this integral is the formalism's reminder that the long-range Coulomb force is always effective, in contradiction with the original assumption that a noninteracting situation can be arranged. As in conventional scattering discussions, an elementary, but not ultimately satisfactory, remedy is to use a weakly screened potential. The momentum-space version of this device is the introduction of $1/[(\mathbf{p}-\mathbf{p}_1)^2+\mu^2]$, which is equivalent to the unphysical use of a small photon mass. It is clear that we have encountered another physical form of the infrared "catastrophe." Since long experience teaches that infrared problems are innocuous when the correct physical questions are asked, we do not want to trouble further with this point, for the moment. The result of the calculation is

$$I^\mu=\frac{1}{4\pi}\left[1-\left(\frac{2m}{M}\right)^2\right]^{-1/2}\left\{-\gamma^\mu f_1(M)+\frac{m}{M^2}i\sigma^{\mu\nu}k_\nu\right\},$$

where

$$f_1(M)=\left(1-2\frac{m^2}{M^2}\right)\ln\frac{M^2-4m^2}{\mu^2}-\tfrac{3}{2}+\left(\frac{2m}{M}\right)^2,$$

but we find it useful to present this in the form

$$I^\mu A_\mu(k)=\frac{1}{4\pi}\left[1-\left(\frac{2m}{M}\right)^2\right]^{-1/2}\frac{1}{M^2}\times\{f_1(M)\gamma J(k)-m\tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}(k)\}.$$

Here $F_{\mu\nu}$ is the field-strength tensor constructed from the vector potential A_μ.

The effect of an extended photon source, including skeletal-interaction modifications, is displayed in the vacuum-amplitude term

$$i\int(dx)\cdots(d\xi)\tfrac{1}{2}\psi_1(x)\gamma^0\left[eq\gamma^\mu F_1(x-\xi)\bar{A}_\mu(\xi)+\frac{eq}{2m}\frac{\alpha}{2\pi}\frac{1}{2}\sigma^{\mu\nu}F_2(x-\xi)\bar{F}_{\mu\nu}(\xi)\right]\psi_1(x),$$

where

$$F_1(k)=1-\frac{\alpha}{\pi}\int_{2m}^{\infty}\frac{dM}{M}\frac{f_1(M)}{[1-(2m/M)^2]^{1/2}}\frac{k^2}{k^2+M^2-i\epsilon},$$

$$F_2(k)=\int_{2m}^{\infty}\frac{dM}{M}\frac{(2m)^2}{[1-(2m/M)^2]^{1/2}}\frac{1}{k^2+M^2-i\epsilon},$$

and

$$F_1(0)=1,\quad F_2(0)=\int_{2m}^{\infty}d[1-(2m/M)^2]^{1/2}=1.$$

We have transcended our results somewhat by using the modified propagation function $\bar{D}_+$ everywhere. There are conceivable alternatives for $F_1(k)$ and $F_2(k)$, as illustrated by the substitution ($-i\epsilon$ is understood)

$$\frac{1}{k^2+M^2}\to\frac{1}{k^2+M^2}\left(-\frac{k^2}{M^2}\right)=\frac{1}{k^2+M^2}-\frac{1}{M^2}.$$

But such additional terms contribute to the primitive interaction and must be rejected. Our results describe those interaction-induced modifications of the primitive interaction, which are not accounted for by the introduction of $\bar{D}_+$, in terms of electric-charge and magnetic-moment form factors. The former maintains the initial identification of the electric charge e and the latter represents a total moment of $\alpha/2\pi$ magnetons. All this is in complete agreement with renormalized perturbative field theory. Incidentally, the field-theoretic results were first stated[4] with the slightly different parametrization illustrated by

$$F_2(k)=\int_0^1\frac{1}{1+(k^2/4m^2)(1-v^2)}.$$

Skeletal electron-photon scattering gives the following vacuum-amplitude contribution:

$$i\int(dx)(dx')A_1^{\mu}(x)\psi_1(x)\gamma^0 eq\gamma_\mu G_+(x-x')eq\gamma_\nu\psi_2(x')\times A_2^{\nu}(x')+i\int(dx)(dx')A_1^{\mu}(x')\psi_1(x)\gamma^0 eq\gamma_\nu\times G_+(x-x')eq\gamma_\mu\psi_2(x')A_2^{\nu}(x),$$

or, equivalently,

$$ie^2\int d\omega_k d\omega_p d\omega_{k'}d\omega_{p'}(2\pi)^4\times\delta(k+p-k'-p')J_1^{\mu}(-k)\eta_1(-p)\gamma^0\times(m-\gamma p)\left[\gamma_\mu\frac{1}{\gamma P+m}\gamma_\nu+\gamma_\nu\frac{1}{\gamma(P-k-k')+m}\gamma_\mu\right]\times(m-\gamma p')\eta_2(p')J_2^{\nu}(k'),$$

in which

$$P^{\mu}=k^{\mu}+p^{\mu}.$$

The primitive interaction describes an extended electron source as the effective electron-photon source

$$\eta_2(p')J_2^{\nu}(k')]_{\text{eff}}=-ieq\left[\gamma^{\nu}\frac{1}{\gamma P+m}-\frac{2P^{\nu}}{P^2+m^2}\right]\eta(P).$$

