Unitary Models of Nuclear Resonance Reactions*†

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While most reaction theories are formally flux conserving, the kinds of models and approximations used to specify the S matrix for resonance reaction processes are often of doubtful unitarity, particularly in the overlapping resonance region. To investigate the consequences of unitarity in this domain, several classes of simple analytically specified unitary S matrices are constructed by means of R-matrix models having various periodic arrangements of poles and residues. The resulting reaction amplitudes have a variety of fluctuating resonance spacings and widths as well as nonresonant direct terms and up to three competing channels. Relationships between resonance parameters, channel transmission coefficients, average cross sections, and cross-section fluctuations are discussed. It is found that contrary to common belief unitarity imposes no restriction on the average ratio of channel width to resonance spacing. In all models investigated having no direct coupling between channels, the transmission coefficients are given by $T_c=1-\exp(-2\pi\bar{\Gamma}_c/D)$. Localized structure in the resonance parameters is investigated, and the effects of a single strong R-matrix pole are compared with those of a "giant-resonance" distribution of R-matrix pole strength. In this way, the shapes of both Robson's analog resonances and Feshbach's doorway state resonances are derived in a different dynamical context. Direct scattering and reaction amplitudes are found to be strongly correlated with resonance amplitudes. Thus, for example, two channels coupled by a direct reaction have correlated resonance width amplitudes, and the pole terms of the resonance reaction amplitude coupling them have a nonzero average. Evidence is found that unitarity imposes resonance-resonance correlations, and these in turn affect the relation between average resonance parameters and average cross sections and even more between these and cross-section fluctuations.

I. INTRODUCTION

THE interpretation of nuclear reactions using the "compound-nucleus" viewpoint has long had to rely on a number of theoretical relationships whose validity could be established only in certain limiting cases or under very special assumptions. Thus, for example, the identification of the channel transmission coefficient T_c with the average channel width to spacing ratio $2\pi\bar{\Gamma}_c/D$ has always been known to be correct only in the limit of very small values of T_c. While it has been possible to express T_c explicitly in terms of the dynamical parameters of resonance formalisms such as R-matrix theory,[1] the corresponding relationships for resonance widths and spacings have been more elusive in the "overlapping-resonance" region, and therefore, no generally applicable relationship between observable resonance parameters and transmission coefficients has been found. Similarly, the widely used "Hauser-Feshbach" formula for average compound nucleus cross sections has had to be accepted largely on the basis of model assumptions regarding "independence of formation and decay on the average".[2–4] More detailed derivations yielding this formula (albeit with modifications) have required either that all competing open channels have small transmission coefficients[5,6] or, more generally, that the S-matrix resonance pole amplitudes are uncorrelated and randomly distributed.[7] This latter "statistical assumption" also lies at the heart of the formulas that have recently been widely used for the interpretation of "cross section fluctuations".[8–11] However, nothing has been known regarding the appropriateness of this statistical assumption or even regarding its consistency with the general requirements of reaction theory. In particular, the requirements of flux conservation, or unitarity of the S matrix, are rather difficult to enforce, or even to check when one is dealing with a reaction amplitude in the form of a general resonance pole expansion. Other general requirements, such as those imposed by time-reversal invariance (symmetry of the S matrix) and causality (analyticity in the physical plane), are fairly easily checked by inspection.

In the present paper we endeavor to answer some of the above questions, particularly insofar as they are affected by the unitarity requirement, for cases of few competing channels. We do this by means of a class of simple analytic unitary models of the S matrix having a sufficient variety of distributions of resonance pole parameters, so as to suggest strongly the general validity of the results obtained.

A. Structure of Resonance Theories

To understand the significance of these models it will be useful to review briefly the logical structure of nuclear-reaction theories. As in all quantum mechanical

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problems, the calculation divides naturally into three stages. In the first stage one puts the Hamiltonian in such a form that it exhibits most advantageously those features of the physical system under study on which one wishes to focus attention. In many cases this consists of writing the Hamiltonian as a sum of a model H_0 and a perturbation H'. In the second stage one solves the Schrödinger equation to obtain the wave function ψ of the system, typically by expanding ψ in the eigenstates of H_0. Finally, one uses ψ to compute the matrix elements of operators corresponding to various measured observables of the system.

In the case of "resonance" or "compound-nucleus" or "statistical" reactions, the first stage consists generally in selecting one of two types of useful forms of the Hamiltonian. The first type comprises the formalisms of Kapur and Peierls,[12] and of Wigner and Eisenbud[13–15] (R-matrix theory) and their generalizations.[16,7] In these theories artificial boundary conditions are imposed at finite surfaces in configuration space and the wave function is expanded in the discrete eigenstates of the finite "interior" region of interaction enclosed by the boundary surfaces. The original purpose of these older formalisms was to give a rigorous theoretical foundation for the "compound-nucleus" resonances described by Bohr,[17] and also to provide a convenient set of parameters for their description, without, however, attempting to give detailed accounts of the values of these parameters in terms of nuclear dynamics. At the same time, by focusing attention on complex closed systems with discrete spectra, these boundary condition theories provided the rationale for the introduction of analogies from equilibrium thermodynamics and statistical mechanics in the description of resonance parameters. Thus, for example, the "Fermi-gas models" of level densities[5,18,19] and the statistical theories of spectra and eigenfunctions[20–22] apply properly to the parameters of the R matrix with constant boundary conditions, and *not* in any direct sense to the resonance parameters of overlapping resonances.[7,11,23] The R-matrix theory is also unique in assuring the unitarity of the S-matrix in a very simple way. Any model or approximation of the R matrix which preserves its easily verified properties of symmetry and reality assures flux-conserving reaction amplitudes. In all other theories, it is far more difficult to estimate the effects on unitarity of using model parameters or approximations. The formalism of Humblet and Rosenfeld is related to the boundary-condition theories.[24]

The second type of resonance reaction model for the Hamiltonian is typefied by the theory of Feshbach,[25,26] the formalism of MacDonald,[27] and Bloch and Gillet's calculations.[28] Here attention is focused on bound and continuum states with boundary conditions at infinity and the aim is to develop connections between aspects of resonance parameters and particular features of nuclear dynamics as embodied in nuclear structure models.

Each of the above formalisms requires the second step of determining the wave function, or in this case its asymptotic behavior as specified by the S matrix or the reaction amplitude. In some formalisms, such as those of Kapur and Peierls[12] and of Humblet and Rosenfeld[24] this stage is atrophied, being essentially completed when the spectrum of the "model Hamiltonian" has been determined. In the case of R-matrix theory,[13] the solution of the Schrödinger equation is performed by "inverting the R matrix." The other theories require the more conventional methods of solving differential or associated integral equations. In discussing resonance reactions, the aim is, however, always to obtain the S matrix in the form of a resonance pole expansion with its parameters of resonance energies, widths, and resonance amplitudes.

Of course, to the extent that they are rigorously correct, *all* of the above-mentioned theories (Wigner-Eisenbud, Kapur-Peierls, Feshbach, MacDonald, etc.) yield the *same* S matrix, or wave function. They differ only in the prescriptions they give for computing the S-matrix parameters from the Hamiltonian.

The final step of determining the observable cross sections reduces to the problem of forming the appropriate bilinear combinations of the S-matrix elements. If one further wishes to give a statistical description of cross sections, such as their energy averages, mean square fluctuations, correlation functions, etc., then the complicated analytic structure of the S-matrix pole expansion can make this final step moderately complex. The statistical theories of fluctuating cross sections deal with the execution of this last step,[7] in general, by making the simplest "statistical" assumptions of randomness and lack of correlations of the S-matrix resonance parameters.[8–11] One object of this study is to examine the validity of such assumptions.

