Kinematic Constraints, Crossing, and the Reggeization of Scattering Amplitudes

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We investigate the constraints among helicity amplitudes which hold at the boundary of the crossed-channel physical region, like the one discovered for nucleon-nucleon scattering by Goldberger, Grisaru, MacDowell, and Wong, and we develop a general scheme for computing these rules with arbitrary external spins. For equal-mass elastic scattering these constraints hold at $s=0$; a catalog of them for several low-spin cases is presented. Complete knowledge of these conditions is needed to discuss, for a general scattering amplitude, a question first raised by Gell-Mann *et al.* in connection with electrodynamics, namely, under what conditions must a given expression for an S matrix (e.g., the perturbation expansion of a field theory) be analytic in the angular momentum? Mandelstam has argued that this "Reggeization" of the vector-spinor amplitude is a consequence of kinematic constraints, and hence independent of the perturbation expansion. He discussed only the equal-mass, $j=\frac{1}{2}$ case. We generalize his argument to unequal masses and to arbitrary spins and angular momenta, and discuss, case by case, the necessity of Reggeization of a list of low-spin amplitudes. We find in particular that under rather general assumptions a spin-$\frac{1}{2}$ particle must lie on a Regge trajectory in a large class of amplitudes, and that the π^+p amplitude is analytic in j in perturbation theory.

I. INTRODUCTION

A FEW years ago, Gell-Mann, Goldberger, Low, Marx, Singh, and Zachariasen,[1–5] in various combinations, studied the conditions under which an elementary particle in conventional field theory can lie on a Regge trajectory for all values of the coupling constant. Following their work, Mandelstam[6] obtained further insight in a beautiful paper which set up a simple criterion for studying this question independently of the perturbation expansion. In particular, his work explained the "miracle" which occurs in electrodynamics, namely, the factorization of Regge residues in low orders of perturbation theory for the spinor-vector scattering amplitude. It also showed why the miracle does not have to happen in some other cases, such as scalar-vector scattering.

The essence of Mandelstam's argument is the counting of kinematical constraints on the several amplitudes, e.g., the helicity amplitudes, which describe the scattering. The result is different for different processes because these kinematical constraints change with the spins of the scattering particles. To simplify this "counting," Mandelstam simplified the kinematics by studying the special case in which the two particles, e.g., vector and spinor, have the same mass.

The motivation of our present work was to answer the following question: Does the "counting" always come out the same whether or not the masses of the two scattering particles are equal? The answer, in general, is that it does not. In the cases studied by Mandelstam, however, it turns out that his conclusions remain unaltered; but we must consider this a minor miracle, for we do not known a general criterion for this accident to happen.

Our technical point, which has other applications also, concerns kinematical constraints on helicity amplitudes $T_{\mu_1\mu_2\mu_3\mu_4}(s,t)$ at $s=0$, $s=(m_1\pm m_2)^2$, and $s=(m_3\pm m_4)^2$. The number of poles or zeroes these amplitudes may have at these points is well known; they follow from the crossing relations, and are systematically documented, for example by Hara,[7] or, more recently, by Wang.[8] Equal mass scattering amplitudes have further $s=0$ constraints, which take the form of relations among several $T_{\mu_1\mu_2;\mu_3\mu_4}$. The first example was discovered by Goldberger, Grisaru, MacDowell, and Wong (GGMW)[9] in their classic paper on nucleon-nucleon scattering. In the present work we find the analogous conditions for more general spin cases. We shall call all the relations we find generalized GGMW, or GGMW-type conditions.

GGMW discovered their condition by writing out the relations connecting the five scalar invariant ampli-

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[1] M. Gell-Mann and M. L. Goldberger, Phys. Rev. Letters **9**, 275 (1962); **10**, 39 (1963).

[2] M. Gell-Mann, M. L. Goldberger, F. E. Low, and F. Zachariasen, Phys. Letters **4**, 265 (1963).

[3] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

[4] M. Gell-Mann, M. L. Goldberger, V. Singh, and F. Zachariasen, Phys. Rev. **133**, B161 (1964).

[5] See also M. Gell-Mann, M. L. Goldberger, and F. E. Low, Rev. Mod. Phys. **36**, 640 (1964); H. Cheng and T. T. Wu, Phys. Rev. **140**, B465 (1956).

[6] S. Mandelstam, Phys. Rev. **137**, B949 (1965).

[7] Y. Hara, Phys. Rev. **136**, B507 (1964).

[8] L. L. C. Wang, Phys. Rev. **142**, 1187 (1966); J. Franklin, Phys. Rev. **152**, 1437 (1966).

[9] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960). We shall refer to this paper as GGMW.

tudes $G_i(s,t,u)$ which describe NN scattering to the five "parity-conserving" helicity amplitudes $f_i(s,z)$. The f_i have no singularities in $z=\cos\theta$ (θ is the scattering angle) and can be expanded in partial waves with energy-independent coefficients. Therefore, constraints among the f_i imply relations among the partial waves. Knowledge of these relations is needed in order to count the number of free parameters, or subtraction constants, in the theory. One of their rules is [Ref. 9, Eq. (4.33a)]

$$G_1=\frac{1}{E^2}\left[f_1+\frac{m^2}{p^2}f_3-zf_4-z\frac{E^2}{m^2}f_5\right], \tag{1.1}$$

where E, p, and z are the center-of-mass energy, momentum, and cosine of the scattering angle. Because $E^2=s/4$ (equal masses), G_1 will have a pole at $s=0$ for fixed t unless the combination in brackets vanishes there. This, therefore, it must do, since the G_i are supposed to have only dynamical singularities.

Mandelstam[6] pointed out the existence of a similar condition for equal-mass scalar-vector scattering. We have exhibited this elsewhere[10] (also see Appendix) and have shown that it comes similarly from the requirement that a scalar amplitude have no kinematical singularities.

As the external spins are increased, this procedure for discovering generalized GGMW conditions, and hence for studying the Reggeization of various theories becomes rapidly very complicated. For the interesting case of spinor-vector scattering there are already twelve independent scalar amplitudes. Few would care to work out their expansion in terms of the twelve independent helicity amplitudes and the partial waves. However, these conditions are, in fact, just consequences of crossing among the helicity amplitudes alone, and do not depend on the identification of the scalar functions.

We develop below a procedure for discovering all the generalized GGMW conditions for any spins from the helicity-amplitude crossing relations, and write them down explicitly for several cases.

These conditions are interesting in themselves, especially for identical-particle scattering where the equal-mass restriction corresponds to the experimental situation. Any dynamical model should satisfy them; otherwise the Mandelstam representation for the scalar amplitudes is violated. They are particularly important in Regge-trajectory exchange models, where they seem to require either bizarre behaviors of the Regge residues or the existence of other trajectories.[11] In Sec. II we explain in detail the general scheme for discovering these conditions, and work out the examples of greatest interest in Sec. III.

[10] E. Abers and V. Teplitz, Nuovo Cimento **39**, 739 (1965).
[11] E. Leader, reported by M. Gell-Mann, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, California, 1966* (University of California Press, Berkeley, California, 1967).

Our principal interest is in the application to the counting of free parameters and the question of the Reggeization of field theories, or, more generally, to decide under what conditions an amplitude cannot differ from its analytic continuation in angular momentum. We discuss the formulation of this problem and compare the equal-mass and unequal-mass cases, in Sec. IV. Several examples are tabulated in Sec. V. Among our results we show that Mandelstam's conclusions about vector-scalar and vector-spinor scattering hold even when the equal-mass restriction is relaxed; and that any amplitude for scattering of charged spin-$\frac{1}{2}$ particles and neutral bosons of any spin must Reggeize, just like the vector-spinor amplitude. We also show that πN scattering is analytic in the angular momentum in any channel which (like π^+p) does not contain a nucleon.

II. GENERALIZED GGMW CONDITIONS

Before developing the machinery needed to compute the GGMW-type conditions, let us establish the notation and list some of the properties of helicity amplitudes we shall need. The S matrix for scattering $1+2\to 3+4$, with momenta and helicities p_i and μ_i, respectively, is

$$S_{\mu_3\mu_4;\mu_1\mu_2}(p_1p_2p_3p_4)=\frac{2iT^s{}_{\mu_3\mu_4;\mu_1\mu_2}}{(16E_1E_2E_3E_4)^{1/2}}\delta_4(p_1+p_2-p_3-p_4), \tag{2.1}$$

which determines the normalization of our helicity amplitudes T^s. The superscript s designates amplitudes physical in the channel for which $s=(p_1+p_2)^2$ is the square of the center-of-mass frame energy, and t and u are the direct and crossed four-momentum transfers. Similarly, the helicity amplitudes for the crossed-channel processes are denoted T^t and T^u.

We define as usual $\lambda=\mu_3-\mu_4$, $\mu=\mu_1-\mu_2$. With our normalization, the partial-wave expansion is[12]

$$T^s{}_{\mu_3\mu_4;\mu_1\mu_2}(s,z)=\frac{1}{\pi}\sum_j(2j+1)t_{\lambda\mu}{}^j(s)d_{\mu\lambda}{}^j(\theta). \tag{2.2}$$

With standard conventions for the d^j, parity conservation implies[12]

$$T_{\mu_3\mu_4;\mu_1\mu_2}=T_{-\mu_3-\mu_4;-\mu_1-\mu_2}(-1)^{(\lambda-\mu)}. \tag{2.3}$$

For elastic scattering, time reversal implies

$$T_{\mu_3\mu_4;\mu_1\mu_2}=T_{\mu_1\mu_2;\mu_3\mu_4}(-1)^{(\lambda-\mu)}. \tag{2.4}$$

And for particle-antiparticle scattering,

$$T_{\mu_3\mu_4;\mu_1\mu_2}=T_{\mu_4\mu_3;\mu_1\mu_2}(-1)^{(\lambda-\mu)}. \tag{2.5}$$

Thus, if the spins of the particles are s_i, there are $(2s_1+1)(2s_2+1)(2s_3+1)(2s_4+1)$ helicity amplitudes,

[12] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959). The $d_{\lambda\mu}{}^j(\theta)$ are the rotation matrices.

but in general they are not all independent because of the above rules. We shall call a given helicity amplitude even or odd according as $\lambda-\mu$ is even or odd.

