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Acoustic Wave Mode in a Weakly Ionized Gas*

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The acoustic wave mode in a weakly ionized gas, which is a perturbed version of an ordinary sound wave in a neutral gas, is investigated. At sufficiently low frequencies, $\omega \ll \omega_c = \Omega_n (N_n/N_i)(T_n/T_e)$, the acoustic oscillations of the electrons, ions, and neutrals are all in phase and have equal amplitudes (N_n = neutral-particle density, N_i = ion-particle density, T_n = neutral temperature, T_e = electron temperature, Ω_n = collision frequency of a neutral particle with charged particles). However, at frequencies of the order of or larger than ω_c, a marked phase difference between the oscillations of the different fluid components occurs. This leads to charge separation and an electro-acoustic effect, i.e., an electric-field perturbation produced by a sound wave. Previously reported wave amplification, predicted on the assumption that all particle species oscillate in phase, is found to be consistent with the dispersion relation derived here at frequencies $\omega \ll \omega_c$. At frequencies $\omega \geq \omega_c$, a reduction of the amplification takes place. As a consequence it is shown that, contrary to what previously has been believed, a decrease of the neutral-gas temperature does not always lead to an increase in the wave amplification.

INTRODUCTION

In a weakly ionized gas such as is encountered in an ordinary laboratory glow discharge at pressures greater than approximately 1 Torr, the electron-plasma oscillations and the ion-acoustic waves are heavily damped as a result of charged-particle collisions with the neutral background gas, and the only truly propagating longitudinal wave is the one that we call the acoustic mode, which is a somewhat perturbed version of the ordinary sound wave in a neutral gas. We wish to investigate this wave mode and determine not only the influence of the electrons and the ions on the dispersion relation but also the possible electric-field perturbation that might be produced by a sound wave entering a weakly ionized gas. Our analysis is based on a three-fluid model of the ionized gas, composed of neutrals, ions, and electrons, and we account for both momentum and energy transfer among the fluid components, as well as entropy production within each component resulting from viscosity and heat conduction.

The three-fluid equations for an ionized gas given, for example, by Fünfer and Lehner,[1] have been used previously in less general form for the study of longitudinal wave motion, but no adequate account of the role played by the electrons and ions on this wave seems to have been given. Sessler[2] considers only momentum transfer among the fluid components, and so obtains only part of the contribution of the electrons and ions to the dispersion relation. Actually, for a weakly ionized gas of the type considered here the energy transfer from the electrons to the neutral-gas component can play a more important role than momentum transfer, and it has been pointed out recently that, as a result of this energy transfer, acoustic amplification can occur.[3–5] This conclusion was reached, however, under the assumption that the neutrals, ions, and electrons all move together, having the same displacement amplitude and phase in the wave. As we shall show here, this assumption is justified only at comparatively low frequencies. In general, a marked difference between the fluid displacements results, both in amplitude and phase, and this difference affects the role of the charged particles in the dispersion relation in an essential way. Furthermore, the difference between the ion and electron displacements leads to a small charge separation which in turn produces an electric-field perturbation that travels along with the acoustic wave.

BASIC EQUATIONS

The linearized equations of motion for the neutrals, ions, and electrons are quite similar, and rather than write them all down we use, when possible, a single representative equation for all components. Thus, if we let the subscripts j and k stand for n (neutrals), i (ions), and e (electrons), the space-time Fourier transforms of the equations for mass and momentum balance can be written

$$-i\omega n_j + ikN_j v_j = 0\,, \tag{1}$$

$$-i\omega v_j + ik\rho_j^{-1} p_j = \omega_{jk}(v_k - v_j) + \omega_{jl}(v_l - v_j) + (q_j/m_j)e - f_{1j}(k,\omega)v_j\,,$$
$$(j, k, l = n, i, e \text{ and permutations})\,. \tag{2}$$

Here N and $\rho = Nm$ are the unperturbed particle and mass densities, n, p, v, and e are the perturbations in particle density, pressure, velocity, and electric field, $\omega/2\pi$ is the acoustic frequency, and k is the propagation constant. The first two terms on the right-hand side in

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[1] E. Fünfer and G. Lehner, in *Ergebnisse der exakten Naturwissenschaften* (Springer-Verlag, Berlin, 1962), Vol. 34, p. 18.

[2] G. M. Sessler, Phys. Fluids 7, 90 (1964).

[3] U. Ingard and K. W. Gentle, Phys. Fluids 8, 1396 (1965).

[4] U. Ingard, Phys. Rev. 145, 41 (1966).

[5] L. D. Tsendin, Zh. Techn. Fiz. 35, 1972 (1965) [English transl.: Soviet Phys.—Tech. Phys. 10, 1514 (1966)].

