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# Integral Equations for Deuteron-Induced Nuclear Reactions*

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A new treatment of deuteron-induced nuclear reactions is presented. Uncoupled Fredholm integral equations for the reaction amplitudes are derived. Approximating the neutron-proton $T$ matrix by a simple separable form gives the optical potential for deuterons, in terms of the neutron-target and proton-target interactions. All reaction amplitudes can be expressed, in this approximation, in terms of the off-shell $(d,d')$ scattering matrix. For the special case of a spinless, inert, infinitely heavy target, the optical potential can be constructed by quadratures. This is done explicitly, and the numerical behavior of the partial-wave deuteron elastic-scattering equations is investigated for a simple model. Finally, the philosophy of this approach to the study of direct reactions is discussed, and extensions of the theory (for the treatment of local neutron-target and proton-target potentials) are suggested.

## I. INTRODUCTION

THE distorted-wave Born-approximation (DWBA) calculation of "direct" reaction amplitudes has recently been criticized on a number of grounds.[1,2] The most serious of these is its inadequate treatment of the three-body aspect of the underlying physical model. (For instance, the DWBA series always diverges in interesting cases and there is thus no inherent error criterion in the DWBA; it is therefore impossible to unequivocally distinguish effects of the model from artifacts of the approximation.) For this reason, a number of authors have felt[1–3] that the best starting point for an improved theory of direct reactions is the correct treatment of the three-body problem.

It is by now well known that the extra degrees of freedom in the three-body problem make the practical application of Faddeev's[4] or Weinberg's[5] equations extremely difficult. The purpose of this article is to discuss the calculation of amplitudes for deuteron-induced reactions using an alternative formulation of the three-body problem which is considerably easier to apply than the Faddeev equations.[1] All the amplitudes may be expressed as quadratures involving the deuteron elastic-scattering operator; this operator in turn may be shown to satisfy an uncoupled Fredholm integral equation. This equation is, in a sense, the "most natural" one to investigate, since it is obtained with minimal effort from the Lippmann-Schwinger equation for this operator.

The paper has the following organization: Section II shows how the Lippmann-Schwinger equations may be transformed into equations with completely continuous ("Fredholm") kernels. In the case of deuteron-induced reactions, approximating the $n$-$p$ scattering matrix by a separable form leads to a set of linear integral equations involving only the deuteron-scattering amplitudes. This reduction is performed in Sec. III. The important physical case of a spinless, recoilless (inert) "core" is especially simple because the kernel of the "reduced" deuteron-scattering equations derived in Sec. III may be constructed exactly by quadratures (involving the $n$-core and $p$-core scattering matrices). The construction of this kernel is carried out explicitly in Sec. IV. [Various elements of the formal approach used here have been previously suggested by other authors. In particular, Eqs. (15), (19), (32), and (34) or their equivalents were also derived by Rosenberg.[6] The simplifications of using the spin-zero, infinite-mass target have also been remarked by numerous other authors.[2,7–9]] An extremely simple case of the recoilless, spinless core model was solved numerically. A description of the case and of the results of the calculation is given in Sec. V. Finally, conclusions and prognostications are presented in Sec. VI.

Appendix A is a straightforward calculation of the kernel of the simple model of Sec. V. Appendix B contains a description of the numerical methods employed to solve this model. Appendix C is a derivation of the functions $\tau(z)$ appearing in (21) and (44), using the two-body off-shell unitarity condition. Finally, Appendix D gives an estimate for the breakup $(d,np)$ cross section in terms of the elastic $(d,d)$ cross section.

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## II. THREE-BODY THEORY OF DIRECT REACTIONS

We consider a model consisting of a neutron, a proton, and a heavy "core," which may have (discrete) internal states. The total Hamiltonian for the system will be written $H=H_0+V_n+V_p+V_{np}$, where $V_n$ and $V_p$ are the neutron-core and proton-core interactions, and $V_{np}$ is the neutron-proton interaction; $H_0$ is the sum of the kinetic energy operators of the three-body system and the "internal" Hamiltonian of the core.

Without justification, define the formal collision operators, describing various reactions which can occur, as follows.[10]

$$U_{pd}(W)=V_p+V_{np}+(V_p+V_{np})(W-H)^{-1}(V_n+V_p)\,, \quad (1)$$

$$U_{dd}(W)=V_n+V_p+(V_n+V_p)(W-H)^{-1}(V_n+V_p)\,, \quad (2)$$

$$U_{0d}(W)=V_n+V_p+V_{np}+(V_n+V_p+V_{np})(W-H)^{-1}(V_n+V_p)\,, \quad (3)$$

$$U_{pp}(W)=V_p+V_{np}+(V_p+V_{np})(W-H)^{-1}(V_p+V_{np})\,. \quad (4)$$

[The subscripts denote which particles are free in the initial (right) and final (left) states; 0 corresponds to all three free.] The operators defined in (1) through (4) above, when their matrix elements are taken between appropriate wave functions and when the limit as $W\to E+i\eta$ is taken, respectively describe $(d,p)$, $(d,d')$, $(d,pn)$, and $(p,p')$ reactions. Consider (1) and (2): in the usual way [that is, by using the identity $A^{-1}-B^{-1}\equiv B^{-1}(B-A)A^{-1}\equiv A^{-1}(B-A)B^{-1}$] we find that they satisfy the Lippmann-Schwinger equations

$$U_{pd}(W)=V_p+V_{np}+U_{pd}(W)[W-H_0-V_{np}]^{-1}(V_n+V_p) \quad (5)$$

and

$$U_{dd}(W)=V_n+V_p+U_{dd}(W)[W-H_0-V_{np}]^{-1}(V_n+V_p)\,. \quad (6)$$

Also, we note that

$$(W-H)^{-1}\equiv(W-H_0-V_{np})^{-1}+(W-H)^{-1}(V_n+V_p)(W-H_0-V_{np})^{-1}\,, \quad (7)$$

so that

$$(W-H)^{-1}(V_n+V_p)=(W-H_0-V_{np})^{-1}U_{dd}(W)\,,$$

and therefore,

$$(W-H)^{-1}\equiv(W-H_0-V_{np})^{-1}+(W-H_0-V_{np})^{-1}U_{dd}(W)(W-H_0-V_{np})^{-1}\,. \quad (8)$$

All the scattering operators can (in principle) therefore be determined by quadrature once $U_{dd}(W)$ is known. [Equation (1) takes a particularly simple form in terms of $U_{dd}(W)$, as we shall eventually see.] We may thus restrict our attention to Eq. (6). In terms of the neutron-proton scattering matrix, defined by

$$t_{np}(W)=V_{np}+V_{np}G_0(W)t_{np}(W)\,, \quad (9)$$

we may write

$$(W-H_0-V_{np})^{-1}\equiv G_0(W)+G_0(W)t_{np}(W)G_0(W)\,. \quad (10)$$

Then Eq. (6) may be rewritten

$$U_{dd}(W)=V_n+V_p+U_{dd}(W)G_0(W)\times[1+t_{np}(W)G_0(W)](V_n+V_p)\,. \quad (11)$$

Clearly, the term $G_0(V_n+V_p)$ of the kernel of Eq. (11) contains "dangerous" delta functions, and as Weinberg has shown,[5] it is for this reason not a completely continuous operator. [Equation (11) is not of Fredholm type.] In order to proceed, we must remove this term from the kernel of (11). If we knew how to construct the operator $[1-G_0(W)(V_n+V_p)]^{-1}$, we could formally eliminate the $G_0(V_n+V_p)$ term from the kernel; this would then leave the kernel $G_0(W)t_{np}(W)G_0(W)\times(V_n+V_p)[1-G_0(W)(V_n+V_p)]^{-1}$, which, under suitable conditions on $V_{np}$, $V_n$, and $V_p$, is presumably the product of a completely continuous[4,5,11,12] operator and a bounded operator, and is thus also completely continuous. One must therefore consider how to construct $[1-G_0(W)(V_n+V_p)]^{-1}$. Note first that this operator may be written

$$[1-G_0(W)(V_n+V_p)]^{-1}G_0(W)G_0^{-1}(W)\equiv(W-H_0-V_n-V_p)^{-1}(W-H_0)\,. \quad (12)$$

Hence

$$G_0(W)(V_n+V_p)[1-G_0(W)(V_n+V_p)]^{-1}\equiv[(W-H_0-V_n-V_p)^{-1}-G_0(W)]G_0^{-1}(W)\,, \quad (13)$$

an identity we shall need later. In the circumstance that $H_0+V_n+V_p$ may be written $H_n+H_p$, with $[H_n,H_p]\equiv 0$, and assuming that the complete spectra of $H_n$ and $H_p$ are known, it is straightforward to construct $H_0+V_n+V_p$ by a convolution[7–9,12,13]

$$(W-H_n-H_p)^{-1}=(2\pi i)^{-1}\oint_\Gamma dZ(Z-H_n)^{-1}(W-Z-H_p)^{-1}\,, \quad (14)$$

where $\Gamma$ and $W$ are chosen so that the contour $\Gamma$ separates the singularities of $(Z-H_n)^{-1}$ and $(W-Z-H_p)^{-1}$, and is taken around the spectrum of $H_n$ in the positive sense. Such a contour (with $\mathrm{Im}W>0$) is illustrated in Fig. 1. This case, with $[H_n,H_p]=0$, is physically quite important. It corresponds (in our model) to a "core" with infinite mass, zero spin, and no internal degrees of freedom.

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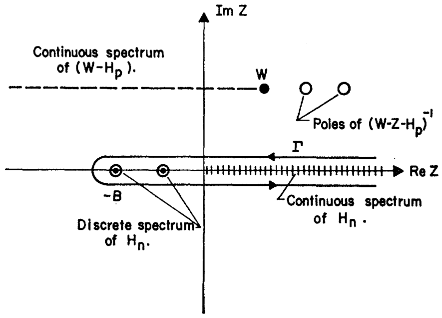


FIG. 1. Appropriate contour for convolution Eq. (14).

In case $[H_n,H_p]\neq 0$, it is necessary to construct $[1-G_0(V_n+V_p)]^{-1}$ in some other manner. Let us define two operators, $\mathbf{X}_n(w)$ and $\mathbf{X}_p(w)$, by the following equations:

$$\mathbf{X}_n=t_n+t_nG_0\mathbf{X}_p\,, \tag{15a}$$

$$\mathbf{X}_p=t_p+t_pG_0\mathbf{X}_n\,, \tag{15b}$$

where $t_n$ and $t_p$ are the $n$-core and $p$-core scattering matrices, respectively, and themselves satisfy

$$t_n=V_n+V_nG_0t_n\,, \tag{16a}$$

$$t_p=V_p+V_pG_0t_p. \tag{16b}$$

By using Eqs. (15) and (16), it is easy to see that

$$[1-G_0(V_n+V_p)][1+G_0(\mathbf{X}_n+\mathbf{X}_p)]=\mathbf{1}\,, \tag{17}$$

and

$$(V_n+V_p)[1+G_0(\mathbf{X}_n+\mathbf{X}_p)]=\mathbf{X}_n+\mathbf{X}_p. \tag{18}$$

Therefore Eq. (11) may be transformed into

$$U_{dd}(W)=\mathbf{X}_n(W)+\mathbf{X}_p(W)+U_{dd}(W)G_0(W)\times t_{np}(W)G_0(W)[\mathbf{X}_n(W)+\mathbf{X}_p(W)]. \tag{19}$$

Faddeev has shown[4] that the kernel of Eq. (15) is completely continuous for reasonable potentials, and so $\mathbf{X}_n$ and $\mathbf{X}_p$ are well defined by Eq. (15). Similarly, since the $\mathbf{X}_n$ and $\mathbf{X}_p$ are bounded operators, the kernel of (19) will also be completely continuous. By comparing (17) with (12) and (13), we see that

$$(W-H_0-V_n-V_p)^{-1}-G_0(W)=G_0(W)[\mathbf{X}_n(W)+\mathbf{X}_p(W)]G_0(W). \tag{20}$$

Equation (20) explicitly demonstrates the connection between the methods for the two cases $[H_n,H_p]=0$ and $[H_n,H_p]\neq 0$. Equation (15) is the more general form, however.

As it stands, Eq. (19) is not a significant improvement over the (coupled) Faddeev-Lovelace[12] equations from the computational point of view. A certain amount of decoupling has been achieved, but only one transition operator at a time is calculated, rather than three simultaneously. The next section introduces an approximation which effects a considerable further simplification.

## III. THE REDUCED EQUATION FOR $(d,d')$ AMPLITUDES

At this point let us introduce the major approximation needed to make Eq. (19), together with Eq. (15), a complete and simple description of deuteron scattering: we write $t_{np}$, in the $n$-$p$ barycentric system, as

$$t_{np}(z)=-|\phi_d\rangle\Lambda_T\tau_d(z)\langle\phi_d|-|\phi_0\rangle\Lambda_S\tau_0(z)\langle\phi_0|\,. \tag{21}$$

In this approximation, the $\tau(z)$ have been chosen to make $t_{np}$ unitary both on and off-the-energy shell.[12] They are derived explicitly in Appendix C. Here $|\phi_d\rangle$ and $|\phi_0\rangle$ are the triplet and single $n$-$p$ form factors, and $\tau_d(z)$ has a pole at $z=-\epsilon_d=-2.225$ MeV. $\Lambda_T$ and $\Lambda_S$ are, respectively, the triplet and singlet spin-projection operators in the $n$-$p$ spin space. At reasonably low energies (in the $n$-$p$ c.m. system), the higher partial waves are not important, so only $s$-wave contributions to $|\phi_d\rangle$ will be considered, although in a more sophisticated treatment it would be necessary to include a $d$-wave term representing the tensor forces.