The even (odd) amplitudes $T^s{}_{\mu_3\mu_4\mu_1\mu_2}$ are even (odd) functions of $W=\sqrt{s}$ for fixed t or u, and therefore the same is true for the helicity partial waves normalized as in Eq. (2.1). Since the partial waves connecting states of given parity are of the form $t_{\lambda\mu}{}^{j\pm}(s)=t_{\lambda\mu}{}^{j}(s)\pm t_{\lambda-\mu}{}^{j}(s)$, this property implies directly the generalized MacDowell principle for fermion channels.[7] The amplitudes $T^s{}_{\mu_3\mu_4\mu_1\mu_2}(s,t)$ as functions of t exhibit the same behavior near $t=0$, $z_s=1$ (even or odd functions of $\sqrt{t}$ according as $\lambda-\mu$ is even or odd). This follows from the crossing relations. It has been shown[3,13] that the amplitudes,

$$T^{s\prime}{}_{\mu_3\mu_4;\,\mu_1\mu_2}=\frac{T^s{}_{\mu_3\mu_4;\,\mu_1\mu_2}}{[1-z]^{|\lambda-\mu|/2}[1+z]^{|\lambda+\mu|/2}}, \qquad (2.6)$$

have no kinematical singularities in z. For elastic scattering, $1-z=-t/2q^2$, where q^2 is the center-of-mass momentum. An identical rule holds, of course, in the t channel near $z_t=1$. For elastic scattering in the t channel (equal-mass-to-equal-mass scattering in the s channel) $z_t=1+s/2q_t^2$.

Consider a t channel scattering amplitude $T^t{}_{\mu_3\mu_1;\,\mu_4\mu_2}$. Then $\lambda=\mu_4-\mu_2$ and $\mu=\mu_3-\mu_1$ are the total angular momenta about the directions of particles 4 and 3. At $s=0$ (forward scattering), these directions are the same, and angular momentum conservation requires the amplitude to vanish unless $\lambda=\mu$. This physical requirement is embodied in the t channel analog of Eq. (2.6). Additional information is also included since this relation says that the helicity amplitude must vanish with s like $s^{|\lambda-\mu|/2}$.

The crossing relations between the T^s and the T^t were discovered by Trueman and Wick,[14] and by Muzinich.[15] It is by using them that we avoid the necessity of finding the Mandelstam scalar amplitudes. The relation between the s channel process $1+2\rightarrow 3+4$ and the t channel process $\bar{4}+2\rightarrow 3+\bar{1}$ is

$$T^t{}_{\mu_3\mu_1;\,\mu_4\mu_2}=\sum_{\mu_1'\mu_2'\mu_3'\mu_4'} d_{\mu_1'\mu_1}{}^{s_1}(\pi-\psi_1)d_{\mu_2'\mu_2}{}^{s_2}(\psi_2) \times d_{\mu_3'\mu_3}{}^{s_4}(\psi_3)d_{\mu_4'\mu_4}{}^{s_4}(\pi-\psi_4)T^s{}_{\mu_3'\mu_4';\,\mu_1'\mu_2'}\,. \qquad (2.7)$$

The angles are defined by[16]

$$\sin\psi_i=\frac{m_i q_{s,i}\sin\theta_s}{q_{ti}\sqrt{t}}\,. \qquad (2.8)$$

We can now compute, for any spins, the generalized GGMW conditions, which all come from requiring that the left-hand side of Eq. (2.7) vanish like the appropriate power of z_t-1, as prescribed by Eq. (2.6). Then, wherever the t channel $\lambda-\mu$ is not zero, there is one relation among the $T^s{}_{\mu_3\mu_4;\,\mu_1\mu_2}$ for each t, at that value of s for which $z_t=1$; that is, along half the boundary of the t-channel physical region. In addition, if $|\lambda-\mu|\geq 3$, T^t must vanish so fast that there are also relations like

$$\partial_s \sum_{\mu_1'\mu_2'\mu_3'\mu_4'} d_{\mu_1'\mu_1}{}^{s_1}d_{\mu_2'\mu_2}{}^{s_2}d_{\mu_3'\mu_3}{}^{s_3} \times d_{\mu_4'\mu_4}{}^{s_4}T^s{}_{\mu_3'\mu_4';\,\mu_1'\mu_2'}=0\,, \qquad (2.9)$$

etc.

When the t channel is elastic scattering, so that the s channel describes equal mass into equal-mass scattering, the point $z_t=1$ is at $s=0$ for all t. Then these rules become constraints among the s-channel amplitudes at $s=0$, and one can derive rules like the original GGMW conditions connecting a finite number of partial waves. For more general masses, $s(z_t=1)$ depends on t, and all we can get in the s channel are energy-dependent rules coupling all partial waves. This will prove to be the key point in Sec. IV, for it distinguishes the nature of the conditions to be imposed on Reggeized theories in equal-mass scattering from unequal-mass scattering. A correct, complete theory, of course, must satisfy these rules in all cases.

When all masses are equal, the crossing rule, Eq. (2.7), is especially simple also. In fact, Eq. (2.8) becomes

$$\sin\psi_i=2m\left[\frac{4m^2-s-t}{(4m^2-t)(4m^2-s)}\right]^{1/2}, \qquad (2.10)$$

so that at $s=0$, $\psi_i=\pi/2$ (independently of t). We therefore need to compute the rotation matrices at this one point only.

Now in this case we can separate the crossing rules into two groups. Recall[12] that $d_{\lambda-\mu}{}^s(\pi/2)=(-1)^{s-\lambda}\times d_{\lambda\mu}(\pi/2)$. This relation, together with the parity conservation rule, Eq. (2.3), can be used to group the terms on the right-hand side of Eq. (2.7) into pairs. If T^t is odd (even), a simple calculation shows that the even (odd) T^s cancel out in pairs; therefore the rule $T^t(s=0)=0$ gives relations among odd (even) $T^s(s=0)$ only.

Let n be the number of independent odd T^t. Then there are exactly n independent odd T^s also. (This is because the number of different parity rules Eq. (2.3) is the same in both channels; whereas each time-reversal rule in the s channel becomes, through the crossing relations, a charge conjugation rule in the t channel.) Since for an odd amplitude, $\lambda-\mu\neq 0$, each of the n odd T^t must vanish at $s=0$. Thus there are n homogeneous linear equations among the n odd T^s. The equations are independent, because the n T^t are independent, and therefore for each odd T^s, $T^s(s=0)=0$.

This result uses up many of the possible relations we can get in this way, and provides no new information, for

[13] F. Calogero and J. Charap, Ann. Phys. (N. Y.) 26, 44 (1964); F. Calogero, J. Charap, and E. Squires, *ibid.* 25, 325 (1963).
[14] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) 26, 322 (1964).
[15] T. Muzinich, J. Math. Phys. 5, 1481 (1964).
[16] The sign of ψ_i is also given in Ref. 14.

the odd T^s are already known to be odd functions of $W=\sqrt{s}$; and since in the equal-mass case no amplitude can have poles in W at $s=0$,[7] they must vanish there.

New conditions can be obtained from the even amplitudes with $\lambda\neq\mu$. When $|\lambda-\mu|\geq 4$, there will also be derivative rules at $s=0$, since the amplitude must vanish like some higher power of s. For high enough spins, there are also new relations among the odd T^s, for each T^t with $|\lambda-\mu|\geq 3$. Remembering that the odd T^s vanish like W at $s=0$, the conditions for $|\lambda-\mu|=3$ have the simple form

$$\frac{\partial}{\partial\sqrt{s}}\sum_{\mu_1'\mu_2'\mu_3'\mu_4'} d_{\mu_1'\mu_1}{}^{s_1}(\pi-\psi)d_{\mu_2'\mu_2}{}^{s_2}(\psi)d_{\mu_3'\mu_3}{}^{s_3}(\psi)\times d_{\mu_4'\mu_4}{}^{s_4}(\pi-\psi)T^s{}_{\mu_3'\mu_4';\mu_1'\mu_2'}=0 \quad (2.11)$$

at $s=0$. Note that for photon amplitudes, we may exhibit GGMW-type rules simply by deleting from Eq. (2.7) or (2.1) amplitudes for zero-helicity photons.

We conclude this section with a discussion of a point we long found confusing. Consider elastic s channel scattering between particles of masses m_1 and m_2. Define $s_t=(m_1+m_2)^2$ and $s_u=(m_1-m_2)^2$; s_t is the physical threshold. The center-of-mass momentum is

$$q^2=(s-s_t)(s-s_u)/4s. \quad (2.12)$$

For any given angular momentum j, the parity-conserving partial-wave amplitudes $t_{\lambda\mu}{}^{j\pm}(s)$ are linear combinations of matrix elements between states of fixed orbital angular momentum L. The amplitudes between states with $L=L_1$ and $L=L_2$ must vanish with q^2 like $q^{(L_1+L_2)}$. Therefore there are threshold conditions among the $t_{\lambda\mu}{}^j(s)$ at $s=s_t$, and, when $m_1\neq m_2$, at $s=s_u$ also. When spins are present, this cannot happen unless the corresponding combinations of the T^s vanish appropriately at s_t and s_u.