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B. Predictions of Resonance Theories

Before turning to the models to be studied, we briefly review the origins of the familiar relations between resonance parameters and their limitations. In the limit of isolated or well-separated resonances (widths $\Gamma \ll$ mean resonance spacings D) and in the absence of nonresonant reactions all of the above mentioned resonance formalisms yield an S matrix of the form

$$S_{cc'}=e^{i(\phi_c+\phi_{c'})}\left[\delta_{cc'}-i\sum_\mu \frac{(\Gamma_{\mu c}\Gamma_{\mu c'})^{1/2}}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu}\right], \tag{1}$$

$$\Gamma_\mu=\sum\nolimits_c \Gamma_{\mu c}, \quad \bar{\Gamma}/D\ll 1$$

where the indices c, c', etc., stand for a particular partial wave and polarization in a particular channel, and all channel or resonance parameters ϕ_c, $\Gamma_{\mu c}$, $\mathcal{E}_\mu$, Γ_μ are real and weakly energy-dependent. The square roots $\Gamma_{\mu c}^{1/2}$ have given signs associated with them. It has been shown by Bethe[5] and by Thomas[6] that this form of the S matrix remains valid provided only that the ratios $\bar{\Gamma}_{\mu c}/D$ of mean partial widths to mean resonance spacings are small compared to unity. The transmission coefficient in channel c is defined by

$$T_c=1-|\bar{S}_{cc}|^2, \tag{2}$$

where $\bar{S}_{cc}$ is the energy average of S_{cc}. Cross sections for transitions from channel c to channel c' are given in units of $\pi\lambda\!\!\!^{-}_c{}^2$ by

$$\sigma_{cc'}=|\delta_{cc'}-S_{cc'}|^2 \tag{3}$$

and the fluctuation cross section or "average compound-nucleus" cross section is given by[29,30]

$$\sigma_{cc'}{}^{\text{fl}}=(|\langle S_{cc'}|^2\rangle-|\bar{S}_{cc'}|^2). \tag{4}$$

Energy averages of continuous functions of the energy, such as $S_{cc'}$, will be indicated either by a bar or by brackets $\langle\ \rangle$.

Ignoring the weak energy dependence of the resonance parameters, one obtains the following results from Eq. (1) by using the customary definitions and methods of averaging[4–8]:

$$T_c=\frac{2\pi\bar{\Gamma}_{\mu c}}{D}-\frac{\pi^2\bar{\Gamma}_{\mu c}{}^2}{D^2}, \tag{5a}$$

$$\sigma_{cc'}{}^{\text{fl}}=\frac{2\pi}{D}\left\langle\frac{\Gamma_{\mu c}\Gamma_{\mu c'}}{\Gamma_\mu}\right\rangle-\frac{\pi^2\langle(\Gamma_{\mu c}\Gamma_{\mu c'})^{1/2}\rangle^2}{D^2}, \quad \frac{\bar{\Gamma}}{D}\ll 1 \tag{5b}$$

Here the bars and brackets $\langle\ \rangle$ indicate averages with respect to the index μ of discrete resonance parameters, such as $\Gamma_{\mu c}$.

The second terms on the right-hand sides of Eqs. (5) are of second order in the small parameters $\bar{\Gamma}_{\mu c}/D$ and are therefore meaningless within the stated domain of validity of Eq. (1). It has therefore been customary to ignore these higher-order terms and to assume the relationships

$$T_c\sim\frac{2\pi\bar{\Gamma}_{\mu c}}{D}, \tag{6a}$$

$$\sigma_{cc'}{}^{\text{fl}}\sim\frac{2\pi}{D}\left\langle\frac{\Gamma_{\mu c}\Gamma_{\mu c'}}{\Gamma_\mu}\right\rangle\sim\frac{T_cT_{c'}}{\sum_{c''}T_{c''}} \times\frac{\langle\Gamma_{\mu c}\Gamma_{\mu c'}/\Gamma_\mu\rangle}{\bar{\Gamma}_{\mu c}\bar{\Gamma}_{\mu c'}/\bar{\Gamma}_\mu}, \quad \frac{\bar{\Gamma}}{D}\ll 1. \tag{6b}$$

Equation (6b) is the Hauser-Feshbach relation with width fluctuation correction.[31] By improperly extending the range of validity of Eq. (6a), one obtains the restriction $2\pi\bar{\Gamma}_{\mu c}/D\leq 1$, since flux conservation restricts the value of T_c to lie between the limits 0 and 1. If one were to assume the general validity of Eq. (5), one would obtain the restriction $2\pi\bar{\Gamma}_{\mu c}/D\leq 4$ which has been stated by Feshbach.[30] All such restrictions on the possible values of $\bar{\Gamma}_{\mu c}/D$ have been very bothersome, because it has not been possible to find any independent dynamical rationale for them. We shall find that no such restrictions do in fact apply.

When Bethe's restrictions ($\bar{\Gamma}_c/D\ll 1$) are removed, then the S matrix may be represented in a finite energy interval by the statistical S matrix[7]

$$S_{cc'}=e^{i(\phi_c+\phi_{c'})}\left[W_{cc'}{}^0-i\sum_\mu\frac{g_{\mu c}g_{\mu c'}}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu}\right], \tag{7}$$

$$\Gamma_\mu=\sum_c \Gamma_{\mu c}, \quad \Gamma_{\mu c}=|g_{\mu c}|^2/N_\mu, \quad N_\mu\geq 1$$

where now the channel resonance amplitudes $g_{\mu c}$ may be complex and $W_{cc'}{}^0$ is a complex matrix. Under appropriate conditions the energy dependences of the parameters in (7) are weak compared to the explicit energy variation of the formula. The evaluation of (2) and (4) from (7) with appropriate definitions of the averages has been shown to give expressions of the form[7]

$$T_c=\langle\Theta_{\mu c}\rangle-\sum\nolimits_{c'}M_{cc'} +\sum_{c'\neq c}\left|W_{cc'}{}^0-\frac{\pi}{D}\langle g_{\mu c}g_{\mu c'}\rangle_\mu\right|^2, \tag{8a}$$

$$\sigma_{cc'}{}^{\text{fl}}=\left\langle\frac{\Theta_{\mu c}\Theta_{\mu c'}}{\sum_{c''}\Theta_{\mu c''}}\right\rangle-M_{cc'}, \tag{8b}$$

where

$$\Theta_{\mu c}=2\pi N_\mu|g_{\mu c}|^2/D, \tag{9a}$$

$$M_{cc'}=\frac{2\pi^2}{D^2}\left[|\langle g_{\mu c}g_{\mu c'}\rangle_\mu|^2 -\left\langle\frac{g_{\mu c}g_{\mu c'}g_{\nu c}{}^*g_{\nu c'}{}^*}{(\mathcal{E}_\mu-\mathcal{E}_\nu)+\frac{1}{2}i(\Gamma_\mu+\Gamma_\nu)}\right\rangle_{\mu\neq\nu}\right]. \tag{9b}$$

[29] H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954).

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The similarity between Eqs. (8) and (9) and the approximations (5) is rather striking, and indicates that in a sense Eqs. (5) are better approximations to the general case than the Eqs. (6).

The symmetry of the S matrix (7) is easily verified and causality is assured if all Γ_μ are positive and all other parameters have no poles in the upper half of the energy plane. However, the relationships required of the parameters in (7) to assure the unitarity of S are extremely difficult to determine. In principle, unitarity can be assured by expressing the parameters in (7) in terms of R-matrix parameters, but these relationships are in general very complex and difficult to state explicitly.[14,15] The same is true for all other formalisms that yield a formally unitary S matrix. We therefore turn to some simple models of the R matrix which permit the explicit determination of unitary S matrices and their resonance parameters.

In this connection, the use of an R matrix need not imply the full dynamical machinery of R-matrix theory. For example, we shall never make explicit reference to a channel radius. Rather, we may view the R-matrix relations here as a convenient formalism for generating a variety of unitary S-matrix models. These models, moreover, are to be interpreted as models of the statistical S-matrix which has been defined in Ref. 7 as a representation of the physical S-matrix within a finite energy interval by means of an expression of the form of Eq. (7) containing a sum over a certain infinite sequence of resonance pole terms. This point will be discussed in greater detail in Sec. II C below.