Let us write (19) in the explicit basis of products of plane waves for the relative motions of $n$, $p$, and the core $c$, and the internal states of $c$. With $m_n=m_p=m$, define

$$M=2m_cm/(m_c+2m) \tag{22}$$

and

$$\text{(a)}\quad \mathbf{K}=\mathbf{k}_n+\mathbf{k}_p\,,\qquad \text{(b)}\quad \mathbf{q}=\tfrac{1}{2}(\mathbf{k}_n-\mathbf{k}_p)\,. \tag{23}$$

In terms of these momenta and the reduced mass $M$, the (diagonal) matrix elements of $H_0$ are given by

$$\langle\mathbf{q}'\mathbf{K}'\alpha'|H_0|\mathbf{q}\mathbf{K}\alpha\rangle=\delta(\mathbf{q}'-\mathbf{q})\delta(\mathbf{K}'-\mathbf{K})\delta_{\alpha'\alpha}\left(\frac{\hbar^2K^2}{2M}+\frac{\hbar^2q^2}{m}+h_\alpha\right), \tag{24}$$

where the $h_\alpha$ are the (discrete) eigenvalues of the internal "core" Hamiltonian, $H_c$. Then Eq. (19) becomes

$$\begin{aligned}\langle\mathbf{K}'\mathbf{q}'\alpha'S'\nu'|[U_{dd}(W)-\mathbf{X}_n(W)-\mathbf{X}_p(W)]|\mathbf{K}\mathbf{q}\alpha S\nu\rangle\\ =-\sum_{S''\nu''\alpha''}\int d\mathbf{K}''\left\{\int d\mathbf{q}''\frac{\langle\mathbf{K}'\mathbf{q}'\alpha'S'\nu'|U_{dd}(W)|\mathbf{K}''\mathbf{q}''\alpha''S''\nu''\rangle\langle\mathbf{q}''|\phi_{S''}\rangle}{W-(\hbar^2K''^2/2M)-(\hbar^2q''^2/m)-h_{\alpha''}}\right\}\\ \times\tau_{S''}\left(W-\frac{\hbar^2K''^2}{2M}-h_{\alpha''}\right)\left\{\int d\mathbf{q}'''\frac{\langle\phi_{S''}|\mathbf{q}'''\rangle\langle\mathbf{K}''\mathbf{q}'''\alpha''S''\nu''|[\mathbf{X}_n(W)+\mathbf{X}_p(W)]|\mathbf{K}\mathbf{q}\alpha S\nu\rangle}{W-(\hbar^2K''^2/2M)-(\hbar^2q'''^2/m)-h_{\alpha''}}\right\}.\end{aligned} \tag{25}$$

Here the neutron and proton spins have been coupled to total spin $S$, with $Z$-projection $\nu$.

To simplify the notation somewhat, define

$$\langle \mathbf{K}'S'\nu'\alpha'|A(W)|\mathbf{K}S\nu\alpha\rangle = \int d\mathbf{q}'\int d\mathbf{q}\left\{\frac{\langle\phi_{S'}|\mathbf{q}'\rangle\langle\mathbf{k}'\mathbf{q}'\alpha'S'\nu'|U_{dd}(W)|\mathbf{K}\mathbf{q}\alpha S\nu\rangle\langle\mathbf{q}|\phi_S\rangle}{(W-(\hbar^2K'^2/2M)-(\hbar^2q'^2/m)-h_{\alpha'})(W-(\hbar^2K^2/2M)-(\hbar^2q^2/m)-h_\alpha)}\right\}, \quad (26)$$

and similarly, define $\langle \mathbf{K}'S'\nu'\alpha'|A_B(W)|\mathbf{K}S\nu\alpha\rangle$ by replacing $U_{dd}(W)$ in (26) by $\mathbf{X}_n(W)+\mathbf{X}_p(W)$. We see that by multiplying Eq. (25) by the appropriate factors, and by integrating over the appropriate indices, we obtain the "reduced" equations

$$\langle \mathbf{K}'S'\nu'\alpha'|[A(W)-A_B(W)]|\mathbf{K}S\nu\alpha\rangle = -\sum_{S''=0}^{1}\sum_{\nu''=-S''}^{S''}\sum_{\alpha''}\int d\mathbf{K}''\langle\mathbf{K}'S'\nu'\alpha'|A(W)|\mathbf{K}''S''\nu''\alpha''\rangle \times\tau_{S''}\left(W-\frac{\hbar^2K''^2}{2M}-h_{\alpha''}\right)\langle\mathbf{K}''S''\nu''\alpha''|A_B(W)|\mathbf{K}S\nu\alpha\rangle. \quad (27)$$

Let us consider the physical significance of Eq. (27): with appropriate normalization of $\langle\phi_d|\mathbf{q}\rangle$, $\tau_d(z)$ reduces to $-(z+\epsilon_d)^{-1}$ near the pole of $t_{np}(z)$ (see Appendix C). By comparison of (21) with the well-known form for the residue of $t_{np}$:

$$t_{np}(z)\xrightarrow[z\to\epsilon_d]{}\Lambda_T V_{np}|\psi_d\rangle(z+\epsilon_d)^{-1}\langle\psi_d|V_{np}\Lambda_T, \quad (28)$$

we may identify $\langle\mathbf{q}|V_{np}{}^T|\psi_d\rangle$ with $-\langle\mathbf{q}|\phi_d\rangle$. However, the Schrödinger equation for the deuteron wave function is, in momentum space,

$$\left(\frac{\hbar^2q^2}{m}+\epsilon_d\right)\langle\mathbf{q}|\psi_d\rangle = -\langle\mathbf{q}|V_{np}{}^T|\psi_d\rangle \quad (29)$$

and so

$$\langle\mathbf{q}|\psi_d\rangle = \frac{\langle\mathbf{q}|\phi_d\rangle}{\epsilon_d+(\hbar^2q^2/m)}. \quad (30)$$

But the "energy shell" is defined by

$$W=(\hbar^2K^2/2M)-\epsilon_d+h_\alpha=(\hbar^2K'^2/2M)-\epsilon_d+h_{\alpha'}, \quad (31)$$

so that Eq. (26) is immediately seen to reduce to the definition of an element of the deuteron-scattering matrix (with $S'=S=1$), on the energy shell. Thus, Eq. (27) has the form of the coupled-channel Lippmann-Schwinger equation for deuteron scattering[14,15] with the role of the effective $d$-core potential being played by $\langle\mathbf{K}'S'\nu'\alpha'|A_B(w)\times|\mathbf{K}S\nu\alpha\rangle$. [The parallel may be made more exact if we write $\tau_d(z)$ as $-S(z)/(z+\epsilon_d)$, and if we symmetrize Eq. (27) by multiplying by $[S(W-(\hbar^2K'^2/2M)-h_{\alpha'})]^{1/2}$ on the left and by $[S(W-(\hbar^2K^2/2M)-h_\alpha)]^{1/2}$ on the right.]

As was previously mentioned, the stripping amplitude can easily be expressed in terms of the elastic-deuteron-scattering amplitude. It is useful to note here how this is done: using the transformation by which Eq. (19) was derived from Eq. (11), we obtain

$$U_{pd}(W)=\mathbf{X}_p(W)+V_{np}[1+G_0(\mathbf{X}_n+\mathbf{X}_p)]+U_{pd}(W)G_0(W)t_{np}(W)G_0(W)[\mathbf{X}_n(W)+\mathbf{X}_p(W)]. \quad (32)$$

On comparing (32) with (19), we find that

$$U_{pd}(W)=\mathbf{X}_p(W)+V_{np}+t_{np}(W)G_0(W)U_{dd}(W)+\mathbf{X}_p(W)G_0(W)t_{np}(W)G_0(W)U_{dd}(W). \quad (33)$$

Bearing in mind the approximation (21) and the relations (26), (29), and (30), we have the result

$$\langle\mathbf{k}_p'\sigma_p'I_fM_f|A_{pd}(W)|\mathbf{K}S\nu\alpha\rangle = \sum_{S''\nu''\alpha''}\int d\mathbf{K}''\langle\mathbf{k}_p'\sigma_p'I_fM_f|A_{pd}{}^{\mathrm{Born}}(W)|\mathbf{K}''S''\nu''\alpha''\rangle \times\left\{\delta_{S''S}\delta_{\nu''\nu}\delta_{\alpha''\alpha}\delta(\mathbf{K}''-\mathbf{K})-\tau_{S''}\left(W-\frac{\hbar^2K''^2}{2M}-h_{\alpha''}\right)\langle\mathbf{K}''S''\nu''\alpha''|A(W)|\mathbf{K}S\nu\alpha\rangle\right\}, \quad (34)$$

[14] J. V. Noble, Phys. Rev. 148, 1528 (1966).
[15] H. Feshbach, Ann. Phys. (N. Y.) 5, 357 (1958).

where

$$\langle \mathbf{k}_p'\sigma_p' I_f M_f | A_{pd}^{\mathrm{Born}}(W) | \mathbf{K} S\nu\alpha\rangle = \sum_{\sigma_n' S'\nu'\alpha'} \int d\mathbf{k}_n' \int d\mathbf{q} \Big\{ \psi_{I_f M_f}^*(\mathbf{k}_n', \sigma_n'\alpha') \langle \tfrac{1}{2}\tfrac{1}{2}\sigma_n'\sigma_p' | S'\nu'\rangle$$
$$\times \langle \mathbf{k}_n'\mathbf{k}_p' S'\nu'\alpha' | \mathbf{X}_p(W) | \tfrac{1}{2}\mathbf{K}+\mathbf{q}, \tfrac{1}{2}\mathbf{K}-\mathbf{q}, S\nu\alpha\rangle \frac{\langle \mathbf{q} | \phi_S\rangle}{W-\hbar^2K^2/2M-\hbar^2q^2/m-h_\alpha}\Big\}$$
$$+\sum_{\sigma_n'} \psi_{I_f M_f}^*(\mathbf{K}-\mathbf{k}_p', \sigma_n'\alpha) \langle \tfrac{1}{2}\tfrac{1}{2}\sigma_n'\sigma_p' | S\nu\rangle \langle \tfrac{1}{2}\mathbf{K}-\mathbf{k}_p' | \phi_S\rangle. \quad (35)$$

Here, $\psi_{I_f M_f}$ is the (bound-state) wave function of the $n$-core final state.

In general, the kernel of Eq. (27) must be constructed by solving Eqs. (15) and taking the appropriate projections as in Eq. (26). This will be a rather laborious procedure even taking separable forms for $t_n$ and $t_p$ and treating $c$ as infinitely massive. A considerable number of physical situations lend themselves to a model with $m_c=\infty$, and $[H_n,H_p]=0$, however.[1,2,7–9] Furthermore, under certain conditions (see Sec. VI) it may make sense to neglect internal excitations of a (spin-zero) core, even when they are quite low lying. Section IV, therefore, indicates how to proceed when $[H_n,H_p]=0$, and $A_B(W)$ may be exactly constructed by quadratures.

## IV. THE RECOILLESS SPIN-ZERO CORE WITH NO INTERNAL STATES

We wish to construct the resolvent $(W-H_n-H_p)^{-1}$ using the convolution (14); using the contour $\Gamma$ given in Fig. 1, and denoting the lowest (discrete) eigenvalue of $H_n$ by $-B$, we may write

$$(W-H_n-H_p)^{-1} = \int_{-B}^{\infty} dx\, (W-x-H_p)^{-1} \hat{\delta}(x-H_n). \quad (36)$$

Here the operator $\hat{\delta}(x-H_n)$ is, as usual, defined in terms of the "spectral family" of $H_n$,[16]

$$\hat{\delta}(x-H_n) = \int dI(E_\lambda)\delta(x-E_\lambda), \quad (37)$$

where the $E_\lambda$ are the eigenvalues of $H_n$, and an ordinary $\delta$ function appears under the integral sign. We may construct the spectral family of $H_n$ from its bound state and continuum eigenfunctions, if they are complete. In any case considered here, they are complete because $H_n$ will always be assumed to be a self-adjoint (unbounded) operator.[16] The bound-state wave functions are straightforward, and the scattering wave functions may be expressed in terms of $t_n$ [see (16a)] by the well-known formula ($m_c=\infty$)

$$\langle \mathbf{p} | \psi_{k_n}^{\pm}\rangle = \delta(\mathbf{p}-\mathbf{k}_n) + \frac{\langle \mathbf{p} | t_n((\hbar^2k_n^2/2m)\pm i\eta) | \mathbf{k}_n\rangle}{(\hbar^2/2m)(k_n^2-k_p^2)\pm i\eta}. \quad (38)$$

We also require the identity[1]

$$\langle \mathbf{k}_p' | (Z-H_p)^{-1} | \mathbf{k}_p\rangle = (Z-\hbar^2k_p^2/2m)^{-1}\delta(\mathbf{k}_p'-\mathbf{k}_p) + (Z-\hbar^2k_p'^2/2m)^{-1} \langle \mathbf{k}_p' | t_p(Z) | \mathbf{k}_p\rangle (Z-\hbar^2k_p^2/2m)^{-1}. \quad (39)$$