Eq. (2) express the rate of momentum transfer between the fluid components. The constant of proportionality ω_{jk} (apart from a factor of the order of unity, or of the mass ratio when $k=e$) is the collision frequency of a particle of the jth type moving through the fluid component k with N_k particles per unit volume. These constants are related as follows: $\omega_{ne}=(N_e m_e/N_n m_n)\omega_{en}$, $\omega_{ie}=(m_e/m_i)\omega_{ei}$, and $\omega_{ni}=(N_i m_i/N_n m_n)\omega_{in}\simeq(N_i/N_n)\omega_{in}$.

Under the conditions of interest, with gas pressures around 1–20 Torr, electron densities around 10^{10}–10^{12} cm^{-3}, and acoustic frequencies $(10\lesssim\omega\lesssim10^5)$, the orders of magnitude of the frequencies involved are $\omega_{ni}, \omega_{ne}\lesssim 1$, $\omega_{en}, \omega_{in}, \omega_{ei}\gtrsim 10^5$, $\omega_{ie}\simeq 10^2$, $\omega_e=(4\pi N_e q^2/m_e)^{1/2}\simeq 10^{10}$–$10^{11}$, $\omega_i=(4\pi N_i q^2/m_i)^{1/2}\simeq 10^8$–$10^9$, $(\omega_{ne},\omega_{ni})\ll\omega\ll(\omega_e,\omega_i)$. Inelastic collisions and ion–neutral-energy transfer have been ignored, since they do not contribute significantly to acoustic dispersion. (The ratio between the energy transfers to the neutrals from the electrons and from the ions is approximately equal to the ratio between the electron and ion mobilities, which is of the order of 1000.)

In Eq. (2) we have also included a term $f_{1j}(k,\omega)v_j$ representing the viscous stress in the jth fluid component. In the absence of boundaries we have, for example, for the neutral component $f_{1n}=(4/3)k^2\eta/\rho$, where η is the coefficient of shear viscosity, and ρ the mass density. For wave propagation in a tube we have in addition a viscous drag force from the walls. When the acoustic viscous boundary layer thickness[6] is small compared with the tube radius, it is possible to consider the wave motion as one dimensional, and the drag force from the wall can be included in the function f_1 with a term proportional to $\sqrt{\omega}$, as indicated later.

The linearized energy balance equation can be expressed as a relation between the temperature perturbation and the density and velocity perturbations,

$$-i\omega c_{vj}\theta_j=\alpha_j(n_e/N_e)+\beta_j(\theta_e/T_e)-ik(KT_j/m_j)v_j -f_{2j}(k,\omega)c_{vj}\theta_j, \quad (j=n,i), \quad (3)$$

$$-i\omega c_{ve}\theta_e=-ik(KT_e/m_e)v_e-f_{2e}(k,\omega)c_{ve}\theta_e -\rho_e^{-1}[(\beta_n\rho_n+\beta_i\rho_i)(\theta_e/T_e) +\alpha_n\rho_n(n_n/N_n)+\alpha_i\rho_i(n_i/N_i)], \quad (4)$$

where T is the unperturbed temperature, c_{vj} is the specific heat per unit mass at constant volume, and K is Boltzmann's constant.

In the energy equations (3) for the neutrals and the ions the terms involving the coefficients α_j and β_j express the energy transfer from the electrons, and the term $f_{2j}(k,\omega)\theta_j$ accounts for the heat conduction in the jth component. For example, for the neutral component we have $f_{2n}\equiv f_2=k^2(K_T/\rho c_{vn})$, where K_T is the heat-conduction coefficient. Analogous to the wall correction to the viscosity term f_{1n} in Eq. (2) the effect of the tube walls on heat conduction in a wave travelling along the tube can be accounted for in a one-dimensional description of motion if the acoustic thermal boundary layer thickness is small compared with the tube radius.[6] The corresponding additional term in f_{2n} is proportional to $\sqrt{\omega}$ [see Eq. (13)].

The energy-transfer coefficients α_j and β_j are obtained from the expression for the rate of energy transfer in the elastic collisions between the electrons and the neutrals. If the "average" collision cross section is σ_{en} and the "average" electron velocity is v_e, the rate of energy transfer per unit volume will be of the form $4(m_e/m_n)(m_e v_e^3/2)N_e N_n\sigma_{en}$, under the assumption that the neutrals initially are at rest in the collision. The meaning of "average" velocity and "average" cross section is obtained in the usual manner from the integral for the energy transfer, involving the electron velocity distribution $f(v_e)$ and the differential elastic cross section, $d\sigma_{en}/d\Omega$. Thus, with