When $m_1=m_2$, q^2 vanishes at $s=s_t$ only, and the $s=s_u$ constraints disappear. But we have seen that new fixed-s constraints, namely the generalized GGMW conditions, appear just in this limit. Since $s_u\to 0$ as $m_1\to m_2$, it is often thought that the GGMW-type constraints are the equal-mass limit of the $s=s_u$ threshold rules.

That this is not so is most easily seen by comparing the partial-wave expansions of the two constraints. The $s=s_u$ rules connect states of the same j only, whereas the the $s=0$ rules in general connect amplitudes of different j. In fact, the constraints are of quite different nature. The $s=0$ rules follow from crossing and angular momentum conservation in the crossed channel. These conditions exist for unequal-mass scattering also, where, however, we have seen that they occur at a t-dependent value of s. They couple an infinite number of partial waves and are for our purposes less interesting. The $s=s_u$ conditions simply disappear at $m_1=m_2$, like the second zero of q^2.

We have no deep explanation for the fact that in some instances the difference between the number of $s=s_u$, $m_1\neq m_2$ conditions and the number of $s=0$, $m_1=m_2$ conditions is the same as the difference in the number of kinematically allowed parameters, or subtraction constants in the two cases. This accident allows the condition counting we perform in Sec. IV, and therefore the conclusions about the Reggeization of perturbation expansions, to be independent of the equality of the masses in those cases. In the Appendix, we illustrate the difference between the two conditions in detail for scalar-vector scattering.

III. CATALOG OF LOW-SPIN GENERALIZED GGMW CONDITIONS

In this section we derive the conditions at $s=0$ in a few cases. We consider only equal-mass elastic scattering, $a+b\to a+b$, without assuming that the spins s_a and s_b are the same. The t channel is $a+\bar{a}\to b+\bar{b}$. The relations (2.3), (2.4), and (2.5), together with the rules

$$d_{\lambda\mu}{}^s(\theta)=(-1)^{(\lambda-\mu)}d_{-\lambda-\mu}{}^s(\theta)=(-1)^{(\lambda-\mu)}d_{\mu\lambda}{}^s(\theta)=(-1)^{(s-\lambda)}d_{\lambda-\mu}{}^s(\pi-\theta), \quad (3.1)$$

reduce the number of independent terms in each crossing rule to the number of independent even or odd helicity amplitudes.

To illustrate the procedure, the first few cases will be discussed in more detail than the subsequent ones.

A. $s_a=0$; $s_b=0$

Scalar-scalar scattering is completely described by one independent amplitude, $T_{00;00}(s,t)$; there is no condition, since $\lambda-\mu=0$ only.

B. $s_a=\frac{1}{2}$; $s_b=0$

Spinor-scalar (e.g., πN) scattering is described by four helicity amplitudes, two of which are independent:

$$T^s{}_{\frac{1}{2}0;\frac{1}{2}0}=T^s{}_{-\frac{1}{2}0;-\frac{1}{2}0}, \quad (3.2a)$$

$$T^s{}_{\frac{1}{2}0;-\frac{1}{2}0}=-T^s{}_{-\frac{1}{2}0;\frac{1}{2}0}. \quad (3.2b)$$

The independent t-channel (e.g., $\pi+\pi\to N+\bar{N}$) amplitudes are $T^t{}_{\frac{1}{2}\frac{1}{2};00}$ and $T^t{}_{\frac{1}{2}-\frac{1}{2};00}$. Their relations to the T^s are obtained from Eq. (2.7) using

$$d^{1/2}(\theta)=\begin{pmatrix}\cos\frac{1}{2}\theta & -\sin\frac{1}{2}\theta\\ \sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta\end{pmatrix}, \quad (3.3)$$

and are

$$T^t{}_{\frac{1}{2}\frac{1}{2};00}=\sin^2(\tfrac{1}{2}\psi)T^s{}_{\frac{1}{2}0;\frac{1}{2}0}+\cos^2(\tfrac{1}{2}\psi)T^s{}_{-\frac{1}{2}0;-\frac{1}{2}0}+\cos\tfrac{1}{2}\psi\sin\tfrac{1}{2}\psi T^s{}_{\frac{1}{2}0;-\frac{1}{2}0}+\cos\tfrac{1}{2}\psi\sin\tfrac{1}{2}\psi T^s{}_{-\frac{1}{2}0;\frac{1}{2}0}=T^t{}_{\frac{1}{2}0;\frac{1}{2}0}, \quad (3.4a)$$

$$T^t{}_{\frac{1}{2}-\frac{1}{2};00}=-\cos\tfrac{1}{2}\psi\sin\tfrac{1}{2}\psi T^s{}_{\frac{1}{2}0;\frac{1}{2}0}+\cos\tfrac{1}{2}\psi\sin\tfrac{1}{2}\psi T^s{}_{-\frac{1}{2}0;-\frac{1}{2}0}+\sin^2(\tfrac{1}{2}\psi)T^s{}_{\frac{1}{2}0;-\frac{1}{2}0}-\cos^2(\tfrac{1}{2}\psi)T^s{}_{-\frac{1}{2}0;\frac{1}{2}0}=T^s{}_{\frac{1}{2}0;-\frac{1}{2}0}. \quad (3.4b)$$

From the vanishing of the left-hand side of Eq. (3.4b) at $s=0$ we obtain $T^s_{\frac{1}{2}0;-\frac{1}{2}0}(0,t)=0$; which, as predicted in general in Sec. II, is just a consequence of the MacDowell symmetry, and so contains no new information.

C. $s_a=1$; $s_b=0$

Scalar-vector scattering is the simplest case which possesses a GGMW condition which is not just a consequence of the oddness of some helicity amplitudes. There are nine amplitudes $T^s_{\lambda 0;\mu 0}(s,t)$, of which four are independent:

$$T^s_{10;10}=T^s_{-10;-10}\,, \qquad (3.5a)$$

$$T^s_{10;-10}=T^s_{-10;10}\,, \qquad (3.5b)$$

$$T^s_{10;00}=-T^s_{-10;00}=-T^s_{00;10}=T^s_{00;-10}\,, \qquad (3.5c)$$

and

$$T^s_{00;00}\,. \qquad (3.5d)$$

The t-channel amplitudes are obtained by exchanging the second and third subscripts. The t-channel versions of Eqs. (3.5a) and (3.5d) provide no rules, since $\lambda-\mu=0$. Equation (3.5c) is an odd amplitude, so gives only $T^s_{10;00}(0,t)=0$, which is the boson version of a MacDowell rule. The vanishing of $T^b_{1-1;00}$ at $s=0$ provides the one generalized GGMW condition for this problem. The crossing relation can be evaluated using

$$d^1(\theta)=\begin{pmatrix}\frac{1}{2}(1+\cos\theta) & -(\sin\theta)/\sqrt{2} & \frac{1}{2}(1-\cos\theta)\\ (\sin\theta)/\sqrt{2} & \cos\theta & -(\sin\theta)/\sqrt{2}\\ \frac{1}{2}(1-\cos\theta) & (\sin\theta)/\sqrt{2} & \frac{1}{2}(1+\cos\theta)\end{pmatrix}. \qquad (3.6)$$

Then, from Eqs. (2.7) and (3.5),

$$T^t_{1-1;00}=\tfrac{1}{2}\sin^2\psi T^s_{10;10}+\tfrac{1}{2}(1+\cos^2\psi)T^s_{10;-10}-\tfrac{1}{2}\sin^2\psi T^s_{00;00}; \qquad (3.7)$$

Hence, at $s=0(\psi=\pi/2)$

$$T^s_{10;10}+T^s_{10;-10}-T^s_{00;00}=0\,. \qquad (3.8)$$

If the vector meson is massless, as in the electrodynamics of scalar mesons, Eq. (3.8) still holds, but with the third term on the left absent.

The rules relating the partial waves may be found by a procedure similar to that originally used by GGMW.[17] The partial waves can be labeled by the parity, angular momentum, and magnitudes of the initial and final helicities.[3] There is a 2×2 matrix $t_{\lambda\mu}{}^{j+}(s)$, with $\lambda=1$ or 0, and parity $(-1)^j$; and a single function $t^{j-}(s)$ for parity $-(-1)^j$. In terms of the amplitudes between states of given helicity which appear[18] in Eq. (2.2),

$$t_{\lambda\mu}{}^{j\pm}=t_{\lambda\mu}{}^j\pm t_{\lambda,-\mu}{}^j\,. \qquad (3.9)$$

To write out Eq. (2.2) explicitly for the three amplitudes appearing in Eq. (3.8), we need only

$$d_{11}{}^j=[j(j+1)(2j+1)]^{-1}\times[j^2P_{j+1}'+(2j+1)P_j'-(j+1)^2P_{j-1}'], \qquad (3.10a)$$

$$d_{1-1}{}^j=-[j(j+1)(2j+1)]^{-1}\times[j^2P_{j+1}'-(2j+1)P_j'-(j+1)^2P_{j-1}'], \qquad (3.10b)$$

$$d_{00}{}^j=P_j=[2j+1]^{-1}[P_{j+1}'-P_{j-1}']\,. \qquad (3.10c)$$

Therefore, when the three amplitudes in Eq. (3.8) are expanded according to Eq. (2.2) using Eq. (3.10), the condition can be written explicitly as an expansion in derivatives of Legendre functions. Since such an expansion is unique, one may compare coefficients of each P_j' to obtain

$$t_{00}{}^{(j-1)+}-t_{00}{}^{(j+1)+}=(j-1/j)t_{11}{}^{(j-1)}-(j+2/j+1)t_{11}{}^{(j+1)-}+[2j+1/j(j+1)]t_{11}{}^{j+} \qquad (3.11)$$

at $s=0$. (For photons the left-hand side is zero.) When projected similarly into partial waves, the odd rule gives, of course, $t_{10}{}^{j+}=0$ at $s=0$.[19]