The implied interaction modifications of the emission from an extended electron source can be presented as

$$\frac{\alpha}{4\pi}\int\frac{(d\mathbf{P})}{(2\pi)^3}\frac{1}{2P^0}dM^2 d\omega_k d\omega_p(2\pi)^4\delta(k+p-P)\times J_1^{\mu}(-k)\eta_1(-p)\gamma^0(m-\gamma p)eq\left(1-\frac{m^2}{M^2}\right)\times\left[\langle E_\mu\rangle_P\frac{1}{\gamma P+m}-\langle S_\mu\rangle\frac{2}{P^2+m^2}\right]\eta(P).$$

Here

$$\begin{pmatrix}E_\mu\\ S_\mu\end{pmatrix}=\left[\gamma_\mu\frac{1}{\gamma P+m}\gamma_\nu+\gamma_\nu\frac{1}{\gamma(P-k-k')+m}\gamma_\mu\right]\times(m-\gamma p')\begin{pmatrix}\gamma^{\nu}\\ P^{\nu}\end{pmatrix}$$

are visualized as describing scattering modifications of the photon that is radiated by the electron, or directly by the source, respectively, and

$$\langle\cdots\rangle_P=\frac{8\pi M^2}{M^2-m^2}\int d\omega_{k'}d\omega_{p'}(2\pi)^4\delta(k'+p'-P)\cdots$$

is an average over all scattering angles.

A calculation in the center-of-mass frame gives

$$\langle E_\mu\rangle_P=-\frac{m}{M^2}\gamma_\mu\gamma k+\left[\frac{1-\chi(M)}{M^2}-\frac{2}{M^2-m^2}\right]\gamma_\mu\gamma k(\gamma P+m)$$

and

$$\langle S_\mu\rangle_P=\frac{m}{2M^2}\chi(M)\gamma_\mu\gamma k(\gamma P+m),$$

where

$$\chi(M)=\left(\frac{M^2}{M^2-m^2}\right)^2\left[\frac{M^2+m^2}{M^2}-\frac{4m^2}{M^2-m^2}\ln\frac{M}{m}\right]$$

varies from a value of $\frac{1}{3}$ at $M=m$ to unity as $M\to\infty$. The projection factor $m-\gamma p$ has been exploited to simplify the results. Note that the relations

$$k^{\mu}\langle E_\mu\rangle_P=0,\quad k^{\mu}\langle S_\mu\rangle_P=0$$

are a property of the combination of interaction processes that occur in the individual matrices. For that

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[4] J. Schwinger, Phys. Rev. **76**, 790 (1949).

reason we have not separated the two processes, despite the possibility of using one of them to identify $\bar{G}_+(P)$.

This account of the propagation of excitations with variable mass M is presented in space-time by the formula

$$i\frac{\alpha}{2\pi}\int(dx)(dx')F_1^{\mu\nu}(x)\psi_1(x)\gamma^0\frac{eq}{2m}\frac{1}{2}\sigma_{\mu\nu}\left[G_1(x-x')\psi(x')-G_2(x-x')\psi_-(x')+\frac{1}{m}G_3(x-x')\eta(x')\right],$$

where

$$[\gamma^\mu(1/i)\partial_\mu+m]\psi(x)=\eta(x),$$
$$[\gamma^\mu(1/i)\partial_\mu-m]\psi_-(x)=\eta(x),$$

and

$$G_1(p)=2m^2\int_m^\infty\frac{dM}{M}\left(1-\frac{m^2}{M^2}\right)\frac{1}{p^2+M^2-i\epsilon},$$
$$G_2(p)=2m^2\int_m^\infty\frac{dM}{M}\left(1-\frac{m^2}{M^2}\right)\chi(M)\frac{1}{p^2+M^2-i\epsilon},$$
$$G_3(p)=-G_1(p)+G_2(p)+4m^2\int_m^\infty\frac{dM}{M}\frac{1}{p^2+M^2-i\epsilon}.$$

We make contact with previous considerations if we extrapolate this formula to those circumstances in which $\eta(p)$ is a simple electron source and $J_1^\mu(k)$ is extended sufficiently ($0<k^2\ll m^2$) to ensure the existence of the interaction term. Since

$$-p^2=m^2\colon G_1(p)=\int_m^\infty d\left(\frac{m^2}{M^2}\right)=1,$$

the contribution that refers specifically to electrons and the extended photon source is

$$i\int(dx)\psi_1(x)\gamma^0\frac{eq}{2m}\frac{\alpha}{2\pi}\frac{1}{2}\sigma_{\mu\nu}F_1^{\mu\nu}(x)\psi(x),$$

which reaffirms the additional magnetic moment of $\alpha/2\pi$ magnetons.