$$\tfrac{3}{2}KT_e=\int\frac{1}{2}m_e v_e^2 f(v_e)d^3v_e,$$

we define

$$\sigma_{en}=\left(\frac{\pi}{8}\right)^{1/2}\left(\frac{KT_e}{m_e}\right)^{-3/2}\int\int_{-1}^{+1}2\pi v_e^3(1-\cos\phi)\times\frac{d\sigma_{en}}{d\Omega}f(v_e)d(\cos\phi)d^3v_e,$$

so that for a hard-sphere Maxwellian electron gas, we obtain $d\sigma_{en}/d\Omega=\sigma_{en}/4\pi$. The rate of energy transfer per unit mass of the neutral gas is given, in general, regardless of the form of $f(v_e)$, by

$$H=(8/\pi)^{1/2}(m_e/m_n)^2(KT_e/m_e)^{3/2}N_e\sigma_{en}. \quad (5)$$

The terms involving α_j and β_j in the energy equations (3) and (4) are the perturbation in this energy transfer resulting from the perturbations θ_e and n_e in T_e and N_e. Thus, it follows from (5) that

$$\alpha_n=H, \quad \beta_n=\alpha_n[\tfrac{3}{2}+d(\ln\sigma_{en})/d(\ln T_e)]. \quad (6)$$

The coefficients α_i and β_i are obtained in a similar manner.

Finally, the perturbation e in the electric field is related to the density perturbations through the Poisson equation

$$ike=4\pi q(n_i-n_e), \quad (7)$$

where q is the absolute value of the electronic charge, and the pressure perturbation p can be expressed in terms of the density and temperature perturbations through the equation of state,

$$p_j=(\partial P/\partial\rho)_T m_j n_j+(\partial P/\partial T)_\rho\theta_j=c_j^2\gamma_j^{-1}m_j n_j+N_j K\theta_j, \quad (j=n,i,e). \quad (8)$$

Here P and ρ are the unperturbed pressure and density, $c=(\gamma KT/m)^{1/2}$ is the adiabatic speed of sound, and $\gamma=c_p/c_v$.

[6] P. M. Morse and U. Ingard, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1961), Vol. XI/1, pp. 14, 22.

This completes the set of linearized equations of motion which is the basis for our analysis. Elimination of all but the velocity variables from these equations leads to three homogeneous equations for v_n, v_i, and v_e, namely,

$$\begin{aligned} Av_n+Bv_i+Cv_e&=0\,,\\ Dv_n+Ev_i+Fv_e&=0\,,\\ Gv_n+Hv_i+Iv_e&=0\,. \end{aligned} \tag{9}$$

The expressions for the coefficients A, B, etc., in their complete form, are rather lengthy and we leave these for the Appendix. Here we present only the approximate expressions that are consistent with our intention to limit the present study to the acoustic mode of motion and to a gas with a comparatively low degree of ionization as encountered in most laboratory glow discharges. Under these conditions, the essential contributions to these coefficients are

$$\begin{aligned} &A\simeq 1-\left(\frac{c_nk}{\omega}\right)^2+i\left(\frac{\omega}{\omega_{nn}}+\frac{\omega_{ni}+\omega_{ne}}{\omega}\right);\quad B\simeq -i\omega_{ni}/\omega;\quad C\simeq -i\frac{\omega_{ne}}{\omega}+\frac{1}{\omega\tau};\\ &D\simeq -i\omega_{in}/\omega;\quad E\simeq -\frac{\omega_i^2}{\omega^2}+i\left(\frac{\omega}{\omega_{ii}}+\frac{\omega_{in}+\omega_{ie}}{\omega}\right);\quad F\simeq\left(\frac{\omega_i}{\omega}\right)^2-i\left(\frac{\omega_{ie}}{\omega}\right);\\ &G\simeq -i\omega_{en}/\omega;\quad H\simeq\left(\frac{\omega_e}{\omega}\right)^2-i\frac{\omega_{ei}}{\omega};\quad I\simeq -\left(\frac{\omega_e}{\omega}\right)^2-\frac{1}{\gamma_e}\left(\frac{c_e}{c_n}\right)^2+i\frac{\omega_{en}+\omega_{ei}}{\omega}. \end{aligned} \tag{10}$$

We have introduced ω_{nn}, which is of the order of the neutral-neutral collision frequency,

$$\frac{\omega}{\omega_{nn}}=\frac{1}{\omega}\left(f_{1n}+\frac{\gamma_n-1}{\gamma_n}f_{2n}\right). \tag{11}$$

This term accounts for the well-known Kirchhoff wave attenuation caused by viscosity and heat conduction in a neutral gas. The characteristic frequency, ω_{ii}, is defined in a similar manner.