D. $s_a=\frac{1}{2}$; $s_b=\frac{1}{2}$

This can be NN or $N\bar{N}$ scattering, for example, and is the amplitude studied exhaustively by GGMW. There are 16 helicity amplitudes $T_{\mu_3\mu_4;\mu_1\mu_2}$, of which six are independent. The μ_i may have the values $\pm\frac{1}{2}$, so we can abbreviate these as $T_{\pm\pm;\pm\pm}$. In either the s or the t channel, the independent amplitudes are[20]

$$\begin{aligned}\varphi_1&=T_{++;++}\,,\\ \varphi_2&=T_{+-;+-}\,,\\ \varphi_3&=T_{+-;+-}\,,\\ \varphi_4&=T_{+-;-+}\,,\\ \varphi_5&=T_{++;+-}\,,\\ \varphi_6&=T_{++;-+}\,.\end{aligned} \qquad (3.12)$$

For NN or $N\bar{N}$ scattering, isotopic spin conservation together with identical particle symmetry or charge parity conservation implies $\phi_5=\phi_6$, so that only five amplitudes are really independent.[9]

The necessary vanishing of the odd amplitudes ϕ_5 and ϕ_6 contains, as always, no new information. Only the t-channel version of ϕ_4 has $\lambda-\mu\neq 0$ and even, so there is only one rule. Putting together the crossing

[17] Reference 9, p. 2269.

[18] Because of our normalization (2.2), the matrices $t^{j\pm}$ are normalized so that, in the elastic unitarity region, $\mathrm{Im}t^{j\pm}=(q/W)t^{j\pm}t^{j\pm}$.

[19] For higher spins, the generalized GGMW conditions must be expanded in higher derivatives of Legendre functions to obtain the partial-wave rules.

[20] The ϕ_i are the ϕ_i defined by GGMW, Ref. 9, Eq. (4.8).

rule (2.7), the parity and time-reversal rules (2.3) and (2.4), and the spin-$\frac{1}{2}$ rotation matrix (3.3) evaluated at $\psi=\pi/2$, one obtains immediately, at $s=0$,

$$T^s{}_{++;++}-T^s{}_{++;--}-T^s{}_{+-;+-}+T^s{}_{+-;-+}=0\,, \quad (3.13)$$

which is relation (7.15a) of GGMW. Their partial-wave rule [Ref. 9, Eq. (7.18)], follows by expansion in terms of derivatives of Legendre polynomials.

E. $s_a=2;\ s_b=0$

The study of the Reggeization of graviton theories suggests itself naturally as an extensoin of the study of electrodynamics, so that the scalar-tensor rules are of some interest. We can study spin-2–spin-0 scattering in close analogy to paragraph C. Let us suppress the scalar helicity label, since it is always zero. There are 25 amplitudes $T_{\lambda\mu}$, of which nine are independent. The six independent even ones are

$$\begin{aligned}&T_{22}=T_{-2-2}\,,\\&T_{20}=T_{02}=T_{0-2}=T_{-20}\,,\\&T_{2-2}=T_{-22}\,,\\&T_{11}=T_{-1-1}\,,\\&T_{1-1}=T_{-21}\,,\\&T_{00}\,.\end{aligned} \quad (3.14a)$$

The three independent odd amplitudes are

$$\begin{aligned}&T_{21}=-T_{12}=-T_{-2-1}=T_{-1-2}\,,\\&T_{2-1}=-T_{-1-2}=-T_{-21}=T_{1-2}\,,\\&T_{10}=-T_{-10}=-T_{01}=T_{0-1}\,.\end{aligned} \quad (3.14b)$$

The crossing relation can be evaluated from these relations and the matrix

$$d^{(2)}(\theta)=\begin{bmatrix}\frac{1}{4}(1+\cos\theta)^2 & -\frac{1}{2}\sin\theta(1+\cos\theta) & \frac{1}{4}\sqrt{6}\sin^2\theta & -\frac{1}{2}\sin\theta(1-\cos\theta) & \frac{1}{4}(1-\cos\theta)^2\\ \frac{1}{2}\sin\theta(1+\cos\theta) & \frac{1}{2}(1+\cos\theta)(2\cos\theta-1) & -(\sqrt{3}/\sqrt{2})\sin\theta\cos\theta & \frac{1}{2}(1-\cos\theta)(1+2\cos\theta) & -\frac{1}{2}\sin\theta(1-\cos\theta)\\ \frac{1}{4}(\sqrt{6})\sin^2\theta & (\sqrt{3}/\sqrt{2})\sin\theta\cos\theta & \frac{1}{2}(3\cos^2\theta-1) & -(\sqrt{3}/\sqrt{2})\sin\theta\cos\theta & \frac{1}{4}(\sqrt{6})\sin^2\theta\\ \frac{1}{2}\sin\theta(1-\cos\theta) & \frac{1}{2}(1-\cos\theta)(1+2\cos\theta) & (\sqrt{3}/\sqrt{2})\sin-\theta\cos\theta & \frac{1}{2}(1+\cos\theta)(2\cos\theta-1) & \frac{1}{2}\sin\theta(1+\cos\theta)\\ \frac{1}{4}(1-\cos\theta)^2 & \frac{1}{2}\sin\theta(1-\cos\theta) & \frac{1}{4}(\sqrt{6})\sin^2\theta & \frac{1}{2}\sin\theta(1+\cos\theta) & \frac{1}{4}(1+\cos\theta)^2\end{bmatrix}. \quad (3.15)$$

The vanishing of $T_{20}{}^t$, $T_{2-2}{}^t$, and $T_{1-1}{}^t$ at $s=0$ provide the following three generalized GGMW conditions among the T^s:

$$(\sqrt{6})T_{22}{}^s+4T_{20}{}^s+(\sqrt{6})T_{2-2}{}^s-(\sqrt{6})T_{00}=0\,, \quad (3.16)$$

$$T_{22}{}^s+2\sqrt{6}T_{20}{}^s+T_{2-2}+4T_{11}{}^s-4T_{1-1}{}^s+3T_{00}{}^s=0\,, \quad (3.17)$$

$$(\sqrt{6})T_{22}{}^s+4T_{20}{}^s+(\sqrt{6})T_{2-2}{}^s-(\sqrt{6})T_{00}{}^s=0\,. \quad (3.18)$$

A relation among the derivatives of the even T^s is obtained by observing that $T_{2-2}{}^t$ must vanish like s^2:

$$\begin{aligned}&T_{22}{}^{s\prime}+(2\sqrt{6})T_{20}{}^{s\prime}+T_{2-2}{}^{s\prime}+4T_{11}{}^{s\prime}\\&\quad-4T_{1-1}{}^{s\prime}+3T_{00}{}^{s\prime}-\frac{t}{4m^2(t-4m^2)}\\&\quad\times[6T_{2-2}{}^s-2T_{22}{}^s-8T_{11}{}^s-6T_{00}]=0\,,\end{aligned} \quad (3.19)$$

where the prime denotes differentiation with respect to s at fixed t. The term in which $t/(t-4m^2)$ appears comes from differentiating the d functions in Eq. (2.7).

Finally, because $T^t{}_{2,-1}$ must vanish like $s^{3/2}$, there is a fifth relation according to the general formula (2.11):

$$\begin{aligned}&\frac{\partial}{\partial\sqrt{s}}[3T_{21}{}^s+T_{2-1}{}^s-(\sqrt{6})T_{10}{}^s]-\left[\frac{t}{4m^2(t-4m^2)}\right]^{1/2}\\&\quad\times[-T_{22}{}^s+3T_{2,-2}+2T_{11}-3T_{00}]=0\,.\end{aligned} \quad (3.20)$$

The second term comes from differentiating the d functions in the crossing relation for $T^t{}_{2,-1}$ at $s=0$. Because $d/d\sqrt{s}T^t{}_{2,-1}$ is an analytic function of s, the left-hand side of Eq. (3.20) must vanish like s, so that no new condition is obtained from requiring $d/dsT^t{}_{2,-1}=0$. The crossing relation from which Eq. (3.20) follows is

$$\begin{aligned}T^t{}_{2-1}(t,s)=&-zy^3T_{22}{}^s+3y^2T_{21}{}^s\\&-2\sqrt{6}yz^3T_{20}{}^s+(1+3z^2)T_{2-1}{}^s\\&+(3+z^2)yzT^s{}_{2-2}+2zy^3T_{11}{}^s-\sqrt{6}y^2T_{10}{}^s\\&+4yz^3T_{1-1}{}^s-3y^3zT_{00}{}^s\,,\end{aligned} \quad (3.21)$$

where $z=[st/16q_s{}^2q_t{}^2]^{1/2}$, and $y^2=1-z^2$.

F. $s_a=\frac{1}{2};\ s_b=1$

This is the case studied by Mandelstam.[6] We shall find three generalized GGMW conditions, hence three restrictions on the amplitude in addition to his six threshold conditions. We shall see in Sec. V that this does not alter the conclusion that the electron must lie on a Regge trajectory. Our result is indeed needed to make the counting of conditions come out the same in the equal-mass as in the unequal-mass case.

There are 36 amplitudes $T^s{}_{\mu_3\mu_4;\mu_1\mu_2}$, of which 12 are independent. As in all fermion channels, half are odd and half are even. The independent even ones are $T^s{}_{\frac{1}{2}1;\frac{1}{2}1}$, $T^s{}_{\frac{1}{2}1;\frac{1}{2}-1}$, $T^s{}_{\frac{1}{2}1;-\frac{1}{2}0}$, $T^s{}_{\frac{1}{2}0;\frac{1}{2}0}$, $T^s{}_{\frac{1}{2}0;-\frac{1}{2}1}$, $T^s{}_{\frac{1}{2}-1;\frac{1}{2}1}$, and $T^s{}_{\frac{1}{2}-1;\frac{1}{2}-1}$. The odd ones have opposite signs for μ_1 and μ_2. The independent t-channel amplitudes are labeled similarly, simply interchanging μ_1 and μ_4.