Two-particle exchange also contributes new processes to the interaction skeleton. The vacuum-amplitude term

$$i\int(dx)(dx')\tfrac{1}{2}\psi(x)\gamma^0eq\gamma A(x)G_+(x-x')eq\gamma A(x')\psi(x')$$

can describe a suitable combination of two-photon sources as an effective electron-positron source. The latter is

$$\eta_a(x)\eta_b(x')]_{\text{eff}}=-i[eq\gamma A(x)G_+(x-x')eq\gamma A(x')\gamma^0]_{ab}.$$

The two-particle exchange between a pair of such sources, designated as J_1 and J_2, is represented by the vacuum-amplitude term

$$-\frac{1}{2}\int(dx)\cdots(dx''')\times[\gamma^0eq\gamma A_1(x'')G_+(x''-x)eq\gamma A_1(x)]_{ca}\times G_+(x-x')_{ab}G_+(x''-x''')_{cd}\times[eq\gamma A_2(x')G_+(x'-x''')eq\gamma A_2(x''')\gamma^0]_{bd},$$

which is equivalent to

$$-\frac{1}{2}\int(dx)\cdots(dx''')\operatorname{tr}[eq\gamma A_1(x)G_+(x-x')\times eq\gamma A_1(x')G_+(x'-x'')eq\gamma A_2(x'')\times G_+(x''-x''')eq\gamma A_2(x''')G_+(x'''-x)].$$

This result represents the physical process in which colliding photon beams create an electron-positron pair which is eventually detected by observation of its two-photon annihilation radiation. But it appears in a form that permits immediate generalization to the wider class of spatio-temporal arrangements in which four photon sources interact. When emission and absorption sources are united in the general photon source, this process can be presented as the following contribution to w:

$$w_{0,4}=i\frac{1}{8}\int(dx)\cdots(dx''')\operatorname{tr}[eq\gamma A(x)G_+(x-x')\cdots\times eq\gamma A(x''')G_+(x'''-x)].$$

Here is the first of an infinite set of interaction skeleton contributions that do not involve charged particles.

Of course, the extrapolation used here is meaningful only if it encounters no barriers. The one situation that is not contemplated in the original physical picture is an overlap of the interaction regions of all four photons. This raises the mathematical question whether the multiple integral exists when all coordinates range over a small common volume. An equivalent question refers to the existence of the momentum integral

$$\int(dp)\frac{1}{(p^2+m^2-i\epsilon)^4}\operatorname{tr}(\gamma A\gamma p)^4.$$

We answer this affirmatively of any integration method that preserves Lorentz covariance, for then, effectively,

$$A_\mu A_\nu A_\lambda A_\kappa\to 1/24(g_{\mu\nu}g_{\lambda\kappa}+g_{\mu\lambda}g_{\nu\kappa}+g_{\mu\kappa}g_{\nu\lambda})(A^2)^2$$

and

$$\operatorname{tr}(\gamma A\gamma p)^4=0.$$

The generalization to all interaction skeletons of this type is easily accomplished. Consider the creation of an electron-positron pair through the collaboration of $m\geq 2$ photons and its detection by annihilation into

$n-m\geq 2$ photons. To present this vacuum amplitude conveniently we extend the trace notation to space-time coordinates and get

$$-\tfrac{1}{2}\,\mathrm{Tr}[(eq\gamma A_1G_+)^{n-m}(eq\gamma A_2G_+)^m].$$

The corresponding united-source description is

$$-\frac{1}{2n}\,\mathrm{Tr}(eq\gamma AG_+)^n,$$

since there are n equivalent positions in the trace for beginning the required sequence of products. All these contributions to the interaction skeleton are combined in

$$w(A)=\tfrac{1}{2}i\sum_{n=4}^{\infty}\frac{1}{n}\,\mathrm{Tr}(eq\gamma AG_+)^n=-\tfrac{1}{2}i\,\mathrm{Tr}\ln'(1-eq\gamma AG_+)$$
$$=-\tfrac{1}{2}i\ln\det{}'(1-eq\gamma AG_+),$$

where ln′, and the associated modified determinant,[5] is defined by omitting the $n=2$ term (the appearance of q assures the vanishing of all odd n terms). As the last reference indicates, this is a known structure in the field-theoretic description of vacuum polarization by an external field. The existence of every term in the series for $w(A)$ gives substance to the formal property of gauge invariance,

$$w(A)=w(A+\partial\lambda).$$

Its differential version

$$\partial^\mu j_\mu(\xi,A)=0,$$

with

$$j_\mu(\xi,A)=\delta w(A)/\delta A^\mu(\xi),$$

validates the identification of the latter as the effective photon source generated by other photons in the absence of charged-particle sources.