Using (37)–(39), we obtain the following matrix elements of $G_0(\mathbf{X}_n+\mathbf{X}_p)G_0$ [see Eq. (20); also note the implicit spin labels in (40)]:

$$\langle \mathbf{k}_n'\mathbf{k}_p' | G_0(W)[\mathbf{X}_n(W)+\mathbf{X}_p(W)]G_0(W) | \mathbf{k}_n\mathbf{k}_p\rangle = \langle \mathbf{k}_n'\mathbf{k}_p' | G_0(W)[t_n(W)+t_p(W)]G_0(W) | \mathbf{k}_n\mathbf{k}_p\rangle$$
$$+\frac{\langle \mathbf{k}_n' | t_n(\hbar^2k_n^2/2m+i\eta) | \mathbf{k}_n\rangle \langle \mathbf{k}_p' | t_p(W-\hbar^2k_n^2/2m) | \mathbf{k}_p\rangle}{[W-(\hbar^2/2m)(k_n^2+k_p'^2)][(\hbar^2/2m)(k_n^2+i\eta-k_n'^2)][W-(\hbar^2/2m)(k_n^2+k_p^2)]}$$
$$+\frac{\langle \mathbf{k}_n' | t_n(\hbar^2k_n'^2/2m-i\eta) | \mathbf{k}_n\rangle \langle \mathbf{k}_p' | t_p(W-\hbar^2k_n'^2/2m) | \mathbf{k}_p\rangle}{[W-(\hbar^2/2m)(k_n'^2+k_p'^2)][(\hbar^2/2m)(k_n'^2-i\eta-k_n^2)][W-(\hbar^2/2m)(k_n'^2+k_p^2)]}$$
$$+\int d\mathbf{k}'' \frac{\langle \mathbf{k}_n' | t_n(\hbar^2k''^2/2m+i\eta) | \mathbf{k}''\rangle \langle \mathbf{k}_p' | t_p(W-(\hbar^2k''^2/2m)) | \mathbf{k}_p\rangle \langle \mathbf{k}'' | t_n(\hbar^2k''^2/2m-i\eta) | \mathbf{k}_n\rangle}{[(\hbar^2/2m)(k''^2+i\eta-k_n'^2)][W-(\hbar^2/2m)(k''^2+k_p'^2)][W-(\hbar^2/2m)(k''^2+k_p^2)][(\hbar^2/2m)(k''^2-i\eta-k_n^2)]}$$
$$+\sum_i \frac{\langle \mathbf{k}_n' | \psi_i\rangle \langle \mathbf{k}_p' | t_p(W+B_i) | \mathbf{k}_p\rangle \langle \psi_i | \mathbf{k}_n\rangle}{[W+B_i-(\hbar^2/2m)k_p'^2][W+B_i-(\hbar^2/2m)k_p^2]}. \quad (40)$$

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[16] R. Riesz and B. Sz-Nagy, *Functional Analysis* (Frederick Ungar Publishing Company, New York, 1955), p. 341.

[$|\psi_i\rangle$ and $-B_i$ are, respectively, the bound-state eigenfunctions and eigenvalues of $H_n$.] Our next task is to take the matrix elements implied by our definition $\langle \mathbf{K}'S'\nu'\alpha'|A_B(W)|\mathbf{K}S\nu\alpha\rangle$.

The partial-wave matrix elements of $A_B(W)$ are required for the numerical solution of Eqs. (27). Let us enumerate some of their properties for the model we have been discussing. The $n$-$p$ scattering matrix, $t_{np}$ [Eq. (21)] has been chosen to be rotation invariant. The rotation invariance of $\mathbf{X}_n$ and $\mathbf{X}_p$ implies that

(1) that angular momentum and its $z$ projection are conserved;

(2) the angular momenta are restricted by triangle inequalities: $L'+S'\geq J\geq |L'-S'|$ and $L+S\geq J\geq |L-S|$; also $J+S'\geq L'\geq |J-S'|$ and $J+S\geq L\geq |J-S|$, where $L$ and $L'$ are the orbital angular momenta of the initial and final $d$-$c$ states, respectively.

Invoking parity conservation, we see that $L'$ and $L$ are further restricted by $(-1)^{L'}=(-1)^L$. The most general form for the partial-wave equations corresponding to Eq. (27) is then (in the $LSJ$ representation)

$$\langle K', J\pm1, 1|[A^J(W)-A_B{}^J(W)]|K, J\pm1, 1\rangle$$
$$=-\int_0^\infty dK''K''^2\langle K', J\pm1, 1|A^J(W)|K'', J+1, 1\rangle\tau_1(W-(\hbar^2K''^2/4m))\langle K'', J+1, 1|A_B{}^J(W)|K, J\pm1, 1\rangle$$
$$-\int_0^\infty dK''K''^2\langle K'J\pm1, 1|A^J(W)|K'', J-1, 1\rangle\tau_1(W-(\hbar^2K''^2/4m))\langle K'', J-1, 1|A_B{}^J(W)|K, J\pm1, 1\rangle, \quad (41a)$$

$$\langle K', J\pm1, 1|[A^J(W)-A_B{}^J(W)]|K, J\mp1, 1\rangle$$
$$=-\int_0^\infty dK''K''^2\langle K', J\pm1, 1|A^J(W)|K'', J+1, 1\rangle\tau_1(W-(\hbar^2K''^2/4m))\langle K'', J+1, 1|A_B{}^J(W)|K, J\mp1, 1\rangle$$
$$-\int_0^\infty dK''K''^2\langle K', J\pm1, 1|A^J(W)|K'', J-1, 1\rangle\tau_1(W-(\hbar^2K''^2/4m))\langle K'', J-1, 1|A_B{}^J(W)|K, J\mp1, 1\rangle. \quad (41b)$$

In Eq. (41), clarity has been sacrificed for brevity; the orbital angular-momentum indices are meant to be taken one at a time, i.e., (41a) and (41b) represents the coupled pair of equations for the amplitudes $J+1\rightarrow J+1$ and $J-1\rightarrow J+1$ (upper sign), and also the independent coupled pair of equations for the $J-1\rightarrow J-1$ and $J+1\rightarrow J-1$ amplitudes (lower sign). There are also coupled equations involving $L'=L=J$, $S=0\rightarrow S=1$ amplitudes:

$$\langle K',J,1|[A^J(W)-A_B{}^J(W)]|K,J,1\rangle$$
$$=-\int_0 dK''K''^2\langle K',J,1|A^J(W)|K'',J,1\rangle\tau_1(W-(\hbar^2K''^2/4m))\langle K'',J,1|A_B{}^J(W)|K,J,1\rangle$$
$$-\int_0 dK''K''^2\langle K',J,1|A^J(W)|K'',J,0\rangle\tau_0(W-(\hbar^2K''^2/4m))\langle K'',J,0|A_B{}^J(W)|K,J,1\rangle, \quad (42a)$$

$$\langle K',J,1|[A^J(W)-A_B{}^J(W)]|K,J,0\rangle$$
$$=-\int_0^\infty dK''K''^2\langle K',J,1|A^J(W)|K'',J,1\rangle\tau_1(W-(\hbar^2K''^2/4m))\langle K'',J,1|A_B{}^J(W)|K,J,0\rangle$$
$$-\int_0^\infty dK''K''^2\langle K',J,1|A^J(W)|K'',J,0\rangle\tau_0(W-(\hbar^2K''^2/4m))\langle K'',J,0|A_B{}^J(W)|K,J,0\rangle. \quad (42b)$$

The conservation of isospin requires the "spin-change" matrix elements of $A_B{}^J$ [i.e., such matrix elements as $\langle K',J,0|A_B{}^J(W)|K,J,1\rangle$] to vanish. (When nuclear cores are sufficiently massive that $m_c=\infty$ gives a sensible approximation, the $p$-core Coulomb repulsion will give rise to "spin-change" forces comparable in magnitude with the deuteron spin-orbit forces used in some optical-model calculations.[1] One may conclude from this that any realistic theory of deuteron-induced reactions must include "spin-change" as well as "spin-flip" of the deuteron if it is to be internally consistent.)

In the most general case, Eqs. (41) and (42) represent three pairs of coupled one-dimensional (singular) integral equations of a type long familiar in coupled-channel potential scattering theory. The set to be solved actually reduces to two pairs, because $A_B{}^J(W)$ is symmetric:

$$\langle K'L'S'|A_B{}^J(W)|KLS\rangle \equiv \langle KLS|A_B{}^J(W)|K'L'S'\rangle. \tag{43}$$

Equation (43) insures that the matrix $\langle K_\mu L'S'|A^J(W)|K_\nu LS\rangle$ and its transpose are simultaneously obtained, reducing the labor by $\frac{1}{3}$. (Here $K_\nu$, $\nu=1,2,\cdots$ is a discrete set of points in $[0,\infty)$, the "mesh" of the numerical solution.) Equations (41)–(43) and the remarks associated with them are, of course, equally applicable to the spin-zero inert core of finite mass.

## V. NUMERICAL RESULTS IN THE STUDY OF A SIMPLE MODEL

As a test of the practicality and tractability of Eq. (27) from the standpoint of numerical approximation, an extremely simple case of Eq. (42a) was solved numerically using an IBM-7044 computer. The $n$-core and $p$-core scattering matrices were taken to be identical and to have one $1s$ separable term:

$$\langle \mathbf{k}'\sigma'|t_n(z)|\mathbf{k}\sigma\rangle \equiv \langle \mathbf{k}'\sigma'|t_p(z)|\mathbf{k}\sigma\rangle = -\langle \mathbf{k}'|f\rangle \bar{\tau}(z)\delta_{\sigma'\sigma}\langle f|\mathbf{k}\rangle. \tag{44}$$

The "form factors" were taken to have the Yamaguchi[17] form:

$$\langle \mathbf{k}|f\rangle = (4\pi)^{-1/2}N_{1/2}/(k^2+\beta^2). \tag{45}$$

With a similar choice for $|\phi_c\rangle$, i.e.,

$$\langle \mathbf{q}|\phi_d\rangle = (4\pi)^{-1/2}N_d/(q^2+\alpha^2), \tag{46}$$

the explicit partial-wave matrix elements of $A_B$ appearing in (42a) are seen to be

$$\langle K'L'S'|A_B{}^J(W)|KLS\rangle = -(K'K)^{-1}|N_d|^2|N_{1/2}|^2\delta_{S'S}\delta_{L'L}2(2m/\hbar^2)^2$$
$$\times\bigg(\int_0^\infty dx\ \bar{\tau}(Q^2+i\eta-x^2)\bigg[f_L(x,K',Q^2)f_L(x,K;Q^2)+\delta_{L,0}|N_{1/2}|^4\pi^2\left(\frac{2m}{\hbar^2}\right)^2\frac{x^2|\bar{\tau}(x^2+i\eta)|^2}{(\beta^2+x^2)^2}g^{(+)}(x,K';Q^2)g^{(-)}(x,K;Q^2)$$
$$+\delta_{L,0}|N_{1/2}|^2\pi\frac{2m}{\hbar^2}\left(\frac{x}{x^2+\beta^2}\right)\{\bar{\tau}(x^2+i\eta)g^{(+)}(x,K';Q^2)f_0(x,K;Q^2)+\bar{\tau}(x^2-i\eta)f_0(x,K';Q^2)g^{(-)}(x,K;Q^2)\}\bigg]$$
$$+\delta_{L,0}|N_{1/2}|^2\pi^2\left(\frac{2m}{\hbar^2}\right)^2\bar{\tau}(Q^2+\bar{\kappa}^2+i\eta)g^{(+)}(i\bar{\kappa},K';Q^2)g^{(+)}(i\bar{\kappa},K;Q^2)\bigg), \tag{47}$$

where $Q^2=(2m/\hbar^2)E$, $\bar{\kappa}^2=(2m/\hbar^2)\epsilon_n$, and $0\leq L-1\leq J\leq L+1$. The functions $f_L(x,K)$ and $g^{(\pm)}(x,K)$ which appear above are defined in Appendix A. We see that the kernel may be defined by one one-dimensional integral. Unfortunately this integral must be done numerically in practice, and turns out to be a major factor in the large amount of computer time required to solve even this simple example. The singularities of the functions $f_L(x,K)$ and $g^{(\pm)}(x,K)$ are at worst logarithmic. They arise from the vanishing of the energy denominators [the $G_0(W)$] of expression (40) [of course, substituted for $G_0U_{dd}G_0$ in (26)] within the range of the various angular integrations. Furthermore, for $K'$ and $K$ real, the singularities can never coincide with the pole of the propagator $\bar{\tau}(Q^2+i\eta-x^2)$ and so the right-hand side of (47) is never singular.