II. SINGLE-CHANNEL MODELS

A. The Picket Fence Model

The models to be employed are generalizations of the picket-fence model of Teichmann,[32] which considers a single-channel R function of equally spaced poles with equal residues.

$$R=\sum_{\mu=-\infty}^{\infty}\frac{\gamma^2}{\mu D-E}=-\frac{\pi\gamma^2}{D}\cot\frac{\pi E}{D}\,. \tag{10}$$

Adopting the natural boundary conditions $L^0=iP$ (details on the R-matrix relations may be found in Refs. 14, 15), we obtain for the S function

$$S=e^{2i\phi}\frac{1+iPR}{1-iPR}, \tag{11}$$

the expression

$$S=e^{2i\phi}\frac{1-it\cot z}{1+it\cot z}, \tag{12}$$

where we have introduced the notation

$$t=\pi P\gamma^2/D=\tau/4\,, \tag{13}$$

$$z=\pi E/D \tag{14}$$

[32] T. Teichmann, Phys. Rev. **77**, 506 (1950).

(the notation τ having been introduced earlier[1] for the quantity $4\pi P\langle\gamma^2\rangle/D$). The expression (12) is now easily rewritten

$$S=e^{2i\phi}\left[\frac{1+t^2}{1-t^2}-\frac{2it}{1-t^2}\cot(z+\tan^{-1}it)\right]. \tag{15}$$

By expanding the contangent, we obtain the single-channel model version of Eq. (7)

$$S=e^{2i\phi}\left[W^0-i\sum_\mu\frac{g_\mu{}^2}{E-\mu D-\mathcal{S}_\mu+\frac{1}{2}i\Gamma_\mu}\right], \tag{16a}$$

with the μ independent parameters

$$W^0=\frac{1+t^2}{1-t^2}, \tag{16b}$$

$$\frac{2\pi g_\mu{}^2}{D}=\frac{4t}{1-t^2}, \tag{16c}$$

$$\begin{aligned}\mathcal{S}_\mu&=0, \quad t<1\\ &=\tfrac{1}{2}D, \quad t>1\,,\end{aligned} \tag{16d}$$

$$\begin{aligned}2\pi\Gamma_\mu/D&=4\ \mathrm{tcth}^{-1}t\equiv 4\tanh^{-1}t, \quad t<1\\ &\equiv 4\coth^{-1}t, \quad t>1\,.\end{aligned} \tag{16e}$$

To obtain the transmission coefficient, we merely average S over an interval π in the variable z and find that

$$\bar{S}=e^{2i\phi}\frac{1-t}{1+t}, \tag{17}$$

and hence

$$T=1-|\bar{S}|^2=\frac{4t}{(1+t)^2}\,. \tag{18}$$

This latter relationship between the transmission coefficient and the average R-matrix parameters has previously been derived for quite general situations involving many competing channels and arbitrary distributions of resonance spacings and resonance amplitudes.[1] In that case t stood for the average ratio $\frac{1}{4}\tau=\pi P\langle\gamma^2\rangle/D$. One obvious consequence of Eq. (18) is that the value of T lies between zero and unity for all positive values of t as it must, since all positive values of t yield a unitary S matrix. There is therefore no restriction on the possible values of the ratio $\langle\gamma^2\rangle/D$ and, in our present model, by virtue of Eq. (16e) there is also no restriction on the possible values of Γ/D. In fact, by eliminating the R-matrix model parameter t between the two "observables" T, and Γ/D as given by Eq. (18) and (16e), we obtain the relationship

$$T=1-\exp(-2\pi\Gamma/D)\,. \tag{19}$$

Comparing this with the optical-model relationship

$$T=1-\exp(-4\eta)\,, \tag{20}$$

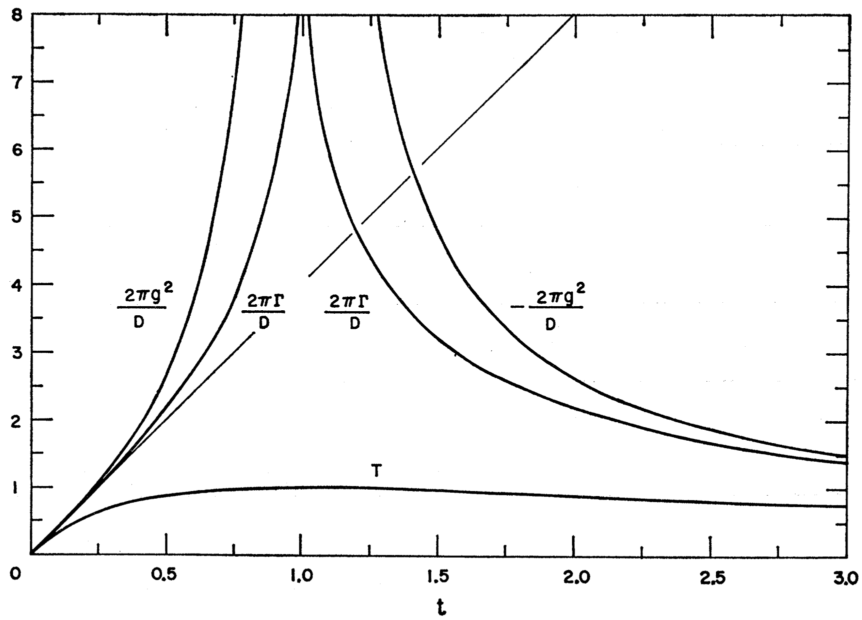

FIG. 1. Dependence of the S-function parameters $2\pi g^2/D$ and $2\pi\Gamma/D$ and of the transmission coefficient T upon the R-function parameter $t=\pi P\gamma^2/D$ for the simple single-channel picket-fence model. The straight line shows the function $\tau=4\pi P\gamma^2/D$.

where η is the imaginary part of the optical-model phase shift,[29,30] we see that

$$\eta=\pi\Gamma/2D. \tag{21}$$

The S-function pole amplitudes g^2 are always larger than the widths Γ, as expected.[7,12,14,24] Their relationship is shown in Fig. 1. Perhaps less expected is the result that W^0 is not unitary except in the isolated resonance limit when $t\rightarrow 0$. This means that in general the representation of the resonance amplitude by a statistical S matrix[7] requires a nonunitary potential scattering amplitude. This point is already implicit in Ref. 7.

It is interesting to note the following symmetry of the S matrix considered as a function of E and t:

$$e^{-2i\phi}S(E,t)=-e^{-2i\phi}S(E+\tfrac{1}{2}D,\,1/t). \tag{22}$$

This probably arises from a symmetry between the "interior" and "channel" regions in the picket fence model.

The most striking feature of this model lies in the singularity of the resonance parameters Γ and g at $t=1$ as shown in Fig. 1. As this value of t is approached the S-matrix poles disappear at infinity in the lower half of the energy plane and simultaneously the pole residues become infinite. There is of course no singularity in the cross sections. The S matrix becomes

$$S=e^{2i(\phi+\pi E/D)},\quad t=1 \tag{23}$$

which has the form of a pure resonance amplitude leading to a cross section whose values vary in energy between zero and $4\pi\lambda\!\!\!^{-2}$ with a period D (and *not* Γ, as might be thought from fluctuation theory!)

$$\sigma=4\pi\lambda\!\!\!^{-2}\sin^2(\phi+\pi E/D),\quad t=1. \tag{24}$$

This behavior, together with the fact that $T=1$ when $t=1$, justifies the description of the $t=1$ situation as pure resonance scattering.

B. Fluctuating Widths and Spacings

The simple picket-fence model (10) is of course unrealistic; both the spacings and widths of real resonances fluctuate and the distribution of resonances is not constant in the infinite energy interval from $-\infty$ to $+\infty$. We now proceed to investigate these effects by means of generalizations of the model (10).