The effect of the heating of the neutrals by the electrons is expressed through the term $1/\omega\tau$ defined by

$$1/\tau=(\gamma_n-1)H/c_n^2. \tag{12}$$

Since H is the rate of energy transferred to the neutrals per unit mass from the electrons, it follows that the characteristic time τ can be interpreted as the time required to approximately double the thermal energy of the neutrals.[7] Using the expression of H given in Eq. (5), we find that for ordinary glow-discharge conditions this time is of the order of 1 sec. The effect of this energy transfer on the state of the electron gas is expressed in the complete form of Eq. (10) as given in the Appendix through the term containing the function R. This term, discussed further there, vanishes if the electron gas is assumed to be isothermal or if the electron heat conductivity is assumed to be infinite. This assumption that the electron gas is an isothermal heat reservoir breaks down at low frequencies, and the problem must then be handled in the manner indicated in the Appendix. The result is an electron temperature perturbation that is out of phase with the gas density and tends to reduce the magnitude of the amplification effect at very low frequencies.

DISPERSION RELATION

Using the expression for the coefficients A, B, etc., as given in Eq. (10), we obtain from Eq. (9) the following dispersion relation for the acoustic wave mode:

$$k^2\simeq\left(\frac{\omega}{c_n}\right)^2\left[1-\left(\frac{\Omega_n}{\omega}+\frac{1}{\omega\tau}\right)\frac{\delta}{1+\delta^2}+i\left(\frac{1}{\omega\tau_n}+\frac{\Omega_n}{\omega}\frac{\delta^2}{1+\delta^2}-\frac{1}{\omega\tau}\frac{1}{1+\delta^2}\right)\right], \tag{13}$$

where

$$\Omega_n=(\omega_{ni}+\omega_{ne})/\omega\,,\quad \delta=\frac{1}{\gamma_n}\frac{T_e}{T_n}\frac{N_i}{N_n}\frac{\omega}{\Omega_n},$$

$$\frac{1}{\tau_n}=f_{1n}+\frac{\gamma_n-1}{\gamma_n}f_{2n}=(\omega/c)^2\left[\frac{4\eta}{3\rho}+(\gamma-1)K_T/\rho c_p\right]+(\omega\eta/2\rho r^2)^{1/2}+(\omega K_T/2\rho c_p r^2)^{1/2}.$$

It should be stressed that this expression for k is valid only as long as the quantities in the second and third terms in the brackets are small compared with unity. Terms of higher order in these quantities have been neglected. In this connection, we are reminded that the continuum description of fluid motion used here is valid only at acoustic frequencies lower than the neutral-neutral collision frequency. At room temperature and pressures $\sim$1 Torr, this means an upper frequency limit $\sim 10^5$ Hz.

We note that there are three terms contributing to the imaginary part of the propagation constant k. The first term $(1/\omega\tau_n)$ represents the damping produced by

[7] Since the major portion of the electrical power supplied to a glow discharge is transferred to the neutrals through elastic collisions, one can identify H with the total power supplied to the discharge per unit mass. Thus if the positive column contains n moles of gas at a temperature T and absorbs a total power P, one finds $\tau=\gamma nRT/P(\gamma-1)$, where R is the gas constant and γ the specific-heat ratio.

viscosity and heat conduction, and the expression given for $(1/\tau_n)$ refers to a wave propagating in a cylindrical tube of radius r. The terms proportional to $\sqrt{\omega}$ and ω^2 correspond to the losses at the walls and in the bulk of the gas, respectively. Thus, at sufficiently low frequencies and tube radii, the boundary losses dominate over the bulk losses. In a typical laboratory discharge at a pressure $\sim$1 Torr and a tube radius $\sim$1 cm, the two loss contributions are equal at a frequency of approximately 1000 Hz.

The second term in Eq. (13) represents the wave damping resulting from the collisions between the neutrals and the charged particles. At low frequencies such that $\delta \lesssim 1$ (typically this means frequencies less than approximately 500–1000 Hz) this term is proportional to the frequency and, consequently, has the same frequency dependence as the viscous and heat-conduction losses, but its magnitude is somewhat smaller. In this low-frequency limit, however, the boundary losses ordinarily dominate so that the bulk loss terms can both be neglected. At high frequencies the viscous bulk and heat-conduction losses will be controlling, since the charged particle-neutral collisional damping decreases with frequency as $1/\omega$ after having reached a maximum value at the frequency $(N_n/N_i) \times (T_n/T_e)\gamma_n\Omega_n$. But even at this optimum frequency, the viscous and heat-conduction losses in the bulk ordinarily dominate, so the charged particle-neutral collisional damping plays a rather insignificant role in the entire frequency range.