It may easily be shown that $\langle K'L'S'|A_B{}^J(W)|KLS\rangle$ is analytic in both $K'$ and $K$ in a finite region of the point $K_0=[(4m/\hbar^2)(W+\epsilon_d)]^{1/2}$, which is the only singularity of the kernels in (41) and (42). Therefore, Rubin's[18] method for analytically continuing potential scattering amplitudes (defined for complex energy) to real, positive energy may be applied without modification to prove the analytic continuation of $\langle K'L'S'|\times A^J(W)|KLS\rangle$, both on- and off-the-energy shell. This result may be generalized in obvious fashion[1] to the case of arbitrary (well-behaved) interactions $V_n$, $V_p$, and $V_{np}$, and of finite $m_c$. Therefore, well-behaved interactions will generate amplitudes [defined by Eqs. (15) and (19)] with well-defined analytic continuations to real positive energies, and which are continuous in their momentum indices. Furthermore, once the partial-wave amplitudes are known, there is no doubt of the existence of the singular integrals which define the various reaction amplitudes in terms of the $\langle K'L'S'|A^J(W)|KLS\rangle$.

[17] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).

[18] M. Rubin, Ph.D. thesis, Princeton University, 1964 (unpublished).

TABLE I. Unitarity sum check.

| Deeuteron energy (MeV) \ n-c binding energy | 8 MeV $-\mathrm{Im}[A^{\dagger}(\mathbf{K},\mathbf{K};E)]$ | 8 MeV Sum rule | 2 MeV $-\mathrm{Im}[A^{\dagger}(\mathbf{K},\mathbf{K};E)]$ | 2 MeV Sum rule |
|---|---|---|---|---|
| 8.44 | 4.07 | 2.95 | 6.00 | 9.75 |
| 9.48 | 4.49 | 6.09 | 6.72 | 6.09 |
| 10.00 | 4.97 | 5.07 | 6.18 | 6.73 |
| 10.52 | 4.66 | 5.69 | 5.68 | 9.45 |
| 11.04 | 4.69 | 5.72 | 6.25 | 10.39 |
| 11.55 | 4.76 | 5.06 | 6.17 | 7.03 |
| 12.07 | 4.79 | 5.41 | 5.52 | 6.65 |
| 12.59 | 4.80 | 5.24 | 6.52 | 3.67 |
| 13.11 | 4.74 | 4.75 | 6.28 | 4.71 |
| 13.63 | 4.74 | 5.07 | 6.09 | 4.83 |
| 14.15 | 4.78 | 5.43 | 5.91 | 6.09 |
| 14.66 | 4.58 | 4.92 | 5.73 | 6.33 |
| 15.70 | 4.53 | 5.70 | 5.41 | 7.00 |
| 16.74 | 4.65 | 5.28 | 5.15 | 7.75 |
| 17.77 | 4.41 | 3.49 | 5.41 | 5.48 |
| 18.81 | 3.86 | 10.15 | 5.12 | 5.28 |
| 19.85 | 4.62 | 7.25 | ... | ... |
| 20.88 | 4.23 | 4.65 | ... | ... |

The partial-wave equations were solved for a range of c.m. energies, for two values of the neutron-binding energy (i.e., for two $Q$ values in the stripping reaction). Differential and total cross sections for stripping and elastic-deuteron scattering were calculated and plotted as functions of the deuteron kinetic energy (Figs. 2–7). The break-up amplitude was not calculated, as it takes a more complicated form in terms of the deuteron-elastic amplitude, and to compute it would have required more computer time than seemed worthwhile. [The total breakup cross section was estimated and found to be of the same order as the elastic cross section; therefore it was small compared to the total cross section (see Appendix D).]

From the unitarity relation and the definition of the cross sections, it is easy to see that with the Goldberger and Watson[10] normalizations used here the imaginary part of the forward elastic-scattering amplitude [$(d,d)$] obeys the following sum rule:

$$-\mathrm{Im}[A_{dd}^{+}(\mathbf{K},\mathbf{K};\hbar^2K^2/4m-\epsilon_d)] = (\hbar^2K/64m\pi^3)[\sigma_{dd}+\sigma_{pd}+\sigma_{np,d}+\sigma_{nd}]. \quad (48)$$

[Note that $\sigma(d+c\rightarrow b+p)\equiv\sigma(d+c\rightarrow b'+n)$ in this model.] When comparing the calculated cross sections with the above sum rule, the following sources of error must be kept in mind:

(i) There is an inherent imprecision of the amplitudes themselves.

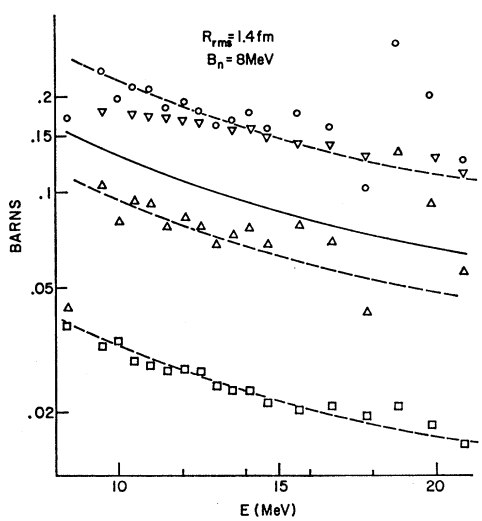



FIG. 2. Deuteron reaction cross sections versus deuteron energy, in the model of Sec. V. The circles are the total cross section, the inverted triangles the unitarity limit, the triangles are the stripping cross section, and the squares are the elastic cross section. The neutron-binding energy is 8 MeV.

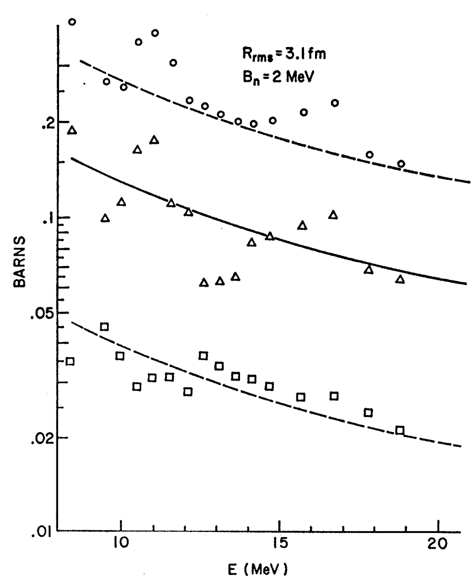
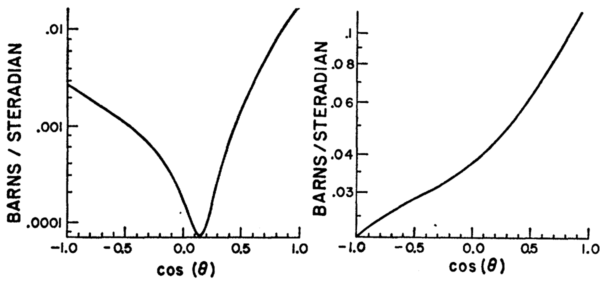
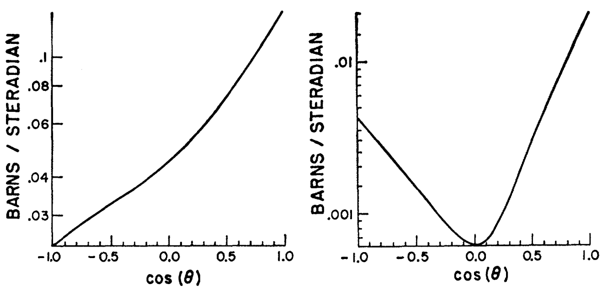



FIG. 3. Same as Fig. 2 except that the unitarity limit was now shown, and the neutron-binding energy is 2 MeV. In both Fig. 2 and Fig. 3, $\beta=4$ fm$^{-1}$.




(a) (b)

FIG. 4. (a) Elastic Born differential cross section at 9.48-MeV deuteron energy, with neutron bound by 8 MeV and $\beta=4$ fm$^{-1}$. (b) elastic differential cross section, with the same parameters as in (a).




(a) (b)

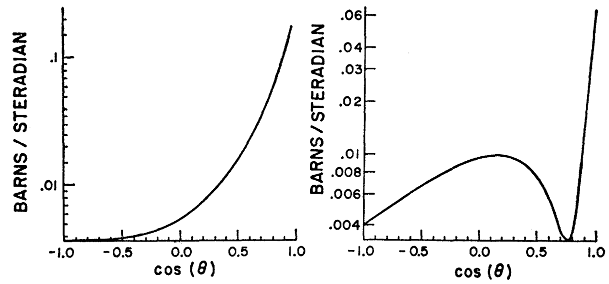
FIG. 5. (a) Elastic Born differential cross section at 9.48-MeV deuteron energy, with neutron bound by 2 MeV and $\beta=4$ fm$^{-1}$. (b) same as (a) except elastic cross section.

(ii) Only the first four partial waves were available to calculate $-\text{Im}[A_{dd}^{+}(E)]$.

(iii) $\sigma_{pd}$ and $\sigma_{nd}$, the stripping cross sections, contained (positive) contributions from all partial waves.

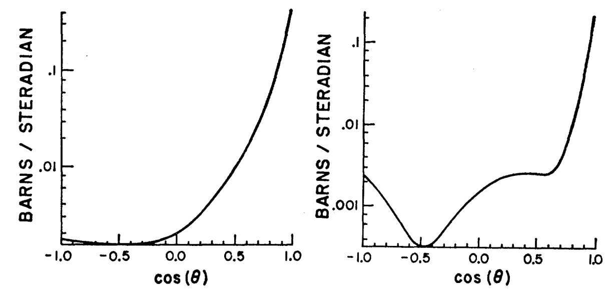
(iv) $\sigma(d+c\rightarrow c+n+p)$ was, of course, not available at all.

In view of the above remarks, the agreement, for the most part appears to be good. In each case, there were only two points (out of eighteen) which badly violated (by a factor of 2) the above sum rule. For these points it was found that the numerical-inversion process had become unstable in calculating the $l=0$ partial-wave amplitude. (The imaginary parts of the $l=0$ on-shell elastic amplitudes had become positive for these bad cases, indicating that these solutions were hardly to be trusted.) This instability was found to result from




(a) (b)

FIG. 6. (a) Stripping Born differential cross section and (b) stripping differential cross section, with the parameters of Fig. 4(a) and 4(b).




(a) (b)

FIG. 7. Stripping cross sections as in Fig. 6(a) and 6(b), with parameters of Figs. 5(a) and 5(b).

certain technical difficulties in the calculation of the elastic Born term, which only showed up for some combinations of deuteron energy and neutron-binding energy. The remainder of the points are in good agreement with the sum rule, and are tabulated in Table I. The cross sections are plotted against deuteron energy in Figs. 2 and 3. To give an idea of the physical magnitudes involved, the "geometrical" cross section, $4\pi/K_d^2$, is also plotted in these figures. The general trend of each set of points was indicated by fitting the curve $C/K_d^2$ to it by eye.

Several qualitative conclusions may be drawn from the calculated results. (Some of them must be regarded as still tentative, owing to the numerical inaccuracies discussed previously.) In the first place, it was found that the effective deuteron-core potential is not definite in sign, even when the $p$-core and $n$-core potentials are both attractive. To see how this can come about, let us again examine Eq. (27):

$$\langle\mathbf{K}'|A(W)|\mathbf{K}\rangle=\langle\mathbf{K}'|A_B(W)|\mathbf{K}\rangle+\frac{4m}{\hbar^2}\int d\mathbf{k}''\frac{\langle\mathbf{K}'|A(W)|\mathbf{K}''\rangle S_d(K_0^2+i\eta-K''^2)\langle\mathbf{K}''|A_B(W)|K\rangle}{K_0^2+i\eta-K''^2}, \tag{49}$$

where we have written

$$\tau_d(W-\hbar^2K''^2/4m)\equiv-\frac{S_d(K_0^2+i\eta-K''^2)}{(K_0^2+i\eta-K''^2)\hbar^2/4m},$$

and plainly, $S_d(0)=1$. If we symmetrize Eq. (44) by multiplying by the appropriate factors of $\sqrt{S}$, we have exactly the Lippmann-Schwinger equation for $d$-core elastic scattering, in which all the inelastic effects have been included in the pseudopotential

$$V_0(\mathbf{K}',\mathbf{K};W)=[S_d(K_0^2+i\eta-K'^2)]^{1/2}\langle\mathbf{K}'|A_B(W)|\mathbf{K}\rangle[S_d(K_0^2+i\eta-K^2)]^{1/2}: \tag{50}$$

$$\tilde{A}(\mathbf{K}',\mathbf{K};W)=V_0(\mathbf{K}',\mathbf{K};W)+\int d\mathbf{k}''\frac{\tilde{A}(\mathbf{K}',\mathbf{K}'';W)V_0(\mathbf{K}'',\mathbf{K};W)}{W-\hbar^2K''^2/4m+\epsilon_d}. \tag{51}$$

Equation (51) may be derived using projective techniques,[14] and gives the equivalent expression for $V_0$: if $\mathbf{B}$ is the projection operator for the deuteron bound state, $\mathbf{B}=|\psi_d\rangle\langle\psi_d|$, and defining $\mathbf{R}=\mathbf{1}-\mathbf{B}$, $V_0$ may be written

$$V_0(W)=\mathbf{B}(V_n+V_p)\mathbf{B}+\mathbf{B}(V_n+V_p)\mathbf{R}[W-\mathbf{R}H\mathbf{R}]^{-1}\mathbf{R}(V_n+V_p)\mathbf{B}. \tag{52}$$

The sign of the imaginary part of $V_0(W)$ is $-\mathrm{sgn}[\mathrm{Im}(W)]$, but above any of the thresholds in $[W-RHR]^{-1}$, the sign of the real part of $V_0(W)$ is not definite even though $V_n$ and $V_p$ are both, say, negative-definite. In this calculation, the matrix elements of $A_B(W)$ were *not* definite in sign. This indicates a serious shortcoming of the usual optical model (which is *not* repaired by such expedients as taking nonlocal potentials[19]), which assumes the real part of the optical potential to be definite in sign. One cause of the low quality of optical-model wave functions in the near zone is undoubtedly the above deficiency.