We first examine the effects of fluctuating widths by considering the model

$$L^0R=-it\cot z-i\delta t\csc z, \tag{25}$$

where again $z=\pi/ED$ and D is the spacings of R-function poles. The R-function residues have alternating values $-\gamma_+^2$ and $-\gamma_-^2$ given by

$$\pi P\gamma_\pm^2/D=t_\pm=t\pm\delta t. \tag{26}$$

Upon calculating the S-matrix and determining its resonance-pole expansion in a similar manner as before, we find again an expression of the form (16a) but now with alternating amplitudes $g_\pm^2$ and alternating values of the width $\Gamma_\pm$ when the mean-square value of t, namely t_+t_-, is less than unity. On the other hand, when t_+t_- is greater than unity, the widths Γ are all equal and the level shifts $\mathcal{S}$ alternate so that the effect is that of resonances with alternating spacings and equal widths. The values of the resonance parameters as functions of t and δt are given in the first column in Table I. It is seen there that all average resonance parameters are the same as in the case of the simple picket fence model (10) and, in particular, we obtain the same relationships

TABLE I. S-matrix resonance parameters for various single-channel models.

L^0R	$-i(t\cot z+\delta t\csc z)$ $\Delta\equiv\dfrac{\delta t}{[1-(t+\delta t)(t-\delta t)]^{1/2}}$	$-\frac{1}{2}it[\cot\frac{1}{2}z+\cot\frac{1}{2}(z+\theta)]$ $\Delta\equiv\dfrac{\cos\frac{1}{2}\theta}{[t^2+\cos^2(\frac{1}{2}\theta)-1]^{1/2}}$	$-(s+it)(\cot z-r)$
W^0	$\dfrac{1+t^2}{1-t^2}$	$\dfrac{1+t^2}{1-t^2}$	$\dfrac{1+(s^2+t^2)(1+r^2)-2itr}{1+(s^2-t^2)(1+r^2)-2sr+2it(s-r+sr^2)}$
$2\pi g^2/D$	$\dfrac{4t}{1-t^2}\left(1\pm\dfrac{\Delta}{t}\right)$	$\dfrac{4t}{1-t^2}\left(1\pm\dfrac{\Delta}{t}\right)$	$\dfrac{4t}{1+(s^2-t^2)(1+r^2)-2sr+2it(s-r+sr^2)}$
$2\pi\Gamma/D$	$4\,\mathrm{tcth}^{-1}t,\quad (t+\delta t)(t-\delta t)>1$ $4\,\mathrm{tcth}^{-1}t\pm 4\,\mathrm{tcth}^{-1}\Delta,\quad (t+\delta t)(t-\delta t)<1$	$4\,\mathrm{tcth}^{-1}t,\quad t^2+\cos^2(\frac{1}{2}\theta)<1$ $4\,\mathrm{tcth}^{-1}t\pm 4\,\mathrm{tcth}^{-1}\Delta,\quad t^2+\cos^2(\frac{1}{2}\theta)>1$	$\ln\dfrac{(1+t-sr)^2+(s+tr)^2}{(1-t-sr)^2+(s-tr)^2}$
$\mathcal{S}/D$	$0,\quad (t+\delta t)(t-\delta t)<1$ $\pm(1/\pi)\cot^{-1}i\Delta,\quad (t+\delta t)(t-\delta t)>1$	$0,\quad t^2+\cos^2(\frac{1}{2}\theta)>1$ $\pm(1/\pi)\cot^{-1}i\Delta,\quad t^2+\cos(\frac{1}{2}\theta)<1$	$\dfrac{1}{2\pi}\tan^{-1}\left(\dfrac{2(s^2+t^2)r-2s}{1+(s^2+t^2)r^2-t^2-s(2r+s)}\right)$ $[+\frac{1}{2}\quad \text{if}\quad 2rs+(1-r^2)(s^2+t^2)>1]$
T	$\dfrac{4t}{(1+t)^2}=1-\exp(-2\pi\bar{\Gamma}/D)$	$\dfrac{4t}{(1+t)^2}=1-\exp(-2\pi\bar{\Gamma}/D)$	$\dfrac{4t}{(1+t-sr)^2+(s+tr)^2}=1-\exp(-2\pi\bar{\Gamma}/D)$

(19) and (21) between average width to spacing ratios and the transmission coefficient and η

$$T=1-\exp(-2\pi\bar{\Gamma}/D),\quad \eta=\pi\bar{\Gamma}/2D \tag{27a}$$

and hence again no restrictions on $\bar{\Gamma}/D$. The resonance amplitudes $g_\mu{}^2$ now have singularities both when the average t and the mean square t_+t_- are unity and they are complex under certain circumstances. This feature has also been discussed previously.[7] Again the cross section is well behaved in consequence of unitarity for all positive values of t_+ and t_-. Also the following relationship is found to hold:

$$\pi\langle|g|^2\rangle/D=\sinh(\pi\bar{\Gamma}/D). \tag{27b}$$

Next we consider the effect of resonance pole spacing fluctuations by means of the R-function model

$$L^0R=-\tfrac{1}{2}it[\cot\tfrac{1}{2}z+\cot\tfrac{1}{2}(z+\theta)], \tag{28}$$

where the spacings of R-function poles have the alternating values,

$$D_1=(\theta/\pi)D,\quad D_2=(1-\theta/\pi)D.$$

The mean spacing is still D and $z=\pi E/D$, and the $\gamma_\mu{}^2$ are all equal to $tD/\pi P$. The parameters of the S-matrix pole expansion (16) are given in the second column in Table I. Again the values of $g_\mu{}^2$ alternate from resonance to resonance and g_μ may be complex. The S-matrix resonance level spacings now alternate in magnitude when $t^2+\cos^2(\frac{1}{2}\theta)<1$. But when that parameter exceeds unity, the spacings are equal and the widths alternate in magnitude. Again the mean spacing, mean widths, and mean amplitudes $\langle g^2\rangle$ and the transmission coefficient T are the same as in the simple picket fence model Eqs. (16). Again Eqs. (27) hold and similar comments apply.

C. Distant Resonances and Boundary Conditions

Another shortcoming of all the models we have discussed so far is that they use uniform distributions of R-matrix poles in the whole doubly infinite energy range from $-\infty$ to $+\infty$, while any physical model of the R matrix will have a lowest pole E_0 corresponding to the ground state and a progressively increasing density of poles E_μ with increasing energy. The proper interpretation of our models is then similar to that of Wigner's statistical R function[14,33,7] which is intended to describe the true R function and therefore also the cross section only in a restricted energy interval ΔE. It does this by using a uniform distribution of pole terms in the infinite energy interval $(-\infty,+\infty)$ which is characteristic of the true R-function poles in ΔE, plus a constant term R^0 to take into account the effect within ΔE of the difference between the true and uniform pole distributions outside ΔE. In a sense R^0 plays the role of direct interaction in the dynamical framework of the R-matrix theory. Assuming that the mean R-function residue $\langle\gamma^2(E)\rangle$ and the mean R-function pole spacing $D(E)$ are sufficiently slowly varying functions of the energy we can represent R^0 by the principal-value integral

$$R^0(E)=\mathrm{Pr}\int_{-\infty}^{\infty}dE'\frac{\langle\gamma^2(E')\rangle}{D(E')}\frac{1}{E'-E}. \tag{29}$$

According to the above interpretation also our models of the S matrix are to be interpreted as representing the S matrix only in ΔE. Therefore, T and η correspond to the local optical-model values of these parameters in ΔE.