Any type of particle exchange among a specified number of sources can be used to infer the general interaction that connects these sources. Concerning $w_{0,4}$, for example, consider the exchange of two electron-positron pairs between two sets of extended photon sources J_1, $J_{1'}$ and J_2, $J_{2'}$. Those processes that can be reduced to single pair exchange between sources restate the modification of the D_+ function. But other processes involve all four sources in an irreducible way. These are the ones in which the two particles emitted by J_2 or $J_{2'}$ are singly absorbed by J_1 and $J_{1'}$. The corresponding vacuum-amplitude term is

$$-\tfrac{1}{4}\,\mathrm{Tr}[eq\gamma A_1G_+eq\gamma A_2G_+eq\gamma A_1G_+eq\gamma A_2G_+],$$

where A_1 and A_2 combine the effects of J_1, $J_{1'}$ and J_2, $J_{2'}$, respectively. Here each G_+ function refers to a particle propagating between a specific pair of sources. The combined-source presentation of this effect is just

$$-\tfrac{1}{8}\,\mathrm{Tr}(eq\gamma AG_+)^4,$$

and its extrapolation to arbitrary spatio-temporal source arrangements recovers $w_{0,4}$. In a variation of this procedure we consider the exchange between extended sources of one particle pair, with the individual particles deflected by two other sources, which supply spacelike momenta.

The physical processes mentioned above have a certain calculational simplicity, since the kinematics of the four-particle states is uniquely determined when the momenta supplied or absorbed by the sources are specified. Accordingly, the required integrations are elementary. The problem is thus posed of producing an extrapolation to more general spatio-temporal circumstances by using the integrated result directly rather than the propagation-function form. We shall only sketch our response here. Consider, for example, the first-described process involving two particle pairs and suppose that each of the four sources $J(k)$ is characterized by $-k^2=M^2$ with a common value of $M>2m$. Apart from monomial functions of the k's that are referred to vectorial-source directions, the integral is an invariant function of the source momenta in two independent combinations. These can be chosen as the positive quantities $-(k_2+k_2')^2$ and $-(k_1+k_2)^2$. Now examine the two independent displacements of the sources for which the effective coordinate dependence is governed by the momenta k_2+k_2' and k_1+k_2. For each of these a space-time extrapolation can be performed by the standard introduction of the function Δ_+ or the equivalent momentum-spectral form. If inspection of the resulting double-spectral structure confirms that reference to extended sources can now be removed, an amplitude for the scattering of light by light is obtained. One will recognize in this account an elementary approach to a calculational procedure that has previously required the relatively elaborate Mandelstam-Cutkosky analytical apparatus.[6]

REPEATED TWO-PARTICLE EXCHANGE

Once experimental circumstances permit particles to interact, there can be no effective control over the number of such interactions. In this section we consider some examples of the unlimited repetition of the two-particle-exchange processes.

During the discussion of the electron-positron interaction that modifies the emission by an extended photon source it was noted that part of the interaction effect could be described as the fusion and reseparation of the two particles. The latter stage can be viewed as a virtual source and the whole process indefinitely iterated. To

[5] J. Schwinger, Phys. Rev. **93**, 615 (1954).

[6] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); R. Cutkosky, J. Math. Phys. **1**, 429 (1960).

formulate this picture we use the general definition

$$\frac{1}{i}\frac{\delta}{\delta J^\mu(\xi)}\langle 0_+|0_-\rangle^{\eta J}=A_\mu(\xi)\langle 0_+|0_-\rangle^{\eta J},$$

which identifies $A_\mu(\xi)$ as a measure of the total effect experienced by a detection source at the given point. The previous discussion of extended photon sources exhibits $A_\mu(\xi)$ as the superposition of effects attributable to photons,

$$A_\mu(\xi)]_{\text{photon}}=\int(d\xi')D_+(\xi-\xi')J_\mu(\xi'),$$

and to electron-positron pairs,

$$A_\mu(\xi)]_{\text{pair}}=\int(d\xi_1)\cdot\cdot(d\xi')D_+(\xi-\xi_1)\times P_{\mu\nu}(\xi_1-\xi_2)D_+(\xi_2-\xi')J^\nu(\xi').$$

In the latter, $P_{\mu\nu}(\xi-\xi')$ is the function identified first, by considering electron-positron propagation, as

$$\tfrac{1}{2}i\ \mathrm{tr}[eq\gamma_\mu G_+(\xi-\xi')eq\gamma_\nu G_+(\xi'-\xi)]$$

and then extrapolated to general space-time circumstances. Its momentum-space description is, effectively,

$$P_{\mu\nu}(k)=g_{\mu\nu}P(k),$$

with

$$P(k)=(k^2)^2\frac{2\alpha}{3\pi}\int_{2m}^\infty\frac{dM}{M}\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2}\times\left(1+\frac{2m^2}{M^2}\right)\frac{1}{k^2+M^2-i\epsilon}.$$