The crossing relations may be evaluated using Eqs. (3.3) and (3.6). Four of the even T^t have $\lambda-\mu=0$. The remaining ones have $|\lambda-\mu|=2$, and give two generalized GGMW conditions at $W=0$:

$$T^s{}_{\frac{1}{2}1;\frac{1}{2}1}+2T^s{}_{\frac{1}{2}1;\frac{1}{2}-1}-2T^s{}_{\frac{1}{2}0;\frac{1}{2}0}+T^s{}_{\frac{1}{2}-1;\frac{1}{2}-1}=0\,, \qquad (3.22a)$$

$$\sqrt{2}T^s{}_{\frac{1}{2}1;\frac{1}{2}1}-2T^s{}_{\frac{1}{2}1;-\frac{1}{2}0}+2T^s{}_{\frac{1}{2}0;\frac{1}{2}-1}-\sqrt{2}T^s{}_{\frac{1}{2}-1;\frac{1}{2}-1}=0\,. \qquad (3.22b)$$

Finally, there is one derivative condition among the odd T^s from the vanishing of $T^t{}_{\frac{1}{2}-\frac{1}{2};-11}$ like $s^{3/2}$, which can be evaluated simply using Eq. (2.11). It is

$$\frac{\partial}{\partial\sqrt{s}}\Big[-2\sqrt{2}T^s{}_{\frac{1}{2}1;\frac{1}{2}0}+2T^s{}_{\frac{1}{2}1;-\frac{1}{2}1}+T^s{}_{\frac{1}{2}1;-\frac{1}{2}-1}+2\sqrt{2}T^s{}_{\frac{1}{2}0;\frac{1}{2}-1}-2T^s{}_{\frac{1}{2}0;-\frac{1}{2}0}+T^s{}_{\frac{1}{2}-1;-\frac{1}{2}1}\Big]_{s=0}=0\,. \qquad (3.23)$$

In Eq. (3.23) there are no terms proportional to derivatives of products of d functions as there were in Eqs. (3.19) and (3.20). The crossing relation for $T^t{}_{\frac{1}{2}-\frac{1}{2},-11}$ is

$$\begin{aligned}T^t{}_{\frac{1}{2}-\frac{1}{2};-11}=&-2\sqrt{2}y^2T^s{}_{\frac{1}{2}1;\frac{1}{2}0}+2y^2T^s{}_{\frac{1}{2}1;-\frac{1}{2}1}\\&+2(1-z^2)T^s{}_{\frac{1}{2}1;-\frac{1}{2}-1}+2\sqrt{2}y^2T^s{}_{\frac{1}{2}0;\frac{1}{2}-1}\\&-2y^2T^s{}_{\frac{1}{2}0;-\frac{1}{2}0}+(1+3z^2)T^s{}_{\frac{1}{2}-1;-\frac{1}{2}1}\,.\end{aligned} \qquad (3.24)$$

The derivatives of all coefficients with respect to $\sqrt{s}$ vanish at $s=0$. In Eq. (3.24), no even T^s appear, whereas in a boson relation like Eq. (3.21) even T^s occur with coefficients vanishing at $z=1$.

G. $s_a=\frac{1}{2}$, $s_b=$ Any Integral Spin

We shall not work these out in detail, since the method should now be clear. For $s_b=2$ (massive graviton-electron scattering) there are 100 amplitudes $T^s{}_{\mu_3\mu_4;\mu_1\mu_2}$, of which 30 are independent, 15 odd, and 15 even. There are clearly an enormous number of conditions.

IV. CONDITION COUNTING AND REGGEIZATION

Mandelstam's criterion[5] for deciding when an amplitude is analytic in the angular momentum can be stated roughly as follows: Consider an elastic scattering amplitude. It is completely described by the two partial-wave matrices $t^{j\pm}(s)$ for all j. The parities of the states are $\pm(-1)^j$. In general, $t^{j\pm}(s)$ is not an analytic function of j but under very general conditions, there always exists an N such that there is a function which interpolates $t^{j\pm}(s)$ and is analytic in j for $\text{Re}j>N$.[21] (There are really two distinct interpolating functions, one for odd j and one for even j. This complication has no effect on anything we have so say, so we shall ignore it in order to avoid having to say everything twice.) Similarly, for integral $j>N$, the size of the matrix is independent of j, but for low j, some configurations become nonsense, and the size of the matrix between physical states is reduced.

For any "real" amplitude whose partial waves are $t^{j\pm}(s)$, one can always define another theory which we will call the Reggeized theory. Its partial-wave matrices $\bar{t}^{j\pm}(s)$ are defined to agree with the original $t^{j\pm}(s)$ for $j>N$, and defined for $j\leq N$ by analytic continuation. By definition, then, the Reggeized theory is analytic in the angular momentum variable. It is known[3] to have kinematic branch points at those values of j and of $-j-1$ for which it connects sense and nonsense states. It presumably has no fixed poles, but only moving (Regge) poles. It almost certainly also has moving cuts,[22] which we at first ignore. We shall indicate later which of our results are independent of them.

Now the Reggeized theory will not, in general, satisfy all the properties the "real" theory does. In particular, it need not satisfy crossing symmetry, since we have selected out one channel for Reggeization. However, the $\bar{t}^{j\pm}$ must satisfy some of the kinematical rules which the $t^{j\pm}$ do; and, when there are enough of these, there will be so many conditions on the $\bar{t}^{j\pm}$ that they cannot be other than the $t^{j\pm}$. In those cases the original theory must have been analytic in j to begin with, and any particles which appear there must in fact lie on Regge trajectories.

Let us enumerate the conditions the partial waves $t^{j\pm}$ must satisfy, before deciding which of these hold for the $\bar{t}^{j\pm}$ also. These follow directly from the properties of the $t^j{}_{\mu_3\mu_4;\mu_1\mu_2}(s)$, which, in turn, follow simply from the properties of the $T_{\mu_3\mu_4;\mu_1\mu_2}(s,t)$, using the expansion (2.2).

First, $t_{\lambda\mu}{}^j(s)$ is an odd or even function of $\sqrt{s}$ according as $\lambda-\mu$ is odd or even. For unequal-mass scattering, it may have kinematical poles at $s=0$ of order $|\lambda-\mu|/2$.[7] (There may be also a dynamical infinity at $s=0$, so that the actual behavior of $t_{\lambda\mu}{}^j$ is faster than $s^{-|\lambda-\mu|/2}$, but this follows from the left-hand discontinuity, and so is not "kinematical" in the sense we are using the term.) Thus the parity-conserving amplitudes $t_{\lambda\mu}{}^{j\pm}=t_{\lambda\mu}{}^j\pm t_{\lambda-\mu}{}^j$ are allowed kinematical poles at $s=0$ of order $||\lambda|+|\mu||/2$, but the residues of $t_{\lambda\mu}{}^{j+}$ and $t_{\lambda\mu}{}^{j-}$ are not independent. There are also the threshold conditions at $s=s_t$ and $s=s_u$.

Next there are all the detailed consequences of crossing symmetry. We know in particular the generalized GGMW conditions we have been discussing.

Finally, there is unitarity. For low enough s,

$$\text{Im}t_{\lambda\mu}{}^{j\pm}=\sum_\sigma (q/\sqrt{s})t_{\lambda\sigma}{}^{j\pm *}t_{\sigma\mu}{}^{j\pm}\,.$$

[21] M. Froissart, La Jolla Conference on Strong and Weak Interactions, 1961 (unpublished). The existence of a Mandelstam representation for the scalar invariants is apparently sufficient. We always refer only to the unique analytic interpolation which permits the Sommerfeld-Watson transform. For details, see, for example, S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963), or E. J. Squires, *Complex Angular Momenta and Particle Physics* (W. A. Benjamin, Inc., New York, 1963).

[22] S. Mandelstam, Nuovo Cimento **30**, 1127 (1963); **30**, 1148 (1963).

For equal-mass scattering, these rules are slightly modified. No poles at $s=0$ are allowed, so the odd amplitudes must vanish there. For boson channels, this implies that $t_{\lambda\mu}{}^{j\pm}(s)$ is odd in $\sqrt{s}$ for odd $\lambda-\mu$. For fermion channels, $t_{\lambda\mu}{}^{j\pm}(s)$ is neither odd nor even, so must be considered a function of $W=\sqrt{s}$. Then, the oddness of the odd $t_{\lambda\mu}{}^{j}$ implies the generalized MacDowell relation, $t_{\lambda\mu}{}^{j+}(W)=t_{\lambda\mu}{}^{j-}(-W)$. Thus equal-mass fermion amplitudes have no further kinematical condition of this type at $s=0$ once the MacDowell symmetry is imposed.

The GGMW-type conditions still hold, with the important difference that they connect only a finite number of partial waves. Threshold conditions at $s=s_t$ and unitarity are the same as for unequal masses, but there is no constraint at $s=s_u$.

Which of these conditions survive Reggeization? Evidently, only those which relate a finite number of partial waves with a j-analytic rule. The threshold conditions, the kinematic poles and zeroes at $s=0$, and unitarity are all of this type. The GGMW conditions are of this type only in the equal-mass case.[23] Therefore, $\bar{t}^{j\pm}(s)$ must satisfy generalized GGMW rules in the equal-mass case, but not otherwise.