Together with the introduction of a constant term R^0 in the R matrix, it is useful to generalize the bound-

[33] E. P. Wigner, Ann. Math. 53, 36 (1951); Proc. Cambridge Phil. Soc. 47, 790 (1951).

ary conditions[13,14] so that

$$L^0=S^0+iP\,, \tag{30}$$

with $S^0=S-B$ no longer necessarily zero. These two effects are clearly coupled, since a change in boundary conditions B will cause a change in the distribution of R-matrix poles and residues and hence a change in R^0. Introducing the notation

$$\pi L^0\gamma^2/D\equiv s+it,\quad R^0D/\pi\gamma^2\equiv r \tag{31}$$

we have for the picket fence model with nonvanishing R^0 and arbitrary boundary conditions

$$L^0R=-(s+it)(\cot z-r)\,. \tag{32}$$

Using this to evaluate the statistical S matrix,[7]

$$S=e^{2i\phi}\frac{1-L^{0*}R}{1-L^0R}\,, \tag{33}$$

we obtain by straightforward calculation again a pole expansion of S of the form (16) with the resonance parameters all equal and independent of μ and given in the last column of Table I. The transmission coefficient is again found to have values restricted to the interval (0,1) for arbitrary values of R^0 and s and for arbitrary positive t. The relation between T, g^2, and Γ/D is again given by Eqs. (27).

We note here that as was true before, the nonresonant part of the S matrix, namely, $e^{2i\phi}W^0$, depends on the R-matrix resonance parameters t and s, and the S-matrix resonance parameters, g^2, Γ, and $\mathcal{S}$, depend on the nonresonant part of R^0 of the R function. This mixing up of the "direct" and "resonance" features of dynamical models in the S matrix is not restricted to R-matrix theory and has also been noted recently by Ratcliff and Austern.[34] To expect the opposite would be equivalent to demanding that separation of the Hamiltonian into two parts H_0 and H' should yield a solution of the Schrödinger equation which can be separated into two terms $\psi_0+\psi'$ each of which is an eigenstate of one of the parts of H separately. But this is clearly impossible.

It is possible, in principle, for R^0 to diverge. In that case we see that g^2, Γ, and T all go to zero and we are left with a smoothly varying nonresonant S function. This, presumably, is the R-matrix interpretation of scattering in the high-energy limit.

D. Structure: Strong Resonances and Giant Resonances

The models we have introduced also permit the discussion of some aspects of isolated features in the S-matrix resonance parameters that are due to strongly energy-dependent properties of the underlying nuclear structure parameters. In the single-channel R-function formalism such features may be described in terms of groups of R-function poles E_μ whose associated values of $\gamma_\mu{}^2$ are very much larger than those of the surrounding resonances. We shall describe two extreme cases here: First the case of the "strong resonance" where only one R-function pole has an anomalously large value, and secondly the case of the "giant resonance" where the anomalous strength is distributed over a very large number of R-function poles but still concentrated in a limited energy region. The "strong-resonance" model has been used by Robson[35] in the description of isobaric analog resonances, while the "giant-resonance" model was first proposed by Lane, Thomas, and Wigner[36] for the interpretation of optical model resonances. The more recently discussed "intermediate-structure" resonances also fall into this general category.[37,38] Between the "strong-resonance" and the "giant-resonance" description there are also possible intermediate distributions of strength among a small number of R-function poles. The precise distribution of such strength will, of course, depend on the choice of R-matrix boundary conditions, however again the resulting S-matrix features must be independent of such boundary conditions.

To describe a "strong resonance" we consider a picket-fence model of the type given by Eq. (32) with all poles having equal strength γ^2, but we consider that the pole at $E=0$ has a very much larger strength $\gamma_0{}^2$ such that $\gamma_0{}^2-\gamma^2\gg D$. Then we can write the complete R-function model in the form

$$L^0R=-(s+it)(\cot z-r+\Delta/z)\,, \tag{34}$$

where

$$\Delta=(\gamma_0{}^2-\gamma^2)/\gamma^2 \tag{35}$$

and the S-matrix is again given by Eq. (33). With our assumptions we may regard Eq. (34) as an R function with the slowly energy-dependent effective background term

$$r_{\text{eff}}=r-\Delta/z \tag{36}$$

and can discuss the S-matrix poles at the energy $E=zD/\pi$ in terms of the last column of Table I using r_{eff} for r. At the energy $E=z=0$ of the strong R-function pole, r_{eff} becomes infinite and, as previously discussed the S-function poles "disappear" there. Both g^2 and Γ may have maxima at another energy which depends on the relationship between the values of s, r, t, and Δ. In the case of the "natural" boundary conditions, $s=0$, g^2, and Γ have maxima when r_{eff} goes to zero, that is when $z=\Delta/r$ which may be positive or negative, depending on the sign of r. The transmission coefficient reflects this same behavior and is given by

$$T=\frac{4t}{(1+t)^2+t^2(r-\Delta/z)^2}\,,\quad s=0 \tag{37}$$

[34] K. F. Ratcliff and N. Austern, *Perspectives in Modern Physics*, edited by R. E. Marshak (Interscience Publishers, Inc., New York, 1966), p. 57.

[35] D. Robson, Phys. Rev. **137**, B535 (1965).

[36] A. M. Lane, R. G. Thomas, and E. P. Wigner, Phys. Rev. **98**, 693 (1955).

[37] A. K. Kerman, L. S. Rodberg and J. E. Young, Phys. Rev. Letters **11**, 422 (1963).

[38] H. Fesbach, *Nuclear Structure Study with Neutrons*, edited by M. Nève de Mévergnies, P. Van Assche, and J. Verview (North-Holland Publishing Company, Amsterdam, 1966), p. 257.

which goes to zero at $E=0$ and to

$$T^{\infty}=\frac{4t}{(1+t)^2+t^2r^2} \quad \text{at} \quad z=\pm\infty\,,$$
$$T^{\max}=\frac{4t}{(1+t)^2} \quad \text{at} \quad rz=\Delta. \tag{38}$$

In Fig. 2, three representative curves are drawn of the transmission coefficients (37) for three different values of R^0. They exhibit the zero at $E=0$ and the maximum which vanishes with vanishing R^0.

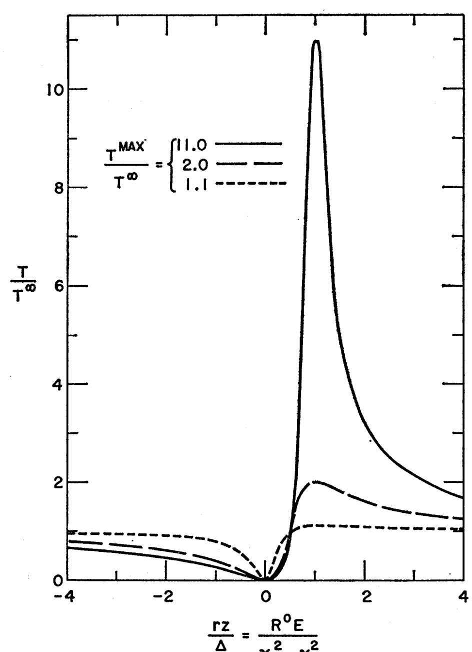

FIG. 2. Energy dependence of the transmission coefficient T in the vicinity of a single strong R-function pole with residue $-\gamma_0^2$, superimposed on a picket fence of poles with residues $-\gamma^2$ and a constant background term R^0; single channel, natural boundary condition case.

If boundary conditions with nonzero s are applicable, the transmission coefficients belong to the same family of curves represented by Fig. 2. In fact, Robson's model for isobaric analog states uses $R^0=0$, but variable shift factors s, and the "enhancement factors" $|f|$ calculated by him,[35] correspond precisely to the transmission coefficients considered here and have precisely the same properties. We conclude therefore that the *qualitative* features of isobaric analog resonances do not depend on the validity of a particular dynamical model, such as Robson's theory of external mixing,[35] but they will apply in any case where boundary conditions can be found that concentrate all of the anomalous resonance strength in a single R-function pole.