The third imaginary term in Eq. (13) accounts for the energy transfer to the acoustic mode from the electrons, and represents a negative damping or amplification. From an experimental standpoint it is interesting to investigate the dependence of this amplification term on the temperatures of the electrons and the neutrals. By introducing the expressions for τ, H, and δ given in Eqs. (12), (5), and (13), we see that the dependence of the amplification term on the neutral gas temperature T_n can be written

$$y \equiv \frac{1}{\tau}\frac{1}{1+\delta^2} = \frac{2y_{\max}}{(T_n/T_n^* + T_n^*/T_n)}, \tag{14}$$

where

$$T_n^* = \frac{\omega}{\Omega_n}\frac{N_i}{N_n}T_e,$$

$$y_{\max} = \frac{(\gamma_n-1)(8/\pi)^{1/2}}{2}\frac{m_e}{m_n}\frac{T_e}{T_n^*}\frac{N_e}{N_n}\omega_{en}.$$

In other words, if we neglect the dependence of Ω_n on T_n, it follows that for a constant electron temperature T_e the amplification term has a maximum value $y_{\max}$ when the neutral gas temperature is T_n^*. This temperature is proportional to the acoustic frequency, and for a typical situation, with ($\omega/\Omega_n \simeq 100$, $N_i/N_n \simeq 10^{-5}$), we see that $T_n^* \simeq 0.01 T_e$. For electron temperatures $\sim$1 eV this temperature is close to room temperature.

Equation (14) shows that if the neutral-gas temperature has been adjusted for maximum amplification of an acoustic frequency ω^* a change of T_n from T_n^* to ϵT_n^* will reduce the amplification of ω^* and transfer the maximum amplification to a frequency $\epsilon\omega^*$. In other words, contrary to what has been previously believed,[4] a decrease of the neutral-gas temperature does not always increase the amplification. The condition for acoustic-wave amplification is $\mathrm{Im}(k)<0$, and if the viscous and heat-conduction losses dominate, which is generally the case, the condition for wave amplification is $\tau(1+\delta^2)<\tau_n$. If we assume that the neutral-gas temperature has been chosen to optimize $\tau(1+\delta^2)$, and if we use this optimum value given in Eq. (14), the criterion for wave amplification can be expressed as

$$(\text{const})(m_e/m_n)(T_e/T_n^*)(N_e/N_n) > \tau_{en}/\tau_n, \tag{15}$$

in which we have introduced $\tau_{en} = 1/\omega_{en}$.

Actually, this criterion can be inferred from qualitative considerations. The mechanism of amplification is related to the fact that the perturbation in gas density produced by an acoustic wave perturbs the electron density, and therefore the rate of energy flow from the electrons to the neutrals. This produces a space-time distribution of energy flow to the neutral gas that corresponds to an acoustic source distribution favoring the growth of the wave. If the electron temperature is considerably higher than the neutral gas temperature, which is the case in most discharges, the rate of energy transfer caused by electron-neutral elastic collisions is of the order of $\tau_{en}^{-1}(m_e/m_n)E_e$ per unit volume of the gas, where (m_e/m_n) is the electron-to-neutral particle mass ratio, $E_e \simeq N_e k T_e$ the thermal electron energy per unit volume, τ_{en} the electron-neutral collision time, and N_e the electron density. Whereas in the absence of a sound wave this transferred energy all goes into thermal energy in the neutral gas, the presence of an acoustic density wave in the gas causes some of the energy to go into the collective motion of the wave. If the wave energy per unit volume is E_w and the thermal energy of the neutral gas is $E_n \simeq N_n k T_n$, it is reasonable to assume that the fraction (E_w/E_n) of the energy flow from the electrons goes into the wave. If this is so, then the wave receives an energy of the order of $(E_w/E_n)(m_e E_e/m_n)$ in time τ_{en}. Competing with this energy transfer is the wave-energy loss through various entropy-producing mechanisms, such as viscosity and heat conduction, and if the corresponding rate of wave-energy loss is E_w/τ_n, where τ_n is the acoustic-wave decay time, the loss during the time τ_{en} is $E_w(\tau_{en}/\tau_n)$. If the energy gain is larger than the loss, then wave amplification should result. Thus, apart from a numerical factor, this condition is satisfied when $(m_e/m_n)(N_e/N_n)(T_e/T_n) > (\tau_{en}/\tau_n)$, in agreement with Eq. (15).

We may be even more explicit about the amplification criterion if we use the expression for $1/\tau_n$ in a cylindrical tube as given in Eq. (13). Then, if we introduce the length $l=\eta/\rho c$, which is of the order of the neutral mean free path, and the neutral collision frequency ω_{nn}, which is of the order of c/l, the general criterion (15) takes on either of two different forms, depending on whether the boundary losses or the bulk losses predominate. Then, omitting the constant of the order of unity, we get

$$(m_e/m_n)(T_e/T_n{}^*)(N_e/N_n)>(l/r)(\omega^*\omega_n)^{1/2}\tau_c \quad (16)$$

for the case in which boundary losses predominate and

$$(m_e/m_n)(T_e/T_n{}^*)(N_e/N_n)>(l/\lambda^*)\omega^*\tau_c \quad (17)$$

for the case of bulk-loss predominance, where λ^* is the wavelength that corresponds to the frequency ω^*.