The generalization that includes endlessly repeated interactions is introduced by writing

$$A_\mu(\xi)=\int(d\xi')D_+(\xi-\xi')J_\mu(\xi')+\int(d\xi_1)(d\xi')D_+(\xi-\xi_1)P(\xi_1-\xi')A_\mu(\xi'),$$

in which P is now considered to describe the last interaction. We continue to define a modified propagation function by

$$A_\mu(\xi)=\int(d\xi')\bar{D}_+(\xi-\xi')J_\mu(\xi'),$$

and thus

$$\bar{D}_+(\xi-\xi')=D_+(\xi-\xi')+\int(d\xi_1)(d\xi_2)\times D_+(\xi-\xi_1)P(\xi_1-\xi_2)\bar{D}_+(\xi_2-\xi')$$

or

$$\bar{D}_+(k)=D_+(k)+D_+(k)P(k)\bar{D}_+(k).$$

The new version of $\bar{D}_+$ is, therefore,

$$\bar{D}_+(k)=\frac{1}{k^2-i\epsilon}\left\{1-k^2\frac{2\alpha}{3\pi}\int_{2m}^\infty\frac{dM}{M}\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2}\times\left(1+\frac{2m^2}{M^2}\right)\frac{1}{k^2+M^2-i\epsilon}\right\}^{-1}$$

$$=\frac{1}{k^2-i\epsilon}+\int dM\frac{a(M)}{k^2+M^2-i\epsilon},$$

in which the real positive function $a(M)$ is easily exhibited. The latter form assumes that $\bar{D}_+(k)$ has only the singularities that express the physical mass spectrum. But it is clear that $\bar{D}_+(k)$ will have a nonphysical singularity for a sufficiently large spacelike value of k^2. This singularity is approximately located at

$$k^2=m^2\exp[(3\pi/\alpha)+5/3],$$

which is fantastically large. Here is no practical limitation to the use of extended sources, and one cannot overlook the possibility that the nonphysical singularity is the formalism's reminder that essential interaction aspects are lacking in this partial result.

The analogous discussion for extended electron sources begins with the definition

$$\frac{1}{i}\frac{\delta_l}{\delta\eta(x)\gamma^0}\langle 0_+|0_-\rangle^{\eta J}=\psi(x)\langle 0_+|0_-\rangle^{\eta J}.$$

The superposition of physical effects is conveyed by

$$\psi(x)=\int(dx')G_+(x-x')\eta(x')+\int(dx_1)(dx')\times G_+(x-x_1)M(x_1-x')\psi(x'),$$

where our previous results are expressed as

$$M(p)=(\gamma p+m)^2\frac{\alpha}{4\pi}\int_m^\infty\frac{dM}{M}\left(1-\frac{m^2}{M^2}\right)\times\left[\frac{1}{\gamma p+M-i\epsilon}+\frac{1}{\gamma p-M+i\epsilon}\right].$$

The modified propagation function defined by

$$\psi(x)=\int(dx')\bar{G}_+(x-x')\eta(x')$$

obeys

$$\bar{G}_+(x-x')=G_+(x-x')+\int(dx_1)(dx_2)\times G_+(x-x_1)M(x_1-x_2)\bar{G}_+(x_2-x')$$

or

$$\bar{G}_+(p)=G_+(p)+G_+(p)M(p)\bar{G}_+(p).$$

Thus

$$\bar{G}_+(p)=\frac{1}{\gamma p+m-i\epsilon}\left\{1-(\gamma p+m)\frac{\alpha}{4\pi}\int_m^\infty\frac{dM}{M}\left(1-\frac{m^2}{M^2}\right)\times\left[\frac{1}{\gamma p+M-i\epsilon}+\frac{1}{\gamma p-M+i\epsilon}\right]\right\}^{-1}$$

$$=\frac{1}{\gamma p+m-i\epsilon}+\int dM\left[\frac{r_+(M)}{\gamma p+M-i\epsilon}+\frac{r_-(M)}{\gamma p-M+i\epsilon}\right],$$

where $r_\pm(M)$ are real positive numbers. Again there is a nonphysical singularity for very large spacelike momenta. It is located near

$$p^2=m^2\exp[(4\pi/\alpha)+1].$$

As a preliminary to discussing another repeated two-particle-exchange process, let us note that the charged-particle structure of the primitive interaction, the effective current

$$\int(dx)(dx')\tfrac{1}{2}\eta(x)\gamma^0G(x,x';\xi)^\mu\eta(x'),$$

can be presented as

$$\tfrac{1}{2}\psi(\xi)\gamma^0eq\gamma^\mu\psi(\xi)+\int(dx)\partial_\nu[\tfrac{1}{2}\psi(x)\gamma^0eq\gamma^\nu\psi(x)]f^\mu(x-\xi),$$

if one uses a form of $f^\mu(x-\xi)$ that does not require detailed reference to an individual source. The skeletal description of coupling among four electron sources can then be converted into

$$w_{4,0}=\frac{1}{2}\int(d\xi)(d\xi')\tfrac{1}{2}\psi(\xi)\gamma^0eq\gamma^\mu\psi(\xi)\times\mathcal{G}_+(\xi-\xi')_{\mu\nu}\tfrac{1}{2}\psi(\xi')\gamma^0eq\gamma^\nu\psi(\xi'),$$

where, in momentum space,

$$\mathcal{G}_+(k)_{\mu\nu}=\mathcal{G}_+(k)_{\nu\mu}=[g_{\mu\lambda}-ik_\mu f_\lambda(k)]g^{\lambda\kappa}D_+(k)[g_{\kappa\nu}-f_\kappa(k)ik_\nu].$$