Next we construct a giant-resonance model by considering an R function with an infinite sequence of equally spaced poles E_μ and pole strengths given by

$$\gamma_\mu^2=\gamma^2+\frac{D\gamma_0^2}{2\pi}\frac{W}{E_\mu^2+\frac{1}{4}W^2}\,, \tag{39}$$

which consists of a constant background of pole strengths γ^2 and an additional combined strength γ_0^2 distributed over a "giant" Lorentzian of half-width $W\gg D$. In accordance with the interpretation given above Eq. (29) we represent the resonance structure in the vicinity of any energy E by constructing an R-function picket-fence model with equal pole strengths everywhere given by Eq. (39) evaluated at the energy $E_\mu=E$, and with a value of R^0 obtained by substituting the functional dependence of γ_μ^2 upon $E_\mu=E$ into the integral (29). In this way, we obtain for the case $s=0$ the following picket fence representation at the energy E of the giant resonance R-function defined by Eq. (39)

$$L^0R=-it_{\text{eff}}(\cot z-r_{\text{eff}})\,, \tag{40}$$

where

$$t_{\text{eff}}=t+\frac{Dt_0}{2\pi}\frac{W}{E^2+\frac{1}{4}W^2}\,, \tag{41a}$$

$$t_{\text{eff}}r_{\text{eff}}=tr-\frac{Dt_0}{\pi}\frac{E}{E^2+\frac{1}{4}W^2}\,, \tag{41b}$$

where $t=\pi\gamma^2/D$, $t_0=\pi\gamma_0^2/D$, and tr is the value of R^0 due to variations in the R-function pole distribution other than that given by the second term of Eq. (39). The energy dependences in Eqs. (41) are, of course, to be considered parametric, rather than functional. We again use the last column in Table I to discuss the S-matrix poles corresponding to the model of Eqs. (40), (41). This discussion is now complicated by the very wide variety of phenomena that can occur. Suffice it to observe that there can occur peaks in the S-matrix widths or in T at values of the energy where t_{eff} has a maximum or where r_{eff} has a minimum; furthermore there may be dips in T or in the values of Γ when r_{eff} has a maximum or when t_{eff} has a maximum value which is greater than unity. As a result the structure of the transmission coefficient at a giant-type resonance may be fairly complex. While the model does not predict the occurrence of a zero in T as in the strong resonance case, such a zero may in practice not always be clearly distinguishable from a dip due to a giant resonance.

We still write down the transmission coefficient for the case when all R-matrix poles are due to the giant resonance only, that is when both t and r vanish in Eqs. (40), (41). Then T is given by a Lorentzian having a half-width equal to the sum of W and $2P\gamma_0^2$:

$$T=\frac{2P\gamma_0^2W}{E^2+\frac{1}{4}(W+2P\gamma_0^2)^2}\,. \tag{42}$$

In the language of Feshbach's doorway state theory, $2P\gamma_0^2$ is here the giant-resonance decay width $\Gamma^\uparrow$, while

W is the width $\Gamma^{\downarrow}$ for compound-nucleus formation. In that notation Eq. (42) has also been given by Feshbach.[38] Again we see that the qualitative features obtained do not depend upon any particular dynamical model for the giant resonance.

The generalization of strong- and giant-resonance structure models to the multichannel situations to be described next is straightforward and leads to no essentially new results.

E. Models with Finite Numbers of Resonances

Before turning to the multichannel case, we interject here some comments on the type of model we are using and on its relation to alternative approaches. We are using a model having an infinite uniformly distributed sequence of poles and a constant term to account for the effect of the actually changing distribution of distant poles. One might wish, however, to consider another type of model in which only the finite number of poles lying in the energy interval of interest are written down explicitly, and the combined effect of distant poles outside the interval of interest is taken into account by a constant term. This latter approach needs to be used with care, because of the occurrence of edge effects which distort the distribution of resonance parameters near the edges of the energy interval containing the poles. As a consequence, only the resonance parameters in the center of the interval are correct and even then only if the interval is large compared to D and Γ. But with that restriction, it is often much more convenient to go to the limit of an infinite interval as we have done here, particularly as it facilitates the definition of averages and permits the easy introduction of statistical methods, such as the ergodic theorem.[7]

We demonstrate the existence of edge effects here by considering the relationship between widths and S-matrix pole residues in the single-channel finite picket-fence model with natural boundary conditions

$$R=R^0+\gamma^2\sum_{n=-N}^{+N}\frac{1}{nD-E}. \tag{43}$$

Introducing the notations

$$PR^0\equiv r,\quad P\gamma^2/D=t/\pi\equiv f,\quad E/D=z/\pi\equiv x, \tag{44}$$

we can write the S matrix as

$$S=e^{2i\phi}\frac{1-ir+if\sum[1/(n-x)]}{1+ir-if\sum[1/(n-x)]}. \tag{45}$$

Let us assume that S has a pole at

$$x=e-ih. \tag{46}$$

Then for this value of x the denominator of S must have a zero and

$$\sum_{n=-N}^{+N}\frac{1}{n-e+ih}=\frac{r-i}{f}. \tag{47}$$

To find the residue of S at $e-ih$, we expand $S(e-ih+\epsilon)$ in powers of ϵ. The coefficient of ϵ^{-1} in this expansion is the residue $-ig^2$ and is found with the help of Eq. (47) to be

$$-ig^2=\frac{-2ih}{1-2fhA}, \tag{48}$$

where

$$A=\sum_{n=-N}^{+N}\frac{(n-e)^2-ih(n-e)}{[(n-e)^2+h^2]^2}. \tag{49}$$

Equation (48) means that the pole expansion of S has a resonance term of the type

$$-i\frac{2h(1-2fhA)^{-1}}{x-e+ih}. \tag{50}$$

If $2fhA$ were to vanish, then the expression (50) would just state the familiar result of single level theory that the resonance-pole amplitude g^2 just equals the pole width $\Gamma=2hD$. However, in the many-level situation, $2fhA$ does not vanish inside the interval $(-ND,\ ND)$. In cases where $h=\frac{1}{2}\Gamma/D$ is larger than unity, the factor A is easily evaluated by replacing the sum in Eq. (49) by an integral. Assuming also that the interval ND is much larger than Γ, one finds that for resonances in the center of the interval

$$A\sim\pi/2h,\quad (1-2fhA)^{-1}\sim1/(1-t)\quad\text{at}\quad e\sim0. \tag{51}$$

This correction factor is to be compared to the ratio of g^2/Γ obtained in our infinite picket-fence model, Eq. (16) and Table I. It has the same property of becoming infinite at $t=1$.

Within a region of the order of Γ around the edges $-ND$ and ND of our interval the real part of A peaks to twice the value (51) and there is also a peak in its imaginary part near the edges. One easily estimates

$$\begin{aligned}&A\sim\frac{1}{h}(\pi\pm\tfrac{1}{2}i),\\&(1-2fhA)^{-1}\sim\frac{1}{1-t(\frac{1}{2}\pm i/\pi)}\quad\text{at}\quad e=\pm N.\end{aligned} \tag{52}$$

We see therefore that not only the magnitudes but also the complex phases of the resonance amplitudes are disturbed near the edges where resonances are cut off. While the factor $|1-2fhA|^{-1}$ is greater than unity at the center of the interval for $t<2$, it may be less than unity for resonances near the edges, effecting there distorted values of $|g|^2$ less than 2Γ. As a result it is essential when averaging over resonances obtained from poles in a finite energy interval to stay several widths Γ away from the edges of the interval.

III. MULTICHANNEL MODELS

A. Resonance Parameters

We first consider the following simple two-channel picket-fence model of the R matrix:

$$R=\begin{pmatrix} 0 & R^0 \\ R^0 & 0 \end{pmatrix}+\sum_{\mu=-\infty}^{\infty}\frac{1}{\mu D-E}\begin{pmatrix} \gamma_1^2 & (-1)^\mu\gamma_1\gamma_2 \\ (-1)^\mu\gamma_1\gamma_2 & \gamma_2^2 \end{pmatrix}$$

$$=\begin{pmatrix} -(\pi\gamma_1^2/D)\cot(\pi E/D) & R^0-(\pi\gamma_1\gamma_2/D)\csc(\pi E/D) \\ R^0-(\pi\gamma_1\gamma_2/D)\csc(\pi E/D) & -(\pi\gamma_2^2/D)\cot(\pi E/D) \end{pmatrix}, \quad (53)$$

which consists of off-diagonal constant matrix elements R^0 (diagonal constant elements add nothing new to what we have learned in the single-channel case and vastly complicate the resulting expressions) and pole terms with constant pole spacings D and constant residues, except that the off-diagonal residues have alternating signs to simulate R-matrix channel amplitudes γ_1 and γ_2 with uncorrelated signs.