Comparing the results of the present analysis with those previously obtained,[4] we note that the general criterion (15) and the criterion (17) are consistent with the past result, but that the criterion (16) is not. The reason for the discrepancy is in part that the criterion formerly used refers to the very low-frequency range in which $\omega\tau \ll 1$. As pointed out in the Appendix, however, the contributions from f_1, f_2, and τ are not additive when $\omega\tau \ll 1$, in contrast with the previous assumption, and the corresponding criterion cannot be correct without further modification. The range of frequencies at which $\omega\tau$ is less than unity is of little practical interest and, furthermore, in this very low-frequency range the acoustic boundary layer may be of the same order of magnitude as the tube radius. Then, even the fundamental wave mode in the tube cannot be treated as one dimensional, and under such conditions a complete reexamination of the problem is required.

The amplification criterion (15), which was derived for the spatial growth of an acoustic wave, applies also for the spontaneous generation of an acoustic eigenmode in a cavity. The attenuation constant $(1/\tau_n)$, which in our analysis has been considered to result from viscous and heat-conduction losses, may, of course, include additional losses resulting, say, from absorptive boundaries. In such cases $(1/\tau_n)$ may be difficult to calculate, but it can always be determined from measurements of the width of the acoustic resonance under consideration in the neutral gas. Clearly, it is possible to obtain experimental evidence of the influence of the amplification mechanism even without satisfying the threshold condition $\tau_n/\tau>1+\delta^2$. The presence of the electron energy transfer to the neutral gas sharpens the acoustic resonance, thereby reducing the width from $(1/\tau_n)$ to

$$(1/\tau_n{}')=[1-(1+\delta^2)^{-1}(\tau_n/\tau)](1/\tau_n).$$

The presence of the charged particles not only affects the attenuation, but also causes a (small) increase in the phase velocity of the wave. It follows from Eq. (13) that the phase velocity in the low-frequency region, $\omega<\Omega_n(N_n/N_i)(T_n/T_e)$, is given by

$$c^2=c_n{}^2\left[1+\frac{1}{\gamma_n}\frac{T_e}{T_n}\frac{N_i}{N_n}\left(1+\frac{1}{\Omega_n\tau}\right)\right]. \quad (18)$$

Apart from the factor $1+(1/\Omega_n\tau)$ expressing the influence of energy transfer from the electrons to the neutrals, this result is identical with the sound speed expected for a gas mixture of an adiabatic neutral-gas component and an isothermal electron gas.

VELOCITY FIELD AND ELECTRO-ACOUSTIC EFFECT

After having discussed the dispersion relation, it remains for us to determine the relationship between the various field components in the acoustic wave mode. The velocity amplitudes of the ions and electrons follow from Eq. (9), $v_i=v_n(DI-FG)/(FH-EI)$ and $v_e=v_n(EG-DH)/(FH-EI)$, and using Eq. (10), we obtain

$$v_i\simeq\frac{v_n}{1+i\delta},\quad n_i\simeq\frac{N_i}{N_n}\frac{n_n}{1+i\delta}, \quad (19)$$

$$v_i-v_e\simeq\frac{(\omega_{in}/\omega)(\omega/\omega_i)^2\delta}{1+i\delta}, \quad (20)$$

where

$$\delta=\frac{1}{\gamma_n}\frac{T_e}{T_n}\frac{N_i}{N_n}\frac{\omega}{\Omega_n}.$$

In other words, the assumption, used in previous studies, that in the acoustic mode the velocities v_i, v_e, and v_n are all the same is good only at frequencies much less than the characteristic value $\omega_c=\gamma_n(T_n/T_e)\times(N_n/N_i)\Omega_n$. As the frequency increases above ω_c, the ion and electron velocities decrease in amplitude and are brought out of phase with the motion of the neutrals.

Since there is a (slight) difference in the ion and electron velocities, the acoustic motion produces a charge separation and therefore an electric-field perturbation. The electric field is obtained from Eq. (10) and by expressing the particle perturbation n_j in terms of the velocity, using Eq. (2),

$$n_j=N_jv_j/(\omega/k), \quad (21)$$

we obtain for the electric field $ike=4\pi q(N_i/c_n)(v_i-v_e)$, which, on account of Eq. (20), leads to

$$e=4\pi qN_i\omega_{in}(\omega/\omega_i)^2\frac{(-i\delta)}{1+i\delta}\simeq\frac{p\sigma_{in}}{q}\frac{(-i\delta)}{1+i\delta}. \quad (22)$$

The last relation in (22), which says that the electric force perturbation qe on an ion is of the order of the sound pressure times the ion-neutral collision cross section, was obtained by setting $\omega_{in}\simeq e_n\sigma_{in}N_n$, $\omega_i^2=4\pi q^2N_i/m_i$, and $N_nm_ic v_n\simeq\rho_n c v_n=p$.