The second qualitative result, which seemed rather surprising, is that in this model, the breakup cross section [as estimated in Appendix D, or from the unitarity sum rule (48)] is almost negligible; most of the absorption comes from transitions to the stripping channels. Also, the total stripping cross sections were considerably larger than the elastic $(d,d)$ cross section. This indicates that the absorption is considerably less than maximum, since for "black" nuclei, $\sigma_{\mathrm{el}}=\sigma_{\mathrm{in}}=\frac{1}{2}\sigma_{\mathrm{tot}}$, as is well known. (One would, of course, expect somewhat different results if there were an asymmetry in the $n$-core and $p$-core forces, and/or if there were many available inelastic channels, so that the absorption approached the blackbody value.) Phillips'[20] results on $n$-$d$ scattering above the breakup threshold are quite similar, qualitatively. Even with only the breakup channel competing with the elastic channel (and with the considerable asymmetry of the $n$-$n$ and $p$-$n$ forces), Phillips finds that the breakup cross section is only a small fraction of the total cross section.

To estimate the effect of the rescattering terms in (40), the calculation was repeated at one energy using only the first term on the right-hand side of Eq. (47); i.e., approximating $\mathbf{X}_n(W)+\mathbf{X}_p(W)$ by $t_n(W)+t_p(W)$. Clearly, in this simple model, the $L>1$ terms are unaltered. However, the $L=0$ elastic Born term, $A_B(W)$, was drastically altered (the real part changed sign), indicating the great importance of rescattering at these energies.

The absorption in the elastic channel shows up in the elastic angular distribution as sharp diffraction minima at $\sim 80°$. The differences between the elastic Born angular distributions (which are not very different from the impulse approximation) and the actual elastic angular distributions are striking. This also indicates that rescattering effects are extremely important. Some typical elastic angular distributions are shown in Figs. 4 and 5.

The stripping angular distributions also show diffraction minima at about $\theta=40°$, and are peaked at $\theta=0°$ (as $s$-wave stripping patterns should be!). Some typical stripping angular distributions are given in Figs. 6 and 7.

It should be noted that it is the $l$-value of the final $n$-core wave function which determines the (forward) angle at which the stripping angular distribution is peaked. This is because the stripping Born term [see Eq. (35)] is dominated (at forward angles) by the term

$$\sum_{\sigma_n'}\psi_{I_fM_f}{}^*(\mathbf{K}-\mathbf{k}_p',\sigma_n')\langle\tfrac{1}{2}\,\tfrac{1}{2}\sigma_n'\sigma_p'|1,\nu\rangle\phi_d(|\tfrac{1}{2}\mathbf{K}-\mathbf{k}_p'|).$$

Obviously this term has a pole in the momentum transfer, $|\mathbf{K}-\mathbf{k}_p'|=Q$, near the physical region (small, imaginary $Q$). Furthermore, if $l_f$ is the $l$ value of the $n$-core state $\psi_{I_fM_f}$, this part of the Born term will have a $Q^{l_f}$ dependence for small $Q$ (forward angles). The situation is complicated by the presence of the $\mathbf{X}_p$ term in the stripping Born term (35); but in circumstances where the $\mathbf{X}_p$ term is small, one may say that the stripping amplitude is dominated by the behavior $Q^{l_f}/(Q^2+K_n^2)$ at small $Q$, and this leads to the characteristic stripping pattern at forward angles. [The characteristic rapid falloff with angle of stripping angular distributions results from increased absorption at higher momentum transfer (i.e., in the low-$l$ partial waves).]

Aaron and Shanley[21] have recently reported the results of a similar investigation, using Amado's stripping model,[22] with no $p$-core interaction. Their stripping and elastic-deuteron angular distributions look qualitatively similar to mine which undoubtedly indicates only that any physically reasonable model must give sensible angular distributions. Neglecting the final $p$-core interaction seems to be a reasonable approximation: in this investigation, it was found that the effect

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of the distortion of the proton final wave function is negligible at these energies; that is, in the Born term for stripping, $V_{np}+\mathbf{X}_p(W)$, the calculated contribution of the $\mathbf{X}_p$ term is negligible ($<1/50$) compared to that of the $V_{np}$ term. This remark to some extent explains the qualitative success of Amado's stripping theory, as well as that of the Butler theory.[23]

Finally, it may be noted that all the total cross sections in the weakly bound case were somewhat larger than those for the strongly bound case, all else being the same. This may undoubtedly be attributed to the fact that the rms radius of the $n$-core state is $\sim 3$ fm for a neutron-binding energy of 2 MeV (with $\beta=4.0\ \mathrm{fm}^{-1}$), and is $\sim 1.4$ fm when the neutron is bound by 8 MeV, leading to a smaller geometrical cross section in the tight-binding case.

## VI. CONCLUSIONS, EXTENSIONS, AND PHILOSOPHY

It is appropriate to conclude by briefly evaluating the progress which has been made, by suggesting extensions of the theory to more realistic cases, and by stating the philosophy of this approach to direct reaction theory, not necessarily in this order. First, the limitations of the theory should be kept in mind. They are just such limitations of the model as the use of single-particle wave functions and treatment of each of the three particles as elementary (thereby neglecting antisymmetrization of the wave functions). These model-dependent approximations are common to all treatments. The chief advantage of three-body theories over, say, the DWBA is that the former have mathematically well-defined solutions, and so one may know *a priori* the accuracy of their numerical approximations. It is thus possible to clearly distinguish the behavior of the model from artifacts of the approximation.

The theory presented in Sec. III is more fundamental than the DWBA in that it contains no undetermined parameters and it predicts more from less data. The input consists of detailed information on the $n$-core and $p$-core interactions, summarized in the $n$-core and $p$-core scattering matrices. From this one calculates the complete behavior of the $n+p+c$ system, including the deuteron-optical potential. Because the three-body theory is far more detailed than the DWBA, calculations are considerably harder. Even the simple model of Sec. V took much more time than that required for a corresponding DWBA fit. The multiparticle scattering approach has been criticized by some proponents of the DWBA on the grounds that it takes too much computer time, and that the models which have so far been examined (using separable potentials) are "unrealistic" because they neglect the higher partial waves in the $n$-core and $p$-core interactions. What is being criticized is the "realism" of separable potentials as compared to local ones. To me this is a matter of personal prejudice and convenience; the paucity of empirical knowledge of, say, $n$-core interactions in the higher partial waves makes it hard to determine whether they are better represented by local potentials than by separable potentials. (These prejudices derive from the fact that local potentials are useful for solving the Schrödinger equation, whereas separable ones are more convenient for solving the Lippmann-Schwinger equation.)

The above remarks notwithstanding, there is some motivation for using local potentials in three-body calculations of direct reactions. A direct comparison between the DWBA and multiparticle reaction theories is desirable because one is responsible for explaining the success of the DWBA, which persists despite its manifold faults. (The resolution of the current ambiguities in DWBA fits should be considered a useful bonus.) Several approaches to practical calculations with local potentials (using the model of Sec. IV) suggest themselves. The first is to employ the approximation of Scadron and Weinberg,[24] which gives the explicit formula

$$t_n(z)\simeq V_n+\sum_{ijk}|v_i\rangle\Delta_{ij}(z)J_{jk}(z)\langle v_k|\,, \tag{53}$$

where

$$J_{ij}(z)=\langle v_i|G_0(z)|v_j\rangle\,, \tag{54}$$

and

$$\Delta_{ij}^{-1}(z)=\delta_{ij}-J_{ij}(z)\,, \tag{55}$$

and similarly for $t_p$. In this (nonunitary) approximation, the higher partial waves are approximated by their Born terms, and the lower partial waves are given by separable forms which have the correct poles and residues. The integrals in (40) involving the purely separable terms are no harder than those of Sec. V, and the use of local forms for $V_n$ and $V_p$ facilitates the calculation of the other terms. This approach is now being investigated in more detail.

Secondly, one could use the methods of Blankenbecler and Sugar[25] to obtain upper and lower bounds for the matrix elements of $A(W)$. Since their technique requires taking matrix elements of $A_B(W)$ with judiciously selected trial functions, the problem then reduces to evaluation of a number of many-dimensional (singular) integrals. Conceivably this could be effected in a practical way by Monte-Carlo methods (even with $t_n$ and $t_p$ derived from local potentials).

Finally, one could think of writing

$$t_n=t_n^{\mathrm{sep}}+[t_n-t_n^{\mathrm{sep}}]=t_n^{\mathrm{sep}}+\Delta t_n\,, \tag{56}$$

where $t_n^{\mathrm{sep}}$ is a separable approximation to $t_n$, and similarly for $t_p$. One could then solve the parts involving the separable approximations $t_n^{\mathrm{sep}}$ and $t_p^{\mathrm{sep}}$ exactly, as described above, and treat the corrections, $\Delta t_n$ and $\Delta t_p$ as perturbations. Presumably all the singularities (and near-singularities) of $t_n$ would be included in $t_n^{\mathrm{sep}}$,

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so that $\Delta t_n$ and $\Delta t_p$ are well behaved, and in some sense "small," corrections. The correction terms would involve multiple quadratures, which again might be performed by Monte-Carlo methods, so that this last approach is sort of hybrid of the first two. A complete theoretical treatment of such an approach would necessarily include some understanding of how to optimally approximate a general two-body off-shell scattering matrix (assuming it were known) by separable terms. (It is clear that this is a somewhat different problem from that of approximating a *potential* by separable terms.)

In all the computational schemes sketched above, it is perfectly straightforward to use local optical potentials to represent the $n$-core and $p$-core interactions. One merely has to be somewhat careful in defining the convolution (14), in order to avoid introducing unphysical singularities.

Despite the enormous practical difficulties which remain to be overcome, it appears likely that it will be possible to do (a few) calculations using local potentials for $V_n$ and $V_p$, at least in the inert, recoilless core model. As was previously mentioned, it may not be a bad approximation to treat the nucleus as inert, when there are many available stripping channels to provide absorption. The reason is, heuristically, that once the nucleus is "sufficiently black," increasing the absorption will not appreciably change the ratio of elastic to inelastic cross sections. Thus the "optical potential" [Eq. (50)] would not be appreciably different from that calculated from Eq. (40).


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## APPENDIX A: DERIVATION OF THE KERNEL OF THE DEUTERON ELASTIC SCATTERING EQUATION, IN THE MODEL OF SECTION V

We had

$$(W-H_n-H_p)^{-1}=\int_{-B}^{\infty} dx\, \hat{\delta}(x-H_n)(W-x-H_p)^{-1}. \tag{36}$$

We now expand the formal relation (37) using the (known) eigenfunctions of $H_n$:

$$\hat{\delta}(x-H_n)=\sum_{\sigma'}\int |\psi_{\mathbf{k},\sigma}{}^{+}\rangle \delta\left(x-\frac{\hbar^2k^2}{2m}\right)\langle\psi_{\mathbf{k},\sigma}{}^{+}|\, d^3k+\sum_i |\psi_i\rangle\delta(x+B_i)\langle\psi_i|\,, \tag{A1}$$

where the $|\psi_i\rangle$ and $-B_i$ are the bound-state wave functions and energies of $H_n$ which appear in (40). We now note that

$$|\psi_{\mathbf{k},\sigma}{}^{+}(n)\rangle=|\mathbf{k},\sigma(n)\rangle+\left(\frac{\hbar^2k^2}{2m}+i\eta-K_n\right)^{-1} t_n\left(\frac{\hbar^2k^2}{2m}+i\eta\right)|\mathbf{k}\sigma\rangle \tag{A2}$$

and

$$(W-x-H_p)^{-1}=(W-x-K_p)^{-1}+(W-x-K_p)^{-1}t_p(W-x)(W-x-K_p)^{-1}. \tag{A3}$$

Using (A1) in (36) and also expanding as in (A2) and (A3), we have matrix elements