To simplify the resulting expressions further we assume "natural" boundary conditions so that $L_j^0=iP_j$ for both channels $j=1, 2$. In that case, the S-matrix can conveniently be written in the form

$$\mathbf{S}=\mathbf{\Omega}[2(\mathbf{1}-i\mathbf{P}^{1/2}\mathbf{R}\mathbf{P}^{1/2})^{-1}-\mathbf{1}]\mathbf{\Omega}, \quad (54)$$

where $\mathbf{\Omega}$ and $\mathbf{P}$ are the diagonal matrices

$$\mathbf{\Omega}=\begin{pmatrix} \exp i\phi_1 & 0 \\ 0 & \exp i\phi_2 \end{pmatrix}, \quad \mathbf{P}=\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}. \quad (55)$$

Using the notation

$$z=\pi E/D, \quad r=(P_1P_2)^{1/2}R^0, \quad t_j=\pi P_j\gamma_j^2/D, \quad (56)$$

we write the matrix to be inverted as

$$(\mathbf{1}-i\mathbf{P}^{1/2}\mathbf{R}\mathbf{P}^{1/2})=\begin{pmatrix} 1+it_1\cot z & -ir+i(t_1t_2)^{1/2}\csc z \\ -ir+i(t_1t_2)^{1/2}\csc z & 1+it_2\cot z \end{pmatrix}. \quad (57)$$

The poles z_μ of the S-matrix are now given by the zeros of the determinant of (57).

$$1+r^2+t_1t_2+i(t_1+t_2)\cot z_\mu-2r(t_1t_2)^{1/2}\csc z_\mu=0, \quad (58)$$

where

$$z_\mu=(\pi/D)(\mathcal{E}_\mu-\tfrac{1}{2}i\Gamma_\mu). \quad (59)$$

The $\mathcal{E}_\mu$ are the S-matrix resonance energies and the Γ_μ are the associated widths.

The properties of the solutions of Eq. (58) depend on the ranges of the two parameters

$$\alpha=(1+r^2+t_1t_2)^2-(t_1+t_2)^2, \quad \beta=(1+r^2-t_1t_2)^2-(t_1-t_2)^2. \quad (60)$$

These solutions are as follows:

$$\mathcal{E}_\mu=\mu D\pm\frac{D}{\pi}\tan^{-1}\left[r\left(\frac{t_1t_2}{\beta}\right)^{1/2}\right], \quad \frac{2\pi\Gamma_\mu}{D}=4\tanh^{-1}\left(\frac{t_1+t_2}{1+t_1t_2+r^2}\right), \quad \alpha>0, \quad \beta>0, \quad (61)$$

$$\mathcal{E}_\mu=2\mu D+\tfrac{1}{2}D$$

$$\frac{2\pi\Gamma_\mu}{D}=4\tanh^{-1}\left(\frac{t_1+t_2}{1+t_1t_2+r^2}\right)\pm4\coth^{-1}\left[2r\left(\frac{t_1t_2}{-\beta}\right)^{1/2}\right], \quad \alpha>0, \quad \beta<0, \quad (62)$$

$$\mathcal{E}_\mu=2\mu D\pm\tfrac{1}{2}D$$

$$\frac{2\pi\Gamma_\mu}{D}=4\coth^{-1}\left(\frac{t_1+t_2}{1+t_1t_2+r^2}\right)\pm4\tanh^{-1}\left[2r\left(\frac{t_1t_2}{-\beta}\right)^{1/2}\right], \quad \alpha<0, \quad \beta<0. \quad (63)$$

In all three ranges the average spacing of resonance energies $\mathcal{E}_\mu$ is D and the average width is given by the tcth^{-1} function [see Eq. (16d)] of the argument of the tanh^{-1} in Eq. (61). The $\pm$ signs indicate that the appropriate sign alternates from one resonance pole to the next. In the range given by Eq. (62), two poles of different widths have the same value of $\mathcal{E}_\mu$. In the case of Eq. (63) the poles are equally spaced with alternating widths, while in Eq. (61) all poles have the same width, but alternating spacings due to the effects of the direct interaction R^0.

When the direct interaction vanishes, then

$$\alpha=\beta=(1-t_1^2)(1-t_2^2), \quad r=0 \quad (64)$$

and only conditions (61) and (63) apply. If both t_1 and t_2 are either greater or less than unity, the poles of the S matrix occur at the same energies $\mathcal{E}_\mu$ as the R-matrix poles. If one of t_1 and t_2 is greater than unity and the other is less than unity, the S-matrix poles are shifted by one-half mean spacing. Further, when $R^0=0$, all widths Γ_μ are the same and their value as given by Eqs.

(61) and (63) can be written as a sum of partial channel widths

$$\Gamma=\Gamma_1+\Gamma_2\,,\qquad 2\pi\Gamma_j/D=4\ \mathrm{tcth}^{-1}t_j,\quad R^0=0. \tag{65}$$

Returning to the general case of the model (53), we can write the elements of the S-matrix (54) in the form of the pole expansion

$$S_{jk}=e^{i(\phi_j+\phi_k)}\left[W_{jk}{}^0-i\sum_\mu \frac{g_{\mu j}g_{\mu k}}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu}\right], \tag{66}$$

where the $\mathcal{E}_\mu$ and Γ_μ are given by the above discussion and the remaining parameters are found by further computation to be

$$W_{11}{}^0=\frac{2}{\alpha}(1+r^2-t_2{}^2)-1\,,\qquad W_{12}{}^0=W_{21}{}^0=\frac{2ir}{\alpha}(1+r^2+t_1t_2)\,, \tag{67}$$

$$2\pi g_1{}^2/D=\frac{4t_1-4t_2(t_1t_2+r^2)}{\alpha}\pm\frac{8ir(t_1t_2)^{1/2}(1+r^2-t_2{}^2)}{\alpha\sqrt{\beta}}\,,$$
$$2\pi g_1g_2/D=\pm\frac{4t_1t_2[(t_1+t_2)^2-(1+r^2+t_1t_2)(1-r^2+t_1t_2)]}{\alpha\sqrt{\beta}}+\frac{2ir(t_1+t_2)}{\alpha}. \tag{68}$$

The expressions for $W_{22}{}^0$ and $g_2{}^2$ are obtained by symmetry from those given for $W_{11}{}^0$ and $g_1{}^2$. It is easily verified that the residues have indeed the form of products of channel amplitudes g_j. They have been written here as products in order to show what parts of the residues are constant from resonance pole to pole and what parts alternate in sign.