The existence of an acoustically driven perturbation in the ion density and velocity, as given by (19), is qualitatively consistent with measurements for a steplike shock wave in a weakly ionized gas,[8] and the electric field given by (22) is qualitatively consistent with the observation by Alexeff and Neidigh[9] of an electrical oscillation at *acoustic* resonance frequencies of a spherical discharge tube during experiments on ion-acoustic waves.

APPENDIX

The complete expressions for the coefficients A, B, C, etc., in Eq. (9) are

$$A=1-\left(\frac{c_nk}{\omega}\right)^2\left[1-\left(\frac{\gamma_n-1}{\gamma_n}\right)\frac{if_{2n}/\omega}{1+i(f_{2n}/\omega)}+\frac{R_n/\omega\tau}{1+i(f_{2n}/\omega)}\right]+i(f_{1n}/\omega)+i(\omega_{ni}+\omega_{ne})/\omega;$$

$$B=-i(\omega_{ni}/\omega)-(c_nk/\omega)^2(R_n/\omega\tau_i)(1+if_{2n}/\omega)^{-1};$$

$$C=-i(\omega_{ne}/\omega)-(c_nk/\omega)^2(i/\omega\tau)[1+(\gamma_e-1)(\beta_n/\alpha_n)R_e]/(1+if_{2n}/\omega);$$

$$D=B^*;\quad E=A^*-(\omega_i/\omega)^2;\quad F=C^*+(\omega_i/\omega)^2;$$

$$G=-i(\omega_{en}/\omega)+(c_nk/\omega)^2(i/\omega\tau)(\rho_n/\rho_e)(\gamma_e-1)(\gamma_n-1)^{-1}R_e;$$

$$H=G^*+(\omega_e/\omega)^2;$$

$$I=1-\frac{1}{\gamma_e}\left(\frac{c_ek}{\omega}\right)^2[1+(\gamma_e-1)R_e]+i\frac{\omega_{en}+\omega_{ei}}{\omega}+i\frac{f_{1e}}{\omega}-\left(\frac{\omega_e}{\omega}\right)^2,$$

where

$$R_e^{-1}=1+if_{2e}/\omega+(i/\omega\rho_ec_{ve}T_e)(\beta_n\rho_n+\beta_i\rho_i)$$

and

$$R_n=(\gamma_e-1)(\beta_n\gamma_n/\omega c_n^2)(P_n/P_e)R_e,$$

and where the asterisk denotes an interchange of roles between the ions and neutrals. One defines R_i similarly. It proves convenient also to define the functions

$$R=R_e-i(\gamma_e-1)^{-1}[(1+ix)(\alpha_n/\beta_n)R_n+(\alpha_i/\beta_i)R_i]$$

and $R'=(\beta_n/\omega c_n^2)R$, and the parameter

$$x=\frac{1}{\gamma_e}\frac{m_e}{m_n}\left(\frac{c_ek}{\omega}\right)^2\frac{[1+(\gamma_e-1)R]\omega}{\omega_{in}+m_e\omega_{en}/m_n}\simeq\left(\frac{T_e}{\gamma_nT_n}\right)\left(\frac{\omega}{\omega_{in}}\right).$$

The dispersion relation $F(k,\omega)=0$ obtained by using these coefficients in Eq. (9) is of third order in k^2 if we neglect heat-conduction and viscosity terms. Of the corresponding three characteristic wave modes, two may be regarded as (degenerate) forms of a plasma oscillation and an ion-acoustic wave, respectively, which are heavily damped in the acoustic frequency range of interest here ($\omega\lesssim10^5$ Hz).[2] In our study of the role of the charged particles on the dispersion relation in the remaining acoustic mode we have limited the calculation to frequencies for which viscous and heat-conduction effects and collisional effects between fluid components are small enough that they can be considered additive. It is not difficult to relax this assumption and to show explicitly how these effects couple with each other. For example, in the low-frequency region in which $\delta\ll1$ (fluid components move together) the complete expression for the dispersion relation is

$$k^2=(\omega/c_n)^2[1+if_{1n}(k,\omega)/\omega][1+if_{2n}(k,\omega)/\omega][1+if_{2n}(k,\omega)/\omega\gamma_n+i/\omega\tau+i(\gamma_n-1)R'(k,\omega)]^{-1}.$$

[8] K. W. Gentle and U. Ingard, Appl. Phys. Letters **5**, 105 (1964).