$$\begin{aligned}
&\langle\mathbf{k}_n'\mathbf{k}_p'|G_0(W)[\mathbf{X}_n(W)+\mathbf{X}_p(W)]G_0(W)|\mathbf{k}_n\mathbf{k}_p\rangle=\frac{\delta(\mathbf{k}_n'-\mathbf{k}_n)\langle\mathbf{k}_p'|t_p(W-\hbar^2k_n{}^2/2m)|\mathbf{k}_p\rangle}{[W-(\hbar^2/2m)(k_n'^2+k_p'^2)][W-(\hbar^2/2m)(k_n{}^2+k_p{}^2)]}\\
&+\frac{\langle\mathbf{k}_n'|t_n((\hbar^2k_n{}^2/2m)+i\eta)|\mathbf{k}_n\rangle\langle\mathbf{k}_p'|t_p(W-(\hbar^2k_n{}^2/2m))|\mathbf{k}_p\rangle}{(\hbar^2/2m)(k_n{}^2+i\eta-k_n'^2)[W-(\hbar^2/2m)(k_n{}^2+k_p'^2)][W-(\hbar^2/2m)(k_n{}^2+k_p{}^2)]}\\
&+\frac{\langle\mathbf{k}_n'|t_n((\hbar^2k_n'^2/2m)-i\eta)|\mathbf{k}_n\rangle\langle\mathbf{k}_p'|t_p(W-\hbar^2k_n'^2/2m)|\mathbf{k}_p\rangle}{(\hbar^2/2m)(k_n'^2-i\eta-k_n{}^2)[W-(\hbar^2/2m)(k_n'^2+k_p'^2)][W-(\hbar^2/2m)(k_n'^2+k_p{}^2)]}\\
&+\sum_i\frac{\langle\mathbf{k}_n'|\psi_i\rangle\langle\mathbf{k}_p'|t_p(W+B_i)|\mathbf{k}_p\rangle\langle\psi_i|\mathbf{k}_n\rangle}{[W+B_i-(\hbar^2k_p'^2/2m)][W+B_i-(\hbar^2k_p{}^2/2m)]}\\
&+\delta(\mathbf{k}_p'-\mathbf{k}_p)\left\{\frac{\langle\mathbf{k}_n'|t_n(\hbar^2k_n{}^2/2m+i\eta)|\mathbf{k}_n\rangle}{(\hbar^2/2m)(k_n{}^2+i\eta-k_n'^2)[W-(\hbar^2/2m)(k_n{}^2+k_p{}^2)]}+\frac{\langle\mathbf{k}_n'|t_n(\hbar^2k_n'^2/2m-i\eta)|\mathbf{k}_n\rangle}{(\hbar^2/2m)(k_n'^2-i\eta-k_n{}^2)[W-(\hbar^2/2m)(k_n'^2+k_p{}^2)]}\right.\\
&\left.+\int d\mathbf{k}''\frac{\langle\mathbf{k}_n'|t_n(\hbar^2k''^2/2m+i\eta)|\mathbf{k}''\rangle\langle\mathbf{k}''|t_n((\hbar^2k''^2/2m)-i\eta)|\mathbf{k}_n\rangle}{(\hbar^2/2m)(k''^2+i\eta-k_n'^2)[W-(\hbar^2/2m)(k''^2+k_p{}^2)](\hbar^2/2m)(k''^2-i\eta-k_n{}^2)}+\sum_i\frac{\langle\mathbf{k}_n'|\psi_i\rangle\langle\psi_i|\mathbf{k}_n\rangle}{W+B_i-(\hbar^2/2m)k_p{}^2}\right\}\\
&+\int d\mathbf{k}''\frac{\langle\mathbf{k}_n'|t_n((\hbar^2k''^2/2m)+i\eta)|\mathbf{k}''\rangle\langle\mathbf{k}_p'|t_p(W-(\hbar^2k''^2/2m))|\mathbf{k}_p\rangle\langle\mathbf{k}''|t_n((\hbar^2k''^2/2m)-i\eta)|\mathbf{k}_n\rangle}{[(\hbar^2/2m)(k''^2+i\eta-k_n'^2)][W-(\hbar^2/2m)(k''^2+k_p'^2)][W-(\hbar^2/2m)(k''^2+k_p{}^2)][(\hbar^2/2m)(k''^2-i\eta-k_n{}^2)]}.
\end{aligned} \tag{A4}$$

The portion of expression (A4) which contains $\delta(\mathbf{k}_p'-\mathbf{k}_p)$ may be shown to be just $G_0(W)t_n(W)G_0(W)$ by three arguments: first, we note that since the contents of the brackets are independent of $V_p$, if we let $V_p\rightarrow 0$, this is the only term left, and is obviously $G_0(W)t_n(W)G_0(W)$. We may also realize that since the $G_0(\mathbf{X}_n+\mathbf{X}_p)G_0$ given by (A4) must be identical with that obtained from Eqs. (15), in which there is only one term containing $\delta(\mathbf{k}_p'-\mathbf{k}_p)$, i.e., $G_0t_nG_0$, the contents of the { } must be what we have said. Finally, we may show directly, using the Low equation for $t_n(W)$, i.e.,

$$\langle\mathbf{k}_n'|t_n(z)|\mathbf{k}_n\rangle=\langle\mathbf{k}_n'|V_n|\mathbf{k}_n\rangle+\sum_i\frac{\langle\mathbf{k}_n'|\psi_i\rangle\langle\psi_i|\mathbf{k}_n\rangle}{z+B_i}+\int d\mathbf{k}''\frac{\langle\mathbf{k}_n'|t_n((\hbar^2k''^2/2m)+i\eta)|\mathbf{k}''\rangle\langle\mathbf{k}''|t_n((\hbar^2k''^2/2m)-i\eta)|\mathbf{k}_n\rangle}{z-(\hbar^2/2m)k''^2}, \quad \text{(A5)}$$

that the expression in brackets is identically

$$\frac{\langle\mathbf{k}_n'|t_n(W-(\hbar^2k_p^2/2m))|\mathbf{k}_n\rangle}{[W-(\hbar^2/2m)(k_n'^2+k_p^2)][W-(\hbar^2/2m)(k_n^2+k_p^2)]}.$$

As the details of this latter direct proof consists only of algebra, we omit it. Substituting expressions (40), (45), and (46) into (26), and integrating over the delta functions which appear, and summing over the spin indices we have

$$\begin{aligned}
\langle\mathbf{K}'S'\nu'|A_B(W)|\mathbf{K}S\nu\rangle=&\frac{|N_d|^2}{4\pi}\delta_{S'S}\delta_{\nu'\nu}\\
&\times\Bigg(-2\left(\frac{2m}{\hbar^2}\right)\frac{|N_{1/2}|^2}{4\pi}\int d\mathbf{x}\frac{[\alpha^2+(\tfrac{1}{2}\mathbf{K}'-\mathbf{x})^2]^{-1}\bar{\tau}(Q^2+i\eta-x^2)[\alpha^2+(\tfrac{1}{2}\mathbf{K}-\mathbf{x})^2]^{-1}}{[\beta^2+(\mathbf{K}'-\mathbf{x})^2][Q^2+i\eta-\tfrac{1}{2}K'^2-2(\tfrac{1}{2}\mathbf{K}'-\mathbf{x})^2][Q^2+i\eta-\tfrac{1}{2}K^2-2(\tfrac{1}{2}\mathbf{K}-\mathbf{x})^2][\beta^2+(\mathbf{K}-\mathbf{x})^2]}\\
&+\frac{|N_{1/2}|^4}{(4\pi)^2}\left(\frac{2m}{\hbar^2}\right)^3\int d\mathbf{k}_n'\int d\mathbf{k}_n\Bigg[[\alpha^2+(\tfrac{1}{2}\mathbf{K}'-\mathbf{k}_n'^2)^2]^{-1}(\beta^2+k_n^2)^{-1}[\beta^2+(\mathbf{K}'-\mathbf{k}_n')^2]^{-1}\\
&\times\Bigg\{\frac{\bar{\tau}(k_n^2+i\eta)\bar{\tau}(Q^2+i\eta-k_n^2)}{[Q^2+i\eta-k_n^2-(\mathbf{K}'-\mathbf{k}_n')^2](k_n^2+i\eta-k_n'^2)[Q^2+i\eta-k_n^2-(\mathbf{K}-\mathbf{k}_n)^2]}\\
&-\frac{\bar{\tau}(k_n'^2-i\eta)\bar{\tau}(Q^2+i\eta-k_n'^2)}{[Q^2+i\eta-k_n'^2-(\mathbf{K}'-\mathbf{k}_n')^2](k_n^2+i\eta-k_n'^2)[Q^2+i\eta-k_n'^2-(\mathbf{K}-\mathbf{k}_n)^2]}\\
&-\frac{|N_{1/2}|^2}{4\pi}\left(\frac{2m}{\hbar^2}\right)\int\frac{d\mathbf{k}''|\bar{\tau}(k''^2+i\eta)|^2\bar{\tau}(Q^2+i\eta-k''^2)}{(\beta^2+k''^2)^2(k''^2+i\eta-k_n'^2)[Q^2+i\eta-k''^2-(\mathbf{K}'-\mathbf{k}_n)^2][Q^2+i\eta-k''^2-(\mathbf{K}-\mathbf{k}_n)^2](k''^2-i\eta-k_n^2)}\\
&-\left(\frac{2m}{\hbar^2}\right)\frac{\bar{\tau}(Q^2+\bar{\kappa}^2+i\eta)}{(k_n'^2+\bar{\kappa}^2)[Q^2+i\eta+\bar{\kappa}^2-(\mathbf{K}'+\mathbf{k}_n')^2][Q^2+i\eta+\bar{\kappa}^2-(\mathbf{K}-\mathbf{k}_n)^2](k_n^2+\bar{\kappa}^2)}\Bigg\}\\
&\times[\alpha^2+(\tfrac{1}{2}\mathbf{K}-\mathbf{k}_n)^2]^{-1}(\beta^2+k_n^2)^{-1}[\beta^2+(\mathbf{K}-\mathbf{k}_n)^2]^{-1}\Bigg]\Bigg), \quad \text{(A6)}
\end{aligned}$$

where $Q^2+i\eta=2mW/\hbar^2$. The conservation of $S$ and its $z$ projection implied by (A6) means that $L$ and its $z$ projection are also conserved. Therefore, for this simple case, we partial wave expand in $L$ rather than $J$.

$$\begin{aligned}
\langle K'L'M'S'\nu'|A_B(W)|KLMS\nu\rangle&=\int d\hat{K}'\int d\hat{K}\; Y_{L'M'}^*(\hat{K}')\langle\mathbf{K}'S'\nu'|A_B(W)|\mathbf{K}S\nu\rangle Y_{LM}(\hat{K})\\
&=\delta_{L'L}\delta_{M'M}\delta_{S'S}\delta_{\nu'\nu}A_B^{L,S}(K',K;W). \quad \text{(A7)}
\end{aligned}$$

We now consider (A6) term by term. The function

$$\mathfrak{D}^{-1}(x,K,Q^2,\hat{x}\cdot\hat{K})=\frac{1}{[\alpha^2+(\frac{1}{2}\mathbf{K}-\mathbf{x})^2][Q^2+i\eta-K^2-2x^2+2\mathbf{x}\cdot\mathbf{K}][\beta^2+(\mathbf{K}-\mathbf{x})^2]} \tag{A8}$$

has the following expansion in Legendre polynomials:

$$\mathfrak{D}^{-1}(x,K,Q^2,\hat{x}\cdot\hat{K})=\sum_{\lambda=0}^{\infty}(2\lambda+1)P_\lambda(\hat{x}\cdot\hat{K})u_\lambda(x,K;Q^2)\,, \tag{A9}$$

where

$$u_\lambda(x,K;Q^2)=\frac{1}{2}\int_{-1}^{1}dt\,P_\lambda(t)\,\mathfrak{D}^{-1}(x,K,Q^2;t)\,. \tag{A10}$$

Clearly, then, using the addition theorem for spherical harmonics, we obtain

$$\int d\hat{x}\;\mathfrak{D}^{-1}(x,K',Q^2;\hat{x}\cdot\hat{K}')\,\mathfrak{D}^{-1}(x,K,Q^2;\hat{x}\cdot\hat{K})=4\pi\sum_{\lambda=0}^{\infty}(2\lambda+1)P_\lambda(\hat{K}'\cdot\hat{K})u_\lambda(x,K';Q^2)u_\lambda(x,K;Q^2)\,. \tag{A11}$$

Inserting the result (A11) into the first term of (A6) and evaluating the angular integrals of (A7), we obtain for the contribution to (A7) of this first term

$$A_B{}^{L,S}(K',K;W)_1=-2\left(\frac{2m}{\hbar^2}\right)^2|N_d|^2|N_{1/2}|^2\int_0^\infty dx\,x^2\bar{\tau}(Q^2+i\eta-x^2)u_\lambda(x,K';Q^2)u_\lambda(x,K;Q^2)\,. \tag{A12}$$

Now noting that

$$\frac{1}{2}\int_{-1}^{1}P_l(t)(z-t)^{-1}dt=Q_l(z)\,,$$

we have

$$\begin{aligned}u_l(x,K;Q^2)&\equiv(xK)^{-1}f_l(x,K;Q^2)\\&=(xK)^{-1}\Bigg\{Q_l\left(\frac{\alpha^2+x^2+\frac{1}{4}K^2}{xK}\right)(\beta^2+\tfrac{1}{2}K^2-2\alpha^2-x^2)^{-1}(\tfrac{1}{2}K^2-Q^2-2\alpha^2)^{-1}\\&\qquad+Q_l\left(\frac{\beta^2+x^2+K^2}{2xK}\right)(x^2-\tfrac{1}{2}K^2-\beta^2+2\alpha^2)^{-1}(x^2-\beta^2-Q^2)^{-1}\\&\qquad-Q_l\left(\frac{2x^2+K^2-Q^2-i\eta}{2xK}\right)(2\alpha^2+Q^2-\tfrac{1}{2}K^2)^{-1}(x^2-\beta^2-Q^2)^{-1}\Bigg\}\,.\end{aligned} \tag{A13}$$

The singularities in the denominators of (A13) are only apparent, as may be seen directly from the definition (A9). However, the argument of the last $Q_l$ can become less than unity, and so this term has a logarithmic singularity for certain values of the parameters $x$ and $K$. This singularity, being square-integrable, does not cause (A12) to diverge.