This function obeys

$$f^\mu(k)\mathcal{G}_+(k)_{\mu\nu}=0$$

in consequence of the defining property of $f^\mu(k)$,

$$ik_\mu f^\mu(k)=1.$$

The behavior of two charged particles will now be examined with the aid of the definition

$$\frac{1}{i}\frac{\delta_l}{\delta\eta(x_1)\gamma^0}\frac{1}{i}\frac{\delta_l}{\delta\eta(x_2)\gamma^0}\langle 0_+|0_-\rangle^{\eta J}=\psi(x_1,x_2)\langle 0_+|0_-\rangle^{\eta J}.$$

The consideration of $w_2+w_{4,0}$ gives the equation

$$\psi(x_1,x_2)=\psi(x_1)\psi(x_2)+\int(dx_1')\cdots(dx_2'')G_+(x_1-x_1')\times G_+(x_2-x_2')I(x_1',x_2';x_1''x_2'')\psi(x_1'',x_2''),$$

where

$$\psi(x_1)\psi(x_2)=\int(dx_1')(dx_2')G_+(x_1-x_1')\times G_+(x_2-x_2')\eta(x_1')\eta(x_2')$$

and

$$I(x_1,x_2;x_1',x_2')=-i(eq\gamma^\mu)_{11'}\delta(x_1-x_1')\times\mathcal{G}_{\mu\nu}(x_1-x_2)(eq\gamma^\nu)_{22'}\delta(x_2-x_2')-i\tfrac{1}{2}(eq\gamma^\mu\gamma^0)_{12}\delta(x_1-x_2)\mathcal{G}_{\mu\nu}(x_1-x_1')\times(\gamma^0\gamma^\nu eq)_{1'2'}\delta(x_1'-x_2').$$

We have recognized that the interaction of the two particles produces a virtual two-particle source, and the iteration of these effects is accomplished by writing $\psi(x_1,x_2)$ in the interaction term, rather than $\psi(x_1)\psi(x_2)$. Thus I appears here to describe the last interaction. Incidentally, a physically acceptable covariant definition for $f^\mu(x-\xi)$ can now be given relative to the timelike direction supplied by the total momentum of both particles.

The equation for $\psi(x_1,x_2)$ can also be presented as

$$\left[\left(\gamma\frac{1}{i}\partial+m\right)_1\left(\gamma\frac{1}{i}\partial+m\right)_2-I\right]\psi(x_1,x_2)=\eta(x_1)\eta(x_2),$$

where I is understood as an integral operator. The modified two-particle propagation function defined by

$$\psi(x_1,x_2)=\frac{1}{2}\int(dx_1')(dx_2')G_+(x_1,x_2;x_1'x_2')\eta(x_1')\eta(x_2')$$

then obeys the (two-particle Green's-function) equation

$$\left[\left(\gamma\frac{1}{i}\partial+m\right)_1\left(\gamma\frac{1}{i}\partial+m\right)_2-I\right]G_+(x_1,x_2;x_1'x_2')=\delta(x_1-x_1')\delta(x_2-x_2')-\delta(x_1-x_2')\delta(x_2-x_1').$$

This familiar structure[7] contains an improved description of electron-electron and electron-positron scattering. But it also alters the nature of the physical system under discussion, for it predicts an infinite set of electrically neutral bosons. These are the positronium states, which are all stable at this stage of the descrip-

[7] As a modest contribution to the history of science I give a time-ordered list of the papers in which differential equations of this form were first proposed; Y. Nambu, Progr. Theoret. Phys. (Kyoto) **5**, 614 (1950); J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951); **37**, 455 (1951); M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951); E. Salpeter and H. Bethe, *ibid.* **84**, 1232 (1951). The discussion of the text emphasizes that the inhomogeneous differential equation of the second reference is essential for a clear physical interpretation of the formalism.

tion. We now discuss the identification of positronium sources.