[9] I. Alexeff and R. V. Neidigh, Phys. Rev. **129**, 516 (1963).

The appropriate generalizations of (13) and (20) would be

$$1-\frac{1}{\gamma_n}\left(\frac{c_nk}{\omega}\right)^2\left[1+(\gamma_n-1)\left(1+i\frac{f_{2n}}{\omega}\right)^{-1}\right]+i\frac{f_{1n}}{\omega}+i\frac{\Omega_n}{\omega}\frac{\delta^2}{1+\delta^2}-\frac{\Omega_n}{\omega}\frac{\delta}{1+\delta^2}-\frac{\delta}{1+\delta^2}\left(\frac{c_nk}{\omega}\right)^2\left(1+i\frac{f_{2n}}{\omega}\right)^{-1}$$
$$\times\left[\frac{1}{\omega\tau}+(\gamma_n-1)R'\right]-\frac{i}{1+\delta^2}\left(\frac{c_nk}{\omega}\right)^2\left(1+i\frac{f_{2n}}{\omega}\right)^{-1}\left[\frac{1}{\omega\tau}+(\gamma_n-1)R'\right]=0$$

and

$$\delta=\frac{1}{\gamma_e}\frac{m_e}{m_n}\left(\frac{c_ek}{\omega}\right)^2\left[1+(\gamma_e-1)R\frac{\omega}{\Omega_n}\frac{N_i}{N_n}\right].$$

If free-space conditions are assumed for the electrons, so that $f_{2e}/\omega\simeq(\omega/\omega_{en})(c_e/c_n)^2$, one can write $R(k,\omega)$ as

$$R\simeq[1-i\gamma_e(c_n/c_e)^2(\rho_n/\rho_e)/(\gamma_n-1)\omega\tau]\{1+i(\omega/\omega_{en})(c_e/c_n)^2+i\gamma_e[1+\tfrac{2}{3}(d\ \ln\sigma/d\ln T_e)](c_n/c_e)^2(\rho_n/\rho_e)/(\gamma_n-1)\omega\tau\}^{-1}.$$

One finds, moreover, that

$$\frac{1}{\omega\tau}+(\gamma_n-1)R'=\frac{1}{\omega\tau}\left\{1+R\left[1+\frac{2}{3}\frac{d(\ln\sigma)}{d(\ln T_e)}\right]\right\},$$

and that the real part of this expression vanishes as $\omega\rightarrow 0$.

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Evaluation of the Pair Distribution Function of a Hard-Sphere Bose Gas at Zero Temperature*

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An explicit evaluation of the pair distribution function of a hard-sphere Bose gas at zero temperature is made. The result correct to first order in a hard-sphere diameter and based on a chain-diagram approximation is plotted, and the corresponding numerical values are given. Approximate analytical expressions for the distribution function at various distances are also reported.

THE pair distribution function $\rho^{(2)}(r)$ of a dilute hard-sphere Bose gas has been studied recently.[1] As is well known, Lee, Huang, and Yang[1] evaluated the low-lying energy states of such a gas and found a phonon spectrum. Also, they found that the pair distribution function decays as $1/r^4$ at large distances. More recently, the distribution function has been evaluated in closed form in chain-diagram approximation.[2] The result correct to first order in the hard-sphere diameter becomes inaccurate at short distances, but since it is expressed in terms of higher transcendental functions we shall report some numerical results.

The pair distribution function in chain-diagram approximation may be expressed by the following integral:

$$\rho^{(2)}(r)=n^2\{1-4\pi^{-2}\gamma^{-1}aI(r)\},$$
$$I(r)=\mathrm{Re}\int\{1-q(q^2+\gamma)^{-1/2}\}\exp 2i\mathbf{q}\cdot\mathbf{r}d\mathbf{q}, \quad (1)$$

where $\gamma=4\pi an$, in which n is the density and a is the hard-sphere diameter. This result is obtained for the lowest temperature and correct to the lowest order in the hard-sphere diameter. The integral on the right-hand side may be evaluated in terms of the modified Bessel $I_\nu(x)$ and Struve functions $\mathbf{L}_\nu(x)$ as follows:

$$\begin{aligned}g(r)&=n^{-2}\rho^{(2)}(r),\\ g(r)-1&=-(4a/r)\psi(x),\\ \psi(x)&=G_0(x)-x^{-1}G_1(x),\\ G_\nu(x)&=I_\nu(x)-\mathbf{L}_\nu(x),\\ x&=2\gamma^{1/2}r.\end{aligned} \quad (2)$$

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[2] A. Isihara and Daniel D. H. Yee, Phys. Rev. **136**, 618 (1964); Chester Nisteruk and A. Isihara, *ibid.* **154**, 150 (1967).