Since these equal mass, $s=0$, conditions are not, as we have pointed out, limits of unequal mass $s=s_u$ conditions, the number of undetermined parameters, and therefore the necessity of Reggeization, need not be the same for equal and for unequal masses. However, this number does turn out to be the same for several cases of interest. We would like to know whether there is a general criterion for this "minor miracle."

Lastly we mention Castillejo-Dalitz-Dyson (CDD) poles. For $j>N$, the Reggeized theory has no CDD poles. For low integer j, $\bar{t}^{j\pm}$ will in general have n CDD poles, where n is the number of nonsense channels at that value of j. Since some of the residues may be zero, the actual number of CDD poles may be any number $\leq n$. CDD poles enter a theory analytic in j when an N-channel problem is replaced by an N-n channel problem either because of the presence of nonsense channels or from approximations.[24]

Now we can set up the general counting scheme. Consider only the sense-sense part of any partial-wave $\bar{t}^{j\pm}$. It has the same left-hand cut as $t^{j\pm}$, may have as many poles at $s=0$, and is unitary. Therefore, as Mandelstam pointed out, it satisfies the same ND^{-1} equations, except for CDD poles and the values of the subtraction parameters.

[23] Apparently these are the only crossing symmetry conditions which couple finite numbers of partial waves with coefficients analytic in j.

[24] CDD poles in channels coupled to nonsense channels occur in the same way they do when a many-channel problem is replaced by a one-channel problem with a given inelasticity factor. See M. Bander, P. Coulter, and G. Shaw, Phys. Rev. Letters **14**, 270 (1965); D. Atkinson, K. Dietz, and D. Morgan, Ann. Phys. (N. Y.) **37**, 77 (1966).

Since the permitted poles at $s=0$ are known, there is a diagonal matrix ρ such that $h_{\lambda\mu}{}^{j\pm}(s)=\rho_\lambda{}^{1/2}t_{\lambda\mu}{}^{j\pm}\rho_\mu{}^{1/2}$ has no kinematical poles or zeros at $s=0$, and is therefore the function to disperse. The great advantage of parity-conserving helicity amplitudes is that $\rho_\lambda(s)$ is simple. In fact, $\rho_\lambda(s)=s^{|\lambda|}$, for unequal-mass scattering.

The analytic amplitude $h_{\lambda\mu}{}^{j\pm}(s)$ is therefore bounded by a constant times $s^{|\lambda+\mu|/2}$, and has as many subtraction constants as there are constants in a polynomial in s (or W for fermion channels) or in $s^{1/2}$ times a polynomial in s (or W) bounded by this power in s.[25] Let us call the total number of independent subtraction constants N_{p}. For equal-mass amplitudes, no poles are allowed and N_{p} is just the number of independent even amplitudes. It is obtained by adding up the number of parameters in all the independent components of $h^{j\pm}$. (Time-reversal symmetry requires h, like t, to be symmetric.) Next, we can count the number of conditions N_C given by the threshold rules and, in the equal-mass case, the generalized-GGMW conditions. Finally, there is a certain number N_{CDD} of parameters describing the permitted CDD poles. If a given amplitude has n sense channels, then m residues and one position describe each CDD pole; if there are n nonsense states for that value of j, the total number of CDD parameters which the Reggeized theory may have is $N_{\mathrm{CDD}}=n(m+1)$.

If the number of CDD poles in $t^{j\pm}$ is greater than n, the two theories certainly cannot agree. If this number is less than or equal to n, the two theories can only differ by varying one of the $N_{\mathrm{CDD}}+N_p$ parameters. But there are N_C conditions on them, so, if $N_{\mathrm{CDD}}+N_p-N_C\leq 0$, $t^{j\pm}=\bar{t}^{j\pm}$. This is always true for sufficiently high j. If it holds for all angular momenta and parities, the entire amplitudes must be identical, i.e., the theory described by $t^{j\pm}$ Reggeizes.

This procedure, then, describes how to count the conditions on the $\bar{t}^{j\pm}$. To conclude, we qualify the applicability of this method by suggesting some ways an amplitude might fail to Reggeize even though $N_{\mathrm{CDD}}+N_p-N_C\leq 0$ for all partial waves.

To begin with, it might be that the Reggeized theory simply does not exist, i.e., that there is no analytic function that interpolates properly the $t^{j\pm}$ for Rej greater than some N. We shall assume that this is not the case.

Most obviously we have ignored cuts and other singularities in the j plane. In their absence, the continuation of $\bar{t}^{j\pm}$ from high j down to the low physical

[25] This is not the way subtraction constants are usually counted, but it always comes out the same. In the ND^{-1} formalism, the number of parameters is the number of subtraction constants in the N equations. Since $\mathrm{Im}N_{\lambda\mu}=\rho_\lambda[(\mathrm{Im}t)D]_{\lambda\mu}$, $N_{\lambda\mu}$ may grow like $s^{|\lambda|}$. The coefficients of the subtraction polynomials must be chosen so that ND^{-1} is symmetric. The general relation between parameters counted this way, and counted by simply adding on polynomials to $h_{\lambda\mu}{}^{j\pm}$ which do not violate the unitarity bound in magnitude, can be seen by expansion in powers of any coupling constant.

values of angular momentum exists and is unambiguous, so that the conditions must hold. But a full amplitude has moving cuts, and what sorts of monsters, such as fixed branch points, lie hidden in the impenetrable jungle to the left of $\mathrm{Re}\,j=N$ is not known. We do not know to what extent these will invalidate the application of our conclusions to a complete scattering amplitude, but clearly something remains to be proved before we can be sure that our condition counting really does imply Reggeization.

However, for amplitudes such as renormalizable field theories which have perturbation expansions, each term of which satisfies the unitarity bound, our counting provides unambiguous statements about the low-order terms. These are known[22] not to have j-plane cuts, and therefore when the parameters are determined by the conditions, their Reggeization is a rigorous conclusion. Thus, for example, the famous factorization of the residues of the vector-spinor Born term[1–3] is really proved by these methods, whereas the Reggeization of the spin-$\frac{1}{2}$ particle to all orders in the coupling constant is only suggestive.

Since the conditions which determine the parameters are not necessarily linear in them, there might be more than one solution. This does not seem to happen in the principal cases of interest, and of course can be checked explicitly in any given problem, but must be borne in mind in general.

Finally, the number of CDD pole parameters N_{CDD} depends on the number of nonsense channels, which in turn depends on the total number of communicating channels considered for high j. In a model, or to a given order of perturbation theory, these are fixed and few, but a "real" amplitude is coupled to an infinite number of channels, and therefore the correct $\bar{t}^{j\pm}$ might have an arbitrarily large number of CDD poles. For convenience, in the next section we include only those channels containing the initial particles in all possible helicity configurations.

Thus we do not know whether any of our results in the next section are statements about anything more general than a few low-order graphs or specific models, such as ND^{-1} with elastic unitarity. All these limitations on the range of applicability of the present work would well bear further investigation.[26]

In the next section we study, case by case, the same amplitudes treated in Sec. III.

[26] The hope that considerations like the present ones might have wider validity than we can prove comes principally from the explicit perturbation-theory computations of Gell-Mann *et al.* (Refs. 1–4) and especially the sixth-order calculation of Cheng and Wu (Ref. 5). They show that the electron in electrodynamics (with massive photons) lies on a Regge trajectory up to sixth order, where already there are graphs with three-particle intermediate states, so that our enumeration of CDD pole parameters (see Sec. V F) does not necessarily hold. Perhaps the answer is that only two-particle nonsense states generate CDD poles in the sense channels.

V. CONDITION COUNTING IN LOW-SPIN AMPLITUDES

A. $s_a=0;\ s_b=0$

For scalar-scalar scattering, $N_{\mathrm{CDD}}=0$ for all j, since there are no physical j for which there are nonsense states. Thus a theory with CDD poles, like ϕ^3 coupling, certainly cannot Reggeize.

1. $m_a=m_b$

There is one state for each j, and consequently $N_p=1$. The orbital angular momentum is $L=j$, and there are no generalized GGMW rules, so $N_C=j$. Hence the number of free parameters $N_{\mathrm{CDD}}+N_p-N_C$ is positive only for S waves. Therefore, $t^0(s)$ and $\bar{t}^0(s)$ can differ by one constant parameter, so the theory need not Reggeize. The free parameter is of course the famous S-wave subtraction constant, or $\lambda\phi^4$ renormalization, which is not determined by the left-hand cut.

2. $m_a\neq m_b$

There are no helicities, so $\lambda=\mu=0$ always, and still $N_p=1$ for each j. There are two thresholds, so the number of conditions doubles for each j:$N_C=2j$. (This is a general relation; the number of threshold conditions for unequal-mass scattering is always just twice this number for equal masses.) N_p-N_C is still positive only for $j=0$, so again there is just one free S-wave parameter. Notice that N_p-N_C is not independent for general j of the equality of the masses, but that the total number of free parameters in the theory nevertheless is.

B. $s_a=\frac{1}{2};\ s_b=0$

In scalar-spinor scattering, we may consider $t^{j+}(W)$ only, remembering that $t^{j-}(W)=t^{j+}(-W)$ so that there are threshold conditions for both positive and negative W. Depending on the charges of the two particles, $t^{1/2+}(W)$ may or may not have to have a CDD pole at the mass of the spinor. There are no nonsense channels, so that $\bar{t}^{1/2+}(W)$ cannot agree with $t^{1/2+}(W)$ when the quantum numbers require the CDD pole[27] (unless the CDD pole in $\bar{t}^{1/2+}(W)$ is provided by coupling to other channels which are nonsense at $j=\frac{1}{2}$).