We consider the next two terms of (A6):

$$\begin{aligned}A_B{}^{L,S}(K',K;W)_2=&\frac{|N_d|^2|N_{1/2}|^4}{(4\pi)^3}\left(\frac{2m}{\hbar^2}\right)^3\int d\hat{K}'Y_{LM}{}^*(\hat{K}')\int d\hat{K}\,Y_{LM}(\hat{K})\\&\times\int d\mathbf{k}'\int d\mathbf{k}\Bigg[[\alpha^2+(\tfrac{1}{2}\mathbf{K}'-\mathbf{k}')^2]^{-1}(\beta^2+k'^2)^{-1}[\beta^2+(\mathbf{K}'-\mathbf{k}')^2]^{-1}\\&\times\Bigg\{\frac{\bar{\tau}^2(k^2+i\eta)\bar{\tau}(Q^2+i\eta-k^2)}{[Q^2+i\eta-k^2-(\mathbf{K}'-\mathbf{k}')^2](k^2+i\eta-k'^2)[Q^2+i\eta-k^2-(\mathbf{K}-\mathbf{k})^2]}\\&-\frac{\bar{\tau}(k'^2-i\eta)\bar{\tau}(Q^2+i\eta-k'^2)}{[Q^2+i\eta-k'^2-(\mathbf{K}'-\mathbf{k}')^2](k^2+i\eta-k'^2)[Q^2+i\eta-k'^2-(\mathbf{K}-\mathbf{k})^2]}\Bigg\}\\&\times[\alpha^2+(\tfrac{1}{2}\mathbf{K}-\mathbf{k})^2]^{-1}(\beta^2+k^2)^{-1}[\beta^2+(\mathbf{K}-\mathbf{k})^2]^{-1}\Bigg]\,.\end{aligned} \tag{A14}$$

We see that the angles involved are $\hat{K}'\cdot\hat{k}'$ and $\hat{K}\cdot\hat{k}$, hence the angular integrations $d\hat{k}'$, $d\hat{k}$ project out all but $L=0$. The angular integrations $d\hat{K}'$ and $d\hat{K}$ contribute a factor of $4\pi$, the $\phi$ integrations in $d\hat{k}'d\hat{k}$ contribute $4\pi^2$, and we have

$$A_B{}^{L,S}(K',K;W)_2=\tfrac{1}{2}|N_d|^2|N_{1/2}|^4\left(\frac{2m}{\hbar^2}\right)^3\int_0^\infty\frac{dk'k'^2}{k'^2+\beta^2}\int_0^\infty dk\,k^2(\beta^2+k^2)^{-1}(k^2+i\eta-k'^2)^{-1}$$
$$\times\left\{\bar{\tau}(k^2+i\eta)\bar{\tau}(Q^2+i\eta-k^2)\int_{-1}^1 dt\,\mathfrak{D}^{-1}(k',K',Q^2+k'^2-k^2;t)\frac{1}{2}\int_{-1}^1 du\,\mathfrak{D}^{-1}(k,K;Q^2;u)\right.$$
$$\left.-\bar{\tau}(k'^2-i\eta)\bar{\tau}(Q^2+i\eta-k'^2)\frac{1}{2}\int_{-1}^1 dt\,\mathfrak{D}^{-1}(k',K',Q^2;t)\int_{-1}^1 du\,\mathfrak{D}^{-1}(k,K,Q^2+k^2-k'^2;u)\right\},\quad\text{(A15)}$$

where $\mathfrak{D}(k,K,Q^2;t)$ has the same meaning as in (A8). Now we define

$$g^{(\pm)}(k,K';Q^2)=\frac{K'}{4\pi}\int_0^\infty\frac{dk'k'^2}{(k'^2+\beta^2)(k'^2\mp i\eta-k^2)}\int_{-1}^1 dt\,\mathfrak{D}^{-1}(k',K';Q^2+k'^2-k^2;t).\quad\text{(A16)}$$

We note that the $t$ integration gives just $2u_0(k',K';Q^2+k'^2-k^2)$; we must now consider integrals of the form

$$\int_0^\infty dk'k'(\beta^2+k'^2)^{-1}(k'^2\mp i\eta-k^2)^{-1}(k'^2+k_1^2)^{-1}(k'^2+k_2^2)^{-1}Q_0\left(\frac{c^2+k'^2+y^2}{2k'y}\right).$$

Noting that

$$Q_0\left(\frac{c^2+k'^2+y^2}{2k'y}\right)=2k'y\int_0^\infty dr\,rj_0(k'r)j_0(yr)e^{-cr},$$

we see that our previous integral becomes

$$2\int_0^\infty\frac{dr}{r}e^{-cr}\sin yr\int_0^\infty dk'k'\sin k'r\,(\beta^2+k'^2)^{-1}(k'^2\mp i\eta-k^2)^{-1}(k'^2+k_1^2)^{-1}(k'^2+k_2^2)^{-1}$$
$$=-i\int_0^\infty dr\,e^{-cr}(\sin yr/r)\int_{-\infty}^\infty dk'k'e^{ik'r}(\beta^2+k'^2)^{-1}(k'^2+k_1^2)^{-1}(k'^2+k_2^2)^{-1}(k'^2\mp i\eta-k^2)^{-1}.$$

Doing the $k'$ integral by contour integration, we get

$$\pi\int_0^\infty\frac{dr}{r}e^{-cr}\sin yr\left\{\frac{-e^{-\beta r}}{(\beta^2+k^2)(k_1^2-\beta^2)(k_2^2-\beta^2)}+\frac{e^{\pm ikr}}{(\beta^2+k^2)(k_1^2+k^2)(k_2^2+k^2)}\right.$$
$$\left.-\frac{e^{-k_1r}}{(\beta^2-k_1^2)(k_2^2-k_1^2)(k^2+k_1^2)}-\frac{e^{-k_2r}}{(\beta^2-k_2^2)(k^2+k_2^2)(k_1^2-k_2^2)}\right\}.$$

The integral

$$\int_0^\infty\frac{dr}{r}\sin yr\,e^{-\gamma r}$$

obviously gives $\tan^{-1}(y/\gamma)$; hence, we obtain for $g^{(\pm)}(x,K;Q^2)$ the rather complicated result:

$$
\begin{aligned}
g^{(\pm)}(x,K;Q^2)=&-\tan^{-1}\left(\frac{K/2}{\alpha+\beta}\right)(x^2+\beta^2)^{-1}(2\alpha^2-2\beta^2-\tfrac{1}{2}K^2)^{-1}(2\alpha^2+Q^2-\tfrac{1}{2}K^2-x^2-\beta^2)^{-1}\\
&+\tan^{-1}\left(\frac{K/2}{\alpha\mp ix}\right)(x^2+\beta^2)^{-1}(2\alpha^2-\beta^2-\tfrac{1}{2}K^2+x^2)^{-1}(2\alpha^2+Q^2-\tfrac{1}{2}K^2)^{-1}\\
&-\tan^{-1}(K/2\beta)(x^2+\beta^2)^{-1}(2\alpha^2-2\beta^2-\tfrac{1}{2}K^2)^{-1}(x^2-Q^2-\beta^2)^{-1}\\
&+\tan^{-1}\left(\frac{K}{\beta\mp ix}\right)(x^2+\beta^2)^{-1}(2\alpha^2-\beta^2-\tfrac{1}{2}K^2+x^2)^{-1}(x^2-Q^2-\beta^2)^{-1}\\
&+\tan^{-1}\left(\frac{K}{\beta+(x^2-Q^2)^{1/2}}\right)(x^2+\beta^2)^{-1}(2\alpha^2-\beta^2-\tfrac{1}{2}K^2+Q^2-x^2)^{-1}(x^2-Q^2-\beta^2)^{-1}\\
&-\tan^{-1}\left(\frac{K}{(x^2-Q^2)^{1/2}\mp ix}\right)(x^2+\beta^2)^{-1}(2\alpha^2-\tfrac{1}{2}K^2+Q^2)^{-1}(x^2-\beta^2-Q^2)^{-1}\\
&-\tan^{-1}\left(\frac{K/2}{\alpha+\beta}\right)(2\beta^2+\tfrac{1}{2}K^2-2\alpha^2)^{-1}(x^2-Q^2-\beta^2)^{-1}(2\alpha^2+x^2-\tfrac{1}{2}K^2-\beta^2)^{-1}\\
&+\tan^{-1}\left(\frac{K/2}{\alpha+(x^2-Q^2)^{1/2}}\right)(x^2-\beta^2-Q^2)^{-1}(2\alpha^2+Q^2-\tfrac{1}{2}K^2)^{-1}(\beta^2-2\alpha^2+\tfrac{1}{2}K^2-Q^2+x^2)^{-1}.
\end{aligned}
\tag{A17}
$$

Here we mean

$$
\begin{aligned}
(x^2-Q^2)^{1/2}&=-i(|x^2-Q^2|)^{1/2}, \quad x^2<Q^2\\
&=(|x^2-Q^2|)^{1/2}, \quad x^2>Q^2.
\end{aligned}
\quad [\text{Recall } \mathrm{Im}(Q^2)>0]
$$

We thus get

$$
\begin{aligned}
A_B{}^{L,S}(K',K;W)_2=&-2|N_d|^2|N_{1/2}|^4\left(\frac{2m}{\hbar^2}\right)^3\frac{\pi}{K'K}\delta_{L,0}\int_0^\infty\frac{dx\,x}{x^2+\beta^2}\bar{\tau}(Q^2+i\eta-x^2)\\
&\times[\bar{\tau}(x^2+i\eta)g^{(+)}(x,K';Q^2)f_0(x,K;Q^2)+\bar{\tau}(x^2-i\eta)f_0(x,K';Q^2)g^{(-)}(x,K;Q^2)].
\end{aligned}
\tag{A18}
$$

The rest of the integrals are just straightforward repetitions of those we have already done, so only the results are presented:

$$
\begin{aligned}
A_B{}^{L,S}(K',K;W)_3=&-|N_d|^2|N_{1/2}|^6\left(\frac{2m}{\hbar^2}\right)^4\frac{4\pi^2}{K'K}\delta_{L,0}\\
&\times\int_0^\infty dx\,x^2\frac{|\bar{\tau}(x^2+i\eta)|^2}{(\beta^2+x^2)^2}\bar{\tau}(Q^2+i\eta-x^2)g^{(+)}(x,K';Q^2)g^{(-)}(x,K;Q^2).
\end{aligned}
\tag{A19}
$$

Equation (A19) is the contribution from the last term of (A4) [that is, from the $\int d\mathbf{k}''$ term in (A6)]. Finally, the contribution from the bound-state term in (A6) is

$$
A_B{}^{L,S}(K',K;W)_4=-|N_d|^2|N_{1/2}|^4\left(\frac{2m}{\hbar^2}\right)^4\delta_{L,0}\frac{4\pi^2}{K'K}\bar{\tau}(Q^2+\bar{\kappa}^2+i\eta)g^{(+)}(i\bar{\kappa},K';Q^2)g^{(+)}(i\bar{\kappa},K;Q^2).
\tag{A20}
$$

## APPENDIX B: ON THE NUMERICAL SOLUTION OF EQUATIONS (41) AND (42)

The singular integral Eqs. (41) and (42) may be reduced to nonsingular Fredholm equations using the method of Noyes.[26] I shall briefly indicate how this is done for the single one-dimensional integral equation arising from the

[26] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965); K. L. Kowalski, *ibid.* **15**, 798 (1965).

model of Sec. V

$$A^{L,S}(K';K;W)=A_B{}^{L,S}(K',K;W)-\int_0^\infty A_B{}^{L,S}(K',K'';W)\tau_S\left(W-\frac{\hbar^2K''^2}{4m}\right)A^{L,S}(K'',K;W)K''^2dK''. \quad \text{(B1)}$$

We specifically require the "half-off-shell" matrix elements of $A(W)$,

$$A^{L,S}(K',K_0;\,\hbar^2K_0{}^2/4m+i\eta-\epsilon_d)\equiv A^{L,S}(K';K_0).$$

Let us now make the ansatz proposed by Noyes

$$A^{L,S}(K';K_0)=a_{L,S}(K';K_0)A^{L,S}(K_0;K_0)\equiv a^{L,S}(K',K_0)A^{L,S}(K_0) \quad \text{(B2)}$$

and find that Eq. (B1) becomes

$$a_{L,S}(K';K_0)=\frac{A_B{}^{L,S}(K';K_0)}{A^{L,S}(K_0)}-\int_0^\infty A_B{}^{L,S}(K',K'';W)\tau_S\left(W-\frac{\hbar^2K''^2}{4m}\right)a_{L,S}(K'',K_0)K''^2dK''. \quad \text{(B3)}$$

Now noting that $a_{L,S}(K_0,K_0)\equiv 1$, we may solve (B3) for $A^{L,S}(K_0)$, and resubstitute it in (B3):

$$A^{L,S}(K_0)=\frac{A_B{}^{L,S}(K_0)}{1+\int_0^\infty A_B{}^{L,S}(K_0,K'';W)\tau_S(W-\hbar^2K''^2/4m)a_{L,S}(K'';K_0)K''^2dK''}. \quad \text{(B4)}$$

Equation (B3) then becomes

$$a_{L,S}(K',K_0)=A_B{}^{L,S}(K';K_0)/A_B{}^{L,S}(K_0)-\int_0^\infty dK''K''^2\left[A_B{}^{L,S}(K',K'';W)-\frac{A_B{}^{L,S}(K';K_0)A_B{}^{L,S}(K'';K_0)}{A_B{}^{L,S}(K_0)}\right]\tau_S\left(W-\frac{\hbar^2K''^2}{4m}\right)a_{L,S}(K'';K_0). \quad \text{(B5)}$$

Clearly, $a_{L,S}(K',K_0)$, defined by (B5), approaches 1 as $K'\to K_0$.