The introduction of relative coordinates x and center-of-mass coordinates X,

$$x=x_1-x_2, \quad X=\tfrac{1}{2}(x_1+x_2),$$

enables us to write

$$G_+(x_1,x_2;x_1',x_2')=\int \frac{(dP)}{(2\pi)^4}e^{iP(X-X')}G_P(x,x'),$$

where

$$\{[\gamma_1(\tfrac{1}{2}P-i\partial)+m][\gamma_2(\tfrac{1}{2}P+i\partial)+m]-I_P\}G_P(x,x') =\delta(x-x')-\delta(x+x').$$

The transposition of discrete indices that accompanies $x_1'\leftrightarrow x_2'$ is understood in the second delta function. The interaction function for a prescribed total momentum is given by

$$I_P(x,x')=-i(eq\gamma^\mu)_{11'}\mathcal{G}_+(x)_{\mu\nu}(eq\gamma^\nu)_{22'}\delta(x-x') -i\tfrac{1}{2}(eq\gamma^\mu\gamma^0)_{12}\delta(x)\mathcal{G}_+(P)_{\mu\nu}(\gamma^0\gamma^\nu eq)_{1'2'}\delta(x').$$

The latter term is specific to positronium. It incorporates the extended photon-source description of the pair-creation and annihilation processes. With regard to $\mathcal{G}_+(P)_{\mu\nu}$, there would seem to be no objection, for the discussion of electrically neutral positronium, to making the simple choice

$$if^\mu(k)=k^\mu/k^2,$$

which gives

$$\mathcal{G}_+(P)_{\mu\nu}=(g_{\mu\nu}-P_\mu P_\nu/P^2)/P^2.$$

In the rest frame of the total momentum, with $P^0=M$, the nonvanishing components of this function are

$$\mathcal{G}_+(P)_{kl}=-\delta_{kl}/M^2,$$

and the factor $(\gamma_k\gamma^0)_{12}\delta(x)$ makes it evident that only the ^3S states are involved.

Eigenfunctions of the internal motion are defined by the homogeneous two-particle equation

$$\{[\gamma_1(\tfrac{1}{2}P'-i\partial)+m][\gamma_2(\tfrac{1}{2}P'+i\partial)+m]-I_{P'}\} \times\varphi_{P'\alpha}(x)=0,$$

where α is intended as a degeneracy index and $-P'^2=M'^2$ refers to a mass eigenvalue. There is an adjoint equation implied by the symmetry property

$$[\gamma_1^0\gamma_2^0I_{-P}(x',x)]^T=\gamma_1^0\gamma_2^0I_P(x,x').$$

Written in terms of left-acting derivatives, it is

$$\varphi_{-P'\alpha}(x)\gamma_1^0\gamma_2^0\{[\gamma_1(\tfrac{1}{2}P'+i\partial)+m] \times[\gamma_2(\tfrac{1}{2}P'-i\partial)+m]-I_{P'}\}=0.$$

The usual subtraction procedure supplies an orthogonality property for two different eigenfunctions, in the center-of-mass frame. We anticipated the correct normalization condition by writing

$$-\tfrac{1}{8}i\int(dx)\varphi_{-M'\alpha}(x)\bigg[\tfrac{1}{2}(M'+M'')-\gamma_1^0(-\gamma_1\cdot i\nabla+m) -\gamma_2^0(\gamma_2\cdot i\nabla+m)-2\frac{M'+M''}{M'^2M''^2}\mathcal{J}\bigg]\varphi_{M'\beta}(x) =M'\delta_{M'M''}\delta_{\alpha\beta},$$

in which the M-dependent part of $\gamma^0{}_1\gamma^0{}_2I_P$ is expressed as $-(1/M^2)\mathcal{J}$. Consideration of the Green's-function equation in the rest frame, with M near a discrete eigenvalue M', gives

$$\tfrac{1}{2}(M-M')\int(dx)\varphi_{-M'\alpha}(x)\bigg[\tfrac{1}{2}(M+M') -\gamma_1^0(-\gamma_1\cdot i\nabla+m)-\gamma_2^0(\gamma_2\cdot i\nabla+m)-2\frac{M+M'}{M^2M'^2}\mathcal{J}\bigg] \times G_M(x,x')=2\varphi_{-M'\alpha}(x')\gamma_1^0\gamma_2^0,$$

from which we infer that

$$M\sim M':G_M(x,x')\sim i\sum_\alpha\frac{\varphi_{M'\alpha}(x)\varphi_{-M'\alpha}(x')\gamma_1^0\gamma_2^0}{M'^2-M^2}.$$

Thus, the singularity structure of the internal Green's function is given by

$$G_P(x,x')\sim i\sum_{M'\alpha}\frac{\varphi_{P'\alpha}(x)\varphi_{-P'\alpha}(x')\gamma_1^0\gamma_2^0}{P^2+M'^2},$$

for $P^0>0$. The similar structure for $P^0<0$ follows from the symmetry property

$$[\gamma_1^0\gamma_2^0G_{-P}(x',x)]^T=\gamma_1^0\gamma_2^0G_P(x,x').$$

One gives a geometrical expression to the bounded nature of the internal motion, in which electron or positron cannot exist freely, by replacing the hyperbolic internal Minkowski space with a Euclidean space through the substitution