1. $m_a=m_b$

Just as for scalar-scalar scattering, $N_p=1$ for each j. There is an $L=j\pm\frac{1}{2}$ threshold at $W=\pm\sqrt{s_t}$; both condi-

[27] This is true only if the spin-$\frac{1}{2}$ object is an "elementary particle," that is to say, when it occurs for all values of the coupling constant. If it is a "bootstrap" particle, then the amplitude exists only for one value of this coupling constant. Then the two amplitudes may agree, and we shall even show that there are no free parameters. However, it is just in this case, where we must consider the whole amplitude rather than the Born term, that all the ambiguities discussed in Sec. IV raise their heads. We would like to think that it makes no difference, but cannot prove it.

tions restrict the same amplitude, so $N_p=(j+\frac{1}{2})+(j-\frac{1}{2})=2j$. Therefore, $N_p-N_C=1-2j$, which is never positive, even for $j=\frac{1}{2}$. Therefore, $t^{j+}(W)$ must be $\bar{t}^{j+}(W)$ for all j, and the theory must Reggeize.

2. $m_a \neq m_b$

This is physically more interesting, since it corresponds to πN scattering. Now $|\lambda|=\frac{1}{2}$, so one kinematic pole in W is allowed, and $N_p=2$. There are two thresholds on the right and two on the left, so $N_C=4j$. Again, $N_p-N_C=2-4j\leq 0$ for all j, and the theory has no free parameters. The rigorous consequence, then, is that low-order terms of, say, π^+p scattering must be analytic in the angular momentum for all values of the pion-nucleon coupling constant.

C. $s_a=1$; $s_b=0$

The partial waves for vector-scalar scattering are described in Sec. IIIC. The $t^{0+}(x)$ matrix connects a sense and a nonsense state, so the sense-sense amplitude $\bar{t}_{00}^{0+}(s)$ may have a CDD pole. Thus there is a chance that a theory like neutral vector-charged scalar scattering, which has a 0^+ CDD pole, namely the charged scalar itself, can be proved to Reggeize.

1. $m_1=m_2$

This example was considered by Mandelstam[6] for $j=0$. For other j, $t^{j+}(s)$ has three independent components, t_{00}^{j+}, t_{10}^{j+}, and t_{11}^{j+}, and $t^{j-}(s)$ has one. To each of these we may add a subtraction constant, but the one added to t_{10}^{j+} is not free, since the $t_{10}^{j+}(0)=0$. (This is always true for odd amplitudes in boson channels.) Therefore, $N_p=3$ for each $j>0$. Now we know that for high enough j, $t^{j\pm}=\bar{t}^{j\pm}$, so the generalized GGMW condition (3.11) is satisfied. Consider the highest j for which $t^{j\pm}$ is suspected of differing from $\bar{t}^{j\pm}$. There is one condition (3.11) (there is one for each j); $t^{j\pm}$ couples $L=j+1$ to $L=j-1$, and t^{j-} has $L=j$, so there are $4j$ threshold conditions. Thus $N_p-N_C=3-(4j+1)$ which is negative for all $j>0$, and there are no free parameters. Only t_{00}^{0+} remains. $N_p=1$, there is one generalized GGMW condition (3.11), and one P-wave threshold condition. There are two CDD pole parameters, and so the total number of parameters by which the Reggeized theory can differ from the original is $N_{\text{CDD}}+N_p-N_C=2+1-2=1$, and we cannot conclude $t^{j\pm}=\bar{t}^{j\pm}$ for all j.

2. $m_a \neq m_b$

For $j>0$, there may be $1/s$ poles in $t_{11}^{j\pm}$ and a $1/W$ pole in t_{10}^{j+}, in addition to the constants. But the poles in t_{11}^{j+} and t_{11}^{j-} are not independent, so $N_p=5$. The number of threshold conditions is $8j$, so $N_p-N_C=5-8j<0$, and there are no free parameters. For $j=0$, $N_p=1$ and there are two P-wave thresholds, so $N_{\text{CDD}}+N_p-N_C=2+1-2=1$. Again there is a free parameter, and the amplitude need not Reggeize.

D. $s_a=\frac{1}{2}$; $s_b=\frac{1}{2}$

Spinor-spinor scattering has a 2×2 matrix $t^{j+}(s)$ with components $t^{j+}{}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}$, $t^{j+}{}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$, and $t^{j+}{}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$; and, for NN or $\bar{N}N$ scattering, two uncoupled amplitudes $t^{j-}{}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}$ and $t^{j-}{}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$. The parities of the states are $\pm(-1)^j$ for $N\bar{N}$ scattering and $\mp(-1)^j$ for NN scattering. At $j=0$, t^{j+} has a nonsense channel coupled to the sense channel. A scalar CDD pole in the $N\bar{N}$ channel therefore has a chance of Reggeizing, but not a pseudoscalar one like the pion. The condition counting is very similar to case C above.

1. $m_a=m_b$

There is one GGMW condition for each j. For $j>0$, $N_p=4$; t^{j+} connects $L=j\pm 1$, whereas both t^{j-} are $L=j$, and so there are $5j$ threshold conditions. Therefore $N_p-N_C=4-(5j+1)<0$. When $j=0$, only the P-wave $t^{0+}{}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}$ and the S-wave $t^{0-}{}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}$ are physical amplitudes; $N_p=2$, there is one GGMW condition, one threshold condition, and one possible CDD pole, so $N_{\text{CDD}}+N_p-N_C=2+2-2=2$. The theory need not Reggeize.

2. $m_a \neq m_b$

The additional kinematical poles allowed bring N_p up to 6 for each $j\neq 0$. Therefore $N_p-N_C=6-10j<0$. At $j=0$, neither $t^{0\pm}{}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}$ has poles at $s=0$, there are two threshold conditions on $t^{0+}{}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}$, so $N_{\text{CDD}}+N_p-N_C=2+2-2=2$ as before.

This unequal-mass case is really not of much interest unless the spin conservation rule characteristic of NN or $N\bar{N}$ scattering is relaxed. Therefore, let us consider two different spin-$\frac{1}{2}$ particles, so that t^{j-} is also a 2×2 matrix, with off-diagonal element $t^{j-}{}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$. At $j=0$, this matrix element now couples a sense to a nonsense state, so the Reggeized theory may have two CDD poles, one of each parity. For $j>0$, we now have $N_p-N_C=7-12j<0$. For $j=0$, $N_{\text{CDD}}+N_p=4+2-2=4$, which as before, is positive.

E. $s_a=2$; $s_b=0$

We suppress the scalar helicity index, and use the notation of Sec. IIIE. For high j, the amplitude is described by a 3×3 matrix t^{j+} and a 2×2 matrix t^{j-}. At $j=1$, t^{j+} has only two sense states, and t^{j-} only 1. At $j=0$, only t_{00}^{j+} is sense, and there are two nonsense channels. This is the channel of interest, since if the spin-2 particle is neutral, e.g., if it is a (massive) graviton, the scalar particle appears as a CDD pole in the 0^+ amplitude, and we wish to know whether it Reggeizes.

We may label the matrix elements $t_{\lambda\mu}^{j\pm}$, where λ and μ are the magnitudes of the helicities of the tensor particle. When λ or μ are 0, only t^{j+} exists.

1. $m_a=m_b$

For $j>1$, $N_p=6$, the number of independent even partial waves. t^{j+} connects $L=j$, $j\pm 2$, and t^{j-} connects

$L=j\pm1$, so there are $9j$ threshold conditions; and we have discovered five generalized GGMW conditions. So $N_p-N_C=6-(9j+5)<0$. When $j=1$, there may be a CDD pole for each parity, described by 4 parameters. $N_p=3$; t^{1+} is coupled P and F wave, while t^{1-} is D wave only. Thus, there are 5 threshold conditions. The partial-wave projection of the GGMW-type rules will not give 5 independent rules, among the elements of t^{1+}, but there are at least 3, so $N_{\rm CDD}+N_p-N_C<0$. Finally consider t^{0+}. The only amplitude is $t_{00}{}^{0+}$, which couples to two nonsense channels; therefore, $N_{\rm CDD}=4$. There are 2 (D-wave) threshold conditions. Three rules, Eqs. (3.16), (3.17) and (3.19) contain T_{00}, so barring some accidental identity, there are 3 GGMW-type conditions. Thus $N_{\rm CDD}+N_p-N_C=4+1-5=0$.

2. $m_a\neq m_b$

For $j>1$, there are 7 pole residues permitted.[28] Thus, $N_p=7+6=13$. $N_C=18j$, and so $N_p-N_C<0$. When $j=1$, there are two poles,[28] and $N_p=5$. The number of undetermined parameters is $N_{\rm CDD}+N_p-N_C=4+5-10<0$. Only T_{00} contributes to $t_{00}{}^{0+}$, so this amplitude may have no poles. Therefore, $N_{\rm CDD}+N_p-N_C=4+1-4=1$; i.e., there is a free parameter.

Thus, the minor miracle has ceased functioning, and the necessity of Reggeization seems to depend on the nonequality of the masses. An example would be interesting.[29]

F. $s_a=\frac{1}{2}$; $s_b=1$

This is the case studied in detail by Mandelstam,[6] and exhaustively in perturbation theory by Gell-Mann *et al.*[1–3] We can now extend Mandelstam's arguments to unequal masses, and to all j. There is one 3×3 matrix t^{j+} for parity $(-1)^{j-1/2}$. The matrix t^{j-} need not be considered separately, because of the generalized MacDowell rule $t_{\lambda\mu}{}^{j+}(-W)=t_{\lambda\mu}{}^{j-}(W)$. Therefore, it suffices to study t^{j+}. When $j=\frac{1}{2}$, one state for each parity is nonsense, so there may be one CDD pole in $t^{1/2+}$. There are two $j=\frac{1}{2}$ sense states, so the CDD pole is described by three parameters.