As Noyes has pointed out, Eq. (B5) is not singular, and the numerical representation of the kernel of (B5) by a finite matrix is then much easier. (The major practical advantage of this trick over distortion of contours à la Rubin[18] is that the intermediate variables are always real, which simplifies the programming.)

Equations (B2), (B3), (B4), and (B5) have obvious generalizations in the case of coupled channels: instead of the scalar $A^{L,S}(K';K_0)$, we deal with finite matrices $\langle K'L'S'|A^J(W)|K_0,L,S\rangle$, and write

$$\langle K'L'S'|A^J(W)|K_0,L,S\rangle=\sum_{L''S''}a_{L'S',L''S''}{}^J(K';K_0)\times\langle K_0L''S''|A^J(W)|K_0LS\rangle. \quad \text{(B6)}$$

The rest of the generalization is obvious, and follows exactly the same pattern, with matrix algebra substituted for scalar algebra.

The Eq. (B5) presents a more stable matrix-inversion problem than does Eq. (B1). This is because any poles or resonances in the physical scattering matrix will appear as zeros (or near zeros) of the denominator of (B4) rather than by the kernel matrix of (B5) becoming ill-conditioned.

It may be of some interest to describe the numerical methods employed in solving the model of Sec. V, in view of the current controversy, over numerical techniques in the 3-body problem. The actual equation which was numerically solved was that suggested by Noyes and rederived above. This method proved stable in practice. A 10-point mesh was used to approximate the integration in the (nonsingular) Noyes equation. The mesh was increased to 12 points for one case, and the resulting amplitudes were stable to within $\sim$10%. (Going from 10 to 12 points more than doubled the running time.)

The most difficult numerical problem encountered was that of calculating the partial-wave elastic-scattering Born term with sufficient accuracy. This term, given by expression (47), required the numerical evaluation of a singular integral. The regions near the logarithmic singularities were handled by Gaussian quadrature formulas with logarithmic weight functions, and the regions between the singularities were handled by a compound Gauss-Legendre quadrature formula. The pole was handled by subtraction, in the usual manner. Between 100 and 200 points per integral were needed to obtain a relative accuracy of $<1\%$. This was because, in the region between two of the logarithmic singularities, the integrand oscillated rapidly (with an average value near zero) with an amplitude two orders of magnitude greater than the size of the integrand outside this region (and outside the regions of singularity, of course). The result was that the answer appeared as small differences between rather large numbers, with a consequent severe loss of precision.

Both the elastic $(d,d)$ and stripping $(d,p)$ on-shell amplitudes were obtained. As a check on the accuracy, the elastic amplitude was compared with the three-particle unitarity relation (48). (It was found that unitarity was satisfied within the over-all accuracy of the numerical methods used.) Only the first four partial waves of the full elastic-scattering matrix were calculated, in order to keep the running time of the program within reason. My best estimate of the over-all accuracy of the solutions is about 15–20%. This uncertainty is compunded of the errors from the use of only 10 mesh points, and the loss of accuracy due to roundoff error in the matrix-inversion process. As the Born amplitude for stripping was known explicitly, I approximated the stripping amplitude in the higher partial waves with the partial-wave Born-stripping amplitude. Thus on adding up the partial waves, we find that the approximate stripping amplitude has the form [see Eq. (34)]

$$A_{pd}^{+}(\mathbf{k}_p',\mathbf{K};E)\simeq A_{pd}^{\mathrm{Born}}(\mathbf{k}_p',\mathbf{K};E) - \sum_{l=0}^{L_{\max}} \frac{(2l+1)}{4\pi} P_l(\hat{k}'\cdot\hat{K}) \int_0^\infty \{A_l^{\mathrm{Born}}(k_p',K'';E) \times \tau_d(E+i\eta-\hbar^2K''^2/4m) A_l^{+}(K'',K;E)\} K''^2 dK''. \tag{B7}$$

The analogous procedure was not possible for the elastic amplitude, since the elastic Born term was not available in closed form. [It should be noted that because the stripping behavior appears through the interference of the "direct" and "rescattering" contributions to Eq. (B7), errors in the elastic amplitude and in the evaluation of the integral in (B7) can be magnified considerably; thus, the uncertainty in the stripping amplitude may be as much as 20–25%.]

## APPENDIX C: CALCULATION OF THE FUNCTIONS $\tau(Z)$

For two-body scattering matrices, the off-shell unitarity condition is

$$\langle \mathbf{q}'|t(Z)|\mathbf{q}\rangle - \langle \mathbf{q}'|t(Z^*)|\mathbf{q}\rangle = -2\pi i \frac{\mu k}{\hbar^2} \int d\hat{k} \langle \mathbf{q}'|t(Z)|\mathbf{k}\rangle\langle \mathbf{k}|t(Z^*)|\mathbf{q}\rangle, \tag{C1}$$

where $Z=(\hbar^2k^2/2\mu)+i\eta$.

Suppose we write

$$\langle \mathbf{q}'|t(Z)|\mathbf{q}\rangle = -\sum_i \langle \mathbf{q}'|i\rangle \tau_i(Z)\langle i|\mathbf{q}\rangle. \tag{C2}$$

Now (C1) leads to the relation

$$0=\frac{\pi\mu k}{\hbar^2}\sum_{i,j}\langle \mathbf{q}'|i\rangle \times \tau_i(Z)[\rho_i(k)\delta_{ij}-f_{ij}(k)]\tau_j(Z^*)\langle j|\mathbf{q}\rangle, \tag{C3}$$

where

$$\frac{\pi\mu k}{\hbar^2}\rho_i(k)=\frac{\mathrm{Im}\,\tau_i((\hbar^2k^2/2\mu)+i\eta)}{|\tau_i((\hbar^2k^2/2\mu)+i\eta)|^2} \tag{C4}$$

and

$$f_{ij}(k)=\int d\hat{k}\langle i|\mathbf{k}\rangle\langle \mathbf{k}|j\rangle \tag{C5}$$

is a Hermitian matrix. But the independence of the coefficients in (C3) leads to

$$\det[\rho_i(k)\delta_{ij}-f_{ij}(k)]=0, \tag{C6}$$

i.e., the $\rho_i(k)$ are the (real) eigenvalues of a Hermitian matrix. The definition (C4), however, together with the proviso that

$$\lim_{|z|\to\infty}\tau_i(Z)=\lambda_i, \tag{C7}$$

gives the formula

$$[\tau_i(E+i\eta)]^{-1}=\frac{1}{\lambda_i}+\int_0^\infty \frac{dk\,k^2\rho_i(k)}{E+i\eta-(\hbar^2k^2/2\mu)}. \tag{C8}$$

For the case of one separable term per channel, as in Eq. (21) and Eq. (44), (C8) reduces to the usual form

$$[\tau(E+i\eta)]^{-1} = \lambda^{-1}+\int d^3k|\phi(\mathbf{k})|^2[E+i\eta-(\hbar^2k^2/2\mu)]^{-1}. \tag{C9}$$

[As in many other situations, the derivation of (C9) was expedited by knowing the answer beforehand.]

For the case of the deuteron, $\mu=m/2$, and since $\lambda$ must produce a pole in $\tau_d(E+i\eta)$ at $E=-\epsilon_d$, we have

$$\tau_d^{-1}(-\epsilon_d)=0=\lambda^{-1} - \int d^3k|\phi_d(\mathbf{k})|^2[\epsilon_d+(\hbar^2k^2/m)]^{-1} \tag{C10}$$

and so

$$\tau_d^{-1}(Z)=-(Z+\epsilon_d)\int d^3k|\phi_d(\mathbf{k})|^2 \times((\hbar^2k^2/m)+\epsilon_d)^{-1}((\hbar^2k^2/m)-Z)^{-1}. \tag{C11}$$

With the deuteron wave function normalized to unity, we see that the residue at the pole, $z=-\epsilon_d$, of $\tau(z)$, is just $-1$. [See Eq. (30).]

## APPENDIX D: ESTIMATING THE $(d,np)$ TOTAL CROSS SECTION

Denoting the elastic $(d,d)$ amplitude by $\langle \mathbf{K}'|A|\mathbf{K}\rangle$, and the breakup $(d,np)$ amplitude by

$$\langle \mathbf{K}'\mathbf{q}'|B(E+i\eta)|\mathbf{K}\rangle,$$

we may write the elastic and breakup total cross sections as

$$\sigma_{\rm el}=N\int d\hat{K}'|\langle K\hat{K}'|A((\hbar^2K^2/4m)-\epsilon_d+i_\eta)|\mathbf{K}\rangle|^2 \quad \text{(D1)}$$

and

$$\sigma_{\rm bu}=\frac{N}{K}\int d\hat{K}'\int_{0\leq q'\leq q_{\max}} d^3q'K_0(q')|\times\langle K_0(q')\hat{K}',\mathbf{q}'|B((\hbar^2K^2/4m)-\epsilon_d+i\eta)|\mathbf{K}\rangle|^2, \quad \text{(D2)}$$

where

$$K_0(q)=2(q_{\max}{}^2-q^2)^{1/2} \quad \text{(D3)}$$

and

$$q_{\max}{}^2=(K^2/4)-(m/\hbar^2)\epsilon_d. \quad \text{(D4)}$$

Now we note that [cf. Eq. (3) and Eq. (32) ff.] the on-shell breakup amplitude may be written

$$\langle \mathbf{K}'\mathbf{q}'|B\left(\frac{\hbar^2K^2}{4m}-\epsilon_d+i\eta\right)|\mathbf{K}\rangle =\int\langle \mathbf{K}'\mathbf{q}'|U_{dd}\left(\frac{\hbar^2K^2}{4m}-\epsilon_d+i\eta\right)|\mathbf{Kq}\rangle\psi_d(\mathbf{q})d\mathbf{q} -\langle\mathbf{q}'|\phi_d\rangle\tau_d\left(\frac{\hbar^2q'^2}{m}+i\eta\right)\times\langle\mathbf{K}'|A\left(\frac{\hbar^2K^2}{4m}-\epsilon_d+i\eta\right)|\mathbf{K}\rangle. \quad \text{(D5)}$$

Now under the assumption that $\langle \mathbf{K}'\mathbf{q}'|U_{dd}(z)|\mathbf{Kq}\rangle$ does not vary much with $q'$ in the range $[0,q_{\max}]$, which is well satisfied in practice, we may replace $\mathbf{q}'$ in the first term of (D5) by 0. But since

$$\langle \mathbf{K}'|A(W)|\mathbf{K}\rangle\equiv\int\psi_d{}^*(\mathbf{q}')\times\langle\mathbf{K}'\mathbf{q}'|U_{dd}(W)|\mathbf{Kq}\rangle\psi_d(\mathbf{q})d^3q'd^3q \quad \text{(D6)}$$

and since $q'^2\psi_d{}^*(q')$ falls off rapidly with $q'$, we replace (D6) by

$$\langle\mathbf{K}'|A(W)|\mathbf{K}\rangle\approx\left[\int\psi_d{}^*(\mathbf{q}')d\mathbf{q}'\right]\times\int\langle\mathbf{K}'0|U_{dd}(W)|\mathbf{Kq}\rangle\psi_d(\mathbf{q})d\mathbf{q}. \quad \text{(D7)}$$

This leads to the result

$$\sigma_{\rm bu}\approx\frac{N}{K}\int d\hat{K}'\left|\langle K\hat{K}'|A\left(\frac{\hbar^2K^2}{4m}-\epsilon_d+i\eta\right)|\mathbf{K}\rangle\right|^2\times\int_{0\leq q'\leq q_{\max}}d^3q'K_0(q')\left|\left[\int\psi_d{}^*(\mathbf{q})d\mathbf{q}\right]^{-1}-\langle\mathbf{q}'|\phi_d\rangle\tau_d((\hbar^2q'^2/m)+i\eta)\right|^2, \quad \text{(D8)}$$

i.e.,

$$\sigma_{\rm bu}(K)\approx\sigma_{\rm el}(K)\frac{8\pi}{K}\int_0^{q_{\max}}dq'q'^2(q_{\max}{}^2-q'^2)^{1/2}\times\left|\left[\int d\mathbf{q}\psi_d{}^*(\mathbf{q})\right]^{-1}+S(q')\psi_d(\mathbf{q}')\right|^2. \quad \text{(D9)}$$

[See Eq. (49)ff.] An upper bound for the factor multiplying $\sigma_{\rm el}/K$ in (D9) is $3K$. Actually, the value of this factor will be much less since (let the phase of $\psi_d$ be real), the phase of $S(q')$ varies rapidly in the interval $[0,q_{\max}]$. In other words, the two terms of the breakup amplitude can interfere considerably, and so the end result is roughly $\sigma_{\rm bu}\lesssim\sigma_{\rm el}$.