$$x^0\rightarrow -ix_4.$$

Related transformations are

$$(dx)\rightarrow -i(dx)_E, \quad \delta(x-x')\rightarrow i\delta(x-x')_E$$

and

$$D_+(x-x')\rightarrow iD(x-x')_E,$$
$$D(x-x')_E=[(2\pi)^2(x-x')^2]^{-1}>0.$$

We suggest the effect of this transformation on $G_M(x,x')$ by the simplified notation $G_M(x_4)$. The Green's-function differential equation obeyed by $-iG_M(x_4)$ implies the following Euclidean reality property:

$$[-iG_M(x_4)]^*=-iG_{-M}(-x_4),$$

in which the spatial coordinates are inert. Accordingly, the Euclidean eigenfunctions can be so chosen that

$$[\varphi_{M'\alpha}(x_4)]^* = \varphi_{-M'\alpha}(-x_4).$$

The orthonormality condition then appears as

$$-\frac{1}{8}\int (dx)_E \varphi_{M'\alpha}(-x_4)^* \Big[\tfrac{1}{2}(M'+M'') - \gamma_1{}^0(-i\gamma_1\cdot\nabla+m) - \gamma_2{}^0(i\gamma_2\cdot\nabla+m) - 2\frac{M'+M''}{M'^2M''^2}\mathcal{G}\Big] \times \varphi_{M''\beta}(x_4) = M'\delta_{M'M''}\delta_{\alpha\beta}.$$

In conformity with the right-hand side, the left-hand member is an element of a Hermitian matrix, labeled by $M'\alpha$ and $M''\beta$. The diagonal elements, in particular, are real. The positiveness of these numbers is required for the particle-source interpretation.

Let production and detection sources be introduced into the defining equation for $\psi(x_1,x_2)$,

$$\frac{1}{i}\frac{\delta_l}{\delta\eta_1(x_1)\gamma^0}\frac{1}{i}\frac{\delta_l}{\delta\eta_1(x_2)\gamma^0}\langle 0_+|0_-\rangle^{\eta J} = \frac{1}{2}\int (dx_1')(dx_2') \times G_+(x_1,x_2;x_1',x_2')\eta_2(x_1')\eta_2(x_2')\langle 0_+|0_-\rangle^{\eta J}.$$

For sufficiently large $X^0 - X^{0\prime} > 0$,

$$-G_+(x_1,x_2;x_1',x_2') \sim \sum_{M'\alpha}\int \frac{(d\mathbf{P})}{(2\pi)^3}\frac{1}{2P^0}e^{iP(X-X')} \times \varphi_{P\alpha}(x)\varphi_{-P\alpha}(x')\gamma_1{}^0\gamma_2{}^0,$$

in which we retain only the discrete mass spectrum. With the aid of the definitions

$$i\zeta_{M'P\alpha} = \left[\frac{(d\mathbf{P})}{(2\pi)^3}\frac{1}{2P^0}\right]^{1/2}\frac{1}{2}\int (dx_1)(dx_2) \times e^{-iPX}\varphi_{-P\alpha}(x)\gamma_1{}^0\gamma_2{}^0\eta(x_1)\eta(x_2)$$

and

$$i\zeta^*{}_{M'P\alpha} = \left[\frac{(d\mathbf{P})}{(2\pi)^3}\frac{1}{2P^0}\right]^{1/2}\frac{1}{2}\int (dx_1)(dx_2) \times \eta(x_2)\eta(x_1)\gamma_1{}^0\gamma_2{}^0\varphi_{P\alpha}(x)e^{iPX}$$

we infer the existence of a contribution to w, w_{pos}, such that

$$w_{\text{pos}} = i\sum_{M'}\sum_{P\alpha}\zeta_{1M'P\alpha}{}^*\zeta_{2M'P\alpha} + \cdots$$

The unwritten terms are the quadratic functions of the individual sources that are implied by the total source concept. This interpretation of the totally commutative quantities $\zeta_{M'P\alpha}$, $\zeta_{M'P\alpha}{}^*$ in relation to sources of the various bosons identified by the mass M', depends upon the validity of the complex-conjugation connection

$$i\int (dx)\varphi_{-P\alpha}(x)\gamma_1{}^0\gamma_2{}^0\eta_P(x) = \left[i\int (dx)\varphi_{P\alpha}(x)\gamma_1{}^0\gamma_2{}^0\eta_P(x)^*\right]^*,$$

where

$$\eta_P(x) = \int (dX)e^{-iPX}\eta(X+\tfrac{1}{2}x)\eta(X-\tfrac{1}{2}x).$$

We refer again to the bounded internal motion of the particles created by these fermion sources. In the rest frame of the boson, the internal Euclidean transformation gives

$$\int (dx)_E[\varphi_{-M'\alpha}(x_4) - \varphi_{M'\alpha}(-x_4)^*]\gamma_1{}^0\gamma_2{}^0\eta_{M'}(x_4) = 0,$$

which is indeed satisfied.

The next stage of dynamical evolution introduces the photon interactions of the positronium particles. Its consequences include the instability of these particles and associated changes in mass values. That discussion will be deferred to another publication.