1. $m_a=m_b$

For $j>\frac{1}{2}$, the 3×3 matrix t^{j+} has six parameters, one for each independent component. There are $6j+1$ threshold conditions on the left, and $6j-1$ on the right. For each j, there are three new generalized GGMW conditions, from the partial-wave expansions of Eqs. (3.21) and (3.22). Therefore, $N_p-N_C=6-(12j+3)<0$ for $j\geq\frac{3}{2}$. For $j=\frac{1}{2}$, $N_p=3$, $N_C=6$, so $N_{\rm CDD}+N_p-N_C=3+3-9=-3$ so the Reggeized theory is completely determined, and must agree with the original theory.

[28] $t_{11}{}^{1\pm}$ and $t_{10}{}^{1+}$ have first-order poles in s and W, respectively. The residues in $t_{11}{}^{1+}$ and $t_{11}{}^{1-}$ are not independent.

[29] Unfortunately, there is no example. Perturbation theory with some kind of tensor-scalar coupling violates the unitarity bound.

2. $m_a\neq m_b$

Ten poles are permitted in the W variable[30] for $j\geq\frac{3}{2}$. Therefore, $N_p-N_C=16-(24j)<0$. For $j=\frac{1}{2}$, each component has $|\lambda|=|\mu|=\frac{1}{2}$, and there may be three poles. Hence $N_{\rm CDD}+N_p-N_C=3+6-12=-3$ as before.

Thus we can state rigorously that to low orders in the coupling constant, vector-spinor scattering with conserved-current coupling[31] is analytic in the angular momentum.

G. $s_a=\frac{1}{2}$, $s_b=s$

The above result can easily be generalized to the scattering of any neutral boson of integral spin s and a spin-$\frac{1}{2}$ particle; $s=2$, of course, is of particular interest. Remember, however, that this is useful only if there exists to begin with an amplitude which satisfies the criteria. The one that is hard to satisfy is the unitarity bound. Nonrenormalizable field theories, which will certainly include first attempts at high spin-spin-$\frac{1}{2}$ amplitudes, have Born terms which violate unitarity. If the divergence of the Born term is known, the calculation can nevertheless be made with a slight change in rules, and a statement made about its Reggeization. We shall not do this, but assume we start with an amplitude bounded by unitarity. Then it is easy to show, subject to our standard qualifications, that the $j=\frac{1}{2}$ amplitude $t^{1/2}$ is the same as $\bar{t}^{1/2}$, i.e., that the spinor particle Reggeizes, as follows.

Unless $s=0$, the scattering is described for $j>s$ by two $(2s+1)\times(2s+1)$ matrices $t^{j\pm}(W)$, related by the MacDowell symmetry. There are always, however, only two physical $j=\frac{1}{2}$ states of either parity, with $L=s$ for $\frac{1}{2}^+$ and with $L=s\pm1$ for $\frac{1}{2}^-$. Therefore, at $j=\frac{1}{2}$, there must be $2s-1$ nonsense channels; hence $3(2s-1)$ CDD pole parameters in the Reggeized theory.

1. $m_i=m_2$

If there are $N_{\rm GGMW}$ generalized GGMW conditions the number of unconstrained parameters in $t^{1/2+}(W)$ is

$$N_{\rm CDD}+N_p-N_C=3(2s-1)+3-(6s+N_{\rm GGMW})=-N_{\rm GGMW}<0.$$

2. $m_a\neq m_b$

The only difference is that $N_{\rm GGMW}=0$, the threshold conditions double, and three poles are permitted. Thus, $N_{\rm CDD}+N_p-N_C=3(2s-1)+6-12s=3-6s<0$.

It is amusing to conjecture that the number $N_{\rm GGMW}$ of generalized GGMW conditions which give inde-

[30] We may label the amplitudes $t_{\frac{1}{2}i;\frac{1}{2}j}{}^{j+}$, where i and j are the vector-meson helicities, and abbreviate this to $t_{ij}{}^{j+}$. Then $\lambda=\frac{1}{2}-i$, $\mu=\frac{1}{2}-j$, and each of the six independent elements may have a pole of order $||\lambda|+|\mu||$ in W at $W=0$.

[31] For another coupling the Born term may violate the unitarity bound.

pendent conditions on the elements of $t^{1/2}(W)$ is just $3-6s$, as it is for $s=1$.

Note added in proof. Since the submission of this paper for publication, a few remarks which are incorrect or misleading have come to our attention. As far as we can tell, none of the calculations or conclusions is affected.

(i) Generalized GGMW conditions do not exist for unequal-mass scattering. This is a consequence of Appendix B of Ref. 8. (ii) The constraints at the unphysical threshold s_u can apparently also be derived from crossing. For inelastic fermion-fermion amplitudes, the number of threshold rules at s_u for a given j may differ from that number at the physical threshold s_t. (iii) In the scalar-vector amplitude discussed in the Appendix, T'_{00} may have a pole at $Q^2=0$, and therefore Eq. (A6) is incorrect. The correct rules must of course imply the correct partial-wave conditions, which is all we have used in Sec. V.

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APPENDIX

Here we write down the kinematics of elastic vector-scalar scattering in some detail to illustrate the difference between the equal-mass and the unequal-mass cases.

Let e and e' be the polarization vectors of the initial and final vector mesons, and p, p', q, q' the 4-momenta of the scalar and vector particles. Let $E=q^0=q'^0$, $q^2=m^2$, and $p^2=\mu^2$. The polarization vectors satisfy the Lorentz condition, $e\cdot q=e'\cdot q'=0$, and the center-of-mass momentum is $Q=[E^2-m^2]^{1/2}$. Then the most general form of the T matrix is[10,32] (for real polarizations)

$$T=T_1e'\cdot e+T_2e\cdot q'e'\cdot q+\tfrac{1}{2}T_3(e\cdot pe'\cdot q+e\cdot q'e'\cdot p)+T_4e'\cdot p'e\cdot p \quad \text{(A1)}$$

In terms of the amplitudes (Eq. 2.6), which contain no kinematical singularities in $z=\cos\theta$, and which are in this case

$$T_{00}'=T_{00;00}, \quad \text{(A2a)}$$

$$T_{01}'=T_{00;10}/\sin\theta, \quad \text{(A2b)}$$

$$T_{11}^{\pm}=\frac{T_{10;10}}{1+z}\pm\frac{T_{-10;10}}{1-z}, \quad \text{(A2c)}$$

the expansion of the T_i into helicity amplitudes is[33]

$$T_1=-T_{11}^{+}-(1+t/2Q^2)T_{11}^{-}, \quad \text{(A3a)}$$

$$T_2=-T_{11}^{-}/Q^2, \quad \text{(A3b)}$$

$$T_3=\frac{2}{Q^2W}[\sqrt{2}mT_{01}'+E(T_{11}^{-}+T_{11}^{+})], \quad \text{(A3c)}$$

$$T_4=\frac{1}{Q^2s}\left[m^2T_{00}'+\frac{\sqrt{2}EmtT_{01}'}{Q^2}+\left(\frac{E^2t}{2Q^2}-m^2\right)T_{11}^{+}+(\tfrac{1}{2}t-m^2)T_{11}^{-}\right]. \quad \text{(A3d)}$$

These relations hold whether or not $m=\mu$, and from them we can get both the $s=s_u$ rules and the GGMW-type condition by requiring that the scalar functions T_i have no poles in s for fixed t.

First consider the unequal-mass case. We require the T_i to be finite at $s=s_u$. From (A3a) or (A3b),

$$T_{11}^{-}(s_u)=0. \quad \text{(A4)}$$

From (A3c),

$$\sqrt{2}T_{01}'(s_u)+T_{11}^{+}(s_u)=0. \quad \text{(A5)}$$

The quantity inside the brace in (A3d) must also vanish at $s=s_u$. In order that this expression not have a pole there,

$$2\sqrt{2}T_{01}'(s_u)+T_{11}^{+}(s_u)=0. \quad \text{(A6)}$$

(A5) and (A6) together imply that $T_{01}'(s_u)$ and $T_{11}^{+}(s_u)$ are both zero. The remaining condition is then for all t,

$$m^2T_{00}'(s_u)+t\lim_{s\to s_u}\frac{1}{Q^2}[\sqrt{2}EmT_{01}'(s)+\tfrac{1}{2}E^2T_{11}^{+}(s)]=0. \quad \text{(A7)}$$

These rules, then, which are consequences of the analyticity of the T_i at $s=s_u$ guarantee the correct partial-wave threshold behavior.

Now let the masses be equal. It is clear that the first three relations (A3a–c), give no constraints on the helicity amplitudes. Neither E^2 nor Q^2 is singular at $s=0$. In fact, $E=W/2$ vanishes there. Rules analogous to (A4–7) still hold at $s=s_t$, of course, guaranteeing the correct behavior at the physical threshold. However, the quantity in braces in (A3d) still must vanish at $s=0$, since $1/s$ appears as a factor. This rule is

$$T_{00}'-T_{11}^{+}+\left(\frac{t}{2m^2}-1\right)T_{11}^{-}=0. \quad \text{(A8)}$$

Since at $s=0$, $t=2m^2(1-z)$, and (A8) becomes

$$T_{00;00}-T_{10;10}-T_{10;-10}=0, \quad \text{(A9)}$$

which is precisely the condition (3.8).

We see then that the equal-mass, zero-energy condition (A9) and the unequal-mass, unphysical threshold conditions, although not entirely unrelated, are quite different in form.

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[32] A. C. Hearn, Nuovo Cimento **21**, 233 (1961).

[33] In Ref. 10, Eq. (A6), the factor $(1-z^2)$ in the second term on the right-hand side of the first line should read $(1-z)^2$.