Again we note the way in which the dynamical resonance parameters t_j and the direct interaction parameter r are inextricably mixed up in both the smooth and resonant parts of the S matrix. Thus all elements of W^0 depend on the t_j. On the other hand, r induces both a fixed sign "correlated" part of g_1g_2 as well as alternating sign "uncorrelated" parts in $g_1{}^2$ and $g_2{}^2$. Thus, we note that even if t_1 vanishes, there will be resonance structure in channel 1:

$$(2\pi/D)^{1/2}g_1=\frac{2irt_2{}^{1/2}}{[(1+r^2)^2-t_2{}^2]^{1/2}},\quad t_1=0\,,\qquad (2\pi/D)^{1/2}g_2=\frac{2(t_2)^{1/2}}{[(1+r^2)^2-t_2{}^2]^{1/2}},\quad t_1=0. \tag{69}$$

Of particular interest is the fact that in order that the S matrix have a nonvanishing smooth off-diagonal element $W_{12}{}^0$, that is exhibit a "direct reaction" between channels 1 and 2, it is necessary that r should not vanish. This in turn means that the resonance amplitude coupling channels 1 and 2 must have pole residues containing terms with "correlated" signs of the type $4ir(t_1+t_2)/\alpha$ occurring in Eq. (68). Such terms contribute to the resonance structure of the reaction cross section but also insure that the resonance reaction amplitude has nonzero average. In view of this one must assume that models having a reaction amplitude which consists of a direct part and a zero average resonance part of the form (66) are incorrect. Such models have commonly been assumed in discussions of cross-section fluctuations.[9–11]

The same mixing of direct and resonance effects is also responsible for the fact that the general relation for the total width given in Eq. (61) does not permit an unambiguous definition of partial widths when r is nonvanishing. Parenthetically we note here that a multichannel picket-fence model with *correlated* resonance amplitudes has been studied by Newton.[39] This can be obtained by using cotangents instead of the cosecants in our Eq. (53). In that case one obtains for the total width the expression $2\pi\Gamma/D=4\tanh^{-1}(t_1+t_2)$ for which there also is no unambiguous definition of partial widths.

Returning once more to our model (53) the transmission coefficients for channels 1 and 2 are easily computed from the energy averages of the diagonal elements of the S matrix. The latter are particularly easy to obtain rigorously in the picket-fence model by the well-known device of calculating the analytic continuation of the S-matrix element far in the upper half of the complex plane. One then finds that

$$T_1=1-|\bar{S}_{11}|^2=\frac{4t_1(1+t_2)^2+4r^2(1+t_2)}{[(1+t_1)(1+t_2)+r^2]^2} \tag{70}$$

and a corresponding expression for T_2. As expected, T_1 measures the total absorption of flux in channel 1, both the part that goes into "compound" resonances, as well as the part that is transferred "directly" into channel 2. If both t_1 and t_2 vanish only the latter process contributes to absorption and we have

$$T_1=4r^2/(1+r^2)^2,\quad t_1=t_2=0. \tag{71}$$

When $r=0$ we obtain again the by now familiar expression

$$T_1=4t_1/(1+t_1)^2,\quad r=0 \tag{72a}$$

which, in conjunction with Eq. (65) again yields the same relation that we found in the single-channel case:

$$T_1=1-\exp(-2\pi\Gamma_1/D),\quad \eta_1=\pi\Gamma_1/2D. \tag{72b}$$

The coupled-channel optical-model transition proba-

[39] T. D. Newton, Can. J. Phys. **30**, 53 (1952).

bility[40] for this model is given by

$$|\bar{S}_{12}|^2=\frac{4r^2}{[(1+t_1)(1+t_2)+r^2]^2}. \qquad (73)$$

One may now generalize the two-channel model (53) by introducing periodic fluctuations in the R-matrix pole spacings and amplitudes γ_i as we did in the single-channel case. Such models hold no surprises. The S-matrix resonance parameters, though complicated in form, are consistent with those of our simple two-channel model (53) and with the various single-channel models we have discussed. In particular, in the absence off-diagonal direct reactions the relationships (72) are satisfied for each channel in the form

$$T_j=1-\exp(-2\pi\bar{\Gamma}_j/D),\quad \eta_j=\pi\bar{\Gamma}_j/2D. \qquad (74)$$

We have also investigated one three-channel model of the form where the symmetric matrix (57) has elements

$$(\mathbf{1}-i\mathbf{P}^{1/2}\mathbf{R}\mathbf{P}^{1/2})=\begin{bmatrix}1+it_1\cot z & i(t_1t_2)^{1/2}c^+z & i(t_1t_2)^{1/2}\csc z\\ & 1+it_2\cot z & i(t_2t_3)^{1/2}c^-z\\ & & 1+it_3\cot z\end{bmatrix}, \qquad (75)$$

where

$$c^\pm z\equiv\tfrac{1}{2}(\csc\tfrac{1}{2}z\pm\sec\tfrac{1}{2}z). \qquad (76)$$

Each of the functions c^+z and c^-z has a pole at $z=n\pi$, as do $\cot z$ and $\csc z$, and like the cosecant they have as many residues of value $+1$ as -1, but they have alternating *pairs* of residues of the same sign in such a way that the R-matrix parameters of the model (75) have the "uncorrelated sign" properties

$$\langle\gamma_j\rangle=\langle\gamma_j\gamma_k\rangle=0,\quad \text{for all}\quad j\neq k \qquad (77)$$

where the angular bracket indicates an average over resonances. Equation (75) is the simplest periodic model for which the property (77) holds. Again the detailed values of the S-matrix resonance parameters are qualitatively consistent with our previous results, and again Eq. (74) holds for each of the three channels of the model (75). In fact, this relation (74) between the transmission coefficient and the average partial width appears to be quite independent of the distribution of R-matrix parameters and was found to hold for every model we investigated whose R-matrix parameters satisfied conditions (77) and where there was no direct coupling between channels. On the other hand, we have not been able to construct a general proof of the relationship (74).

B. Average Cross Sections and Cross-Section Fluctuations

The purpose of our periodic resonance models is to investigate general relationships imposed on resonances by the unitarity requirement. It is not particularly interesting to write down detailed cross sections arising from our models. Actual resonances are not equally spaced or even periodically spaced with periodic resonance amplitudes. It is, however, worthwhile to consider average properties of our model cross sections, such as average cross section, mean square fluctuations, etc. In a sense, we have already done this by considering the transmission coefficient which in the single-channel case is proportional to the fluctuation cross section or the "average compound-nucleus cross section" which in units of $\pi\lambda\!\!\!^{-2}$ is defined by

$$\sigma_{jk}^{\text{fl}}=\langle|S_{jk}|^2\rangle-|\langle S_{jk}\rangle|^2. \qquad (78)$$

To calculate the fluctuation cross sections for the two-channel model (53) with $R^0=0$, we write the S-matrix elements in the form

$$S_{11}=e^{2i\phi_1}\left[\frac{1+t_1^2}{1-t_1^2}+\frac{2it_1}{1-t_1^2}\cot(z+iy)\right], \qquad (79)$$

$$S_{12}=e^{i(\phi_1+\phi_2)}\frac{2i(t_1t_2)^{1/2}}{[(1-t_1^2)(1-t_2^2)]^{1/2}}\csc(z+iy), \qquad (80)$$

where

$$y=\text{tcth}^{-1}t_1+\text{tcth}^{-1}t_2,$$
$$t_j=\pi P\gamma_j^2/D.$$

Straightforward integration over an interval of 2π in z yields the following expressions for the fluctuation cross sections defined by Eq. (78)

$$\sigma_{11}^{\text{fl}}=\frac{4t_1^2}{(t_1+t_2)(1+t_1t_2)}\left(\frac{1-t_2}{1+t_1}\right)^2, \qquad (81)$$

$$\sigma_{12}^{\text{fl}}=\frac{4t_1t_2}{(t_1+t_2)(1+t_1t_2)}, \qquad (82)$$

which, as required add up to the total average compound-nucleus absorption cross section in channel 1:

$$\sigma_{11}^{\text{fl}}+\sigma_{12}^{\text{fl}}=T_1=\frac{4t_1}{(1+t_1)^2}. \qquad (83)$$

It is now interesting to compare the results (81) and (82) with the predictions of the statistical theory[7] summarized in Eqs. (8) and (9). We rewrite Eq. (8b) here in the form

$$\sigma_{jk}^{\text{fl}}=\frac{2\pi}{D}\left\langle\frac{|g_{\mu j}|^2|g_{\mu k}|^2}{\Gamma_\mu}\right\rangle_\mu-M_{jk}, \qquad (84)$$

where the term M_{jk} is defined in Eq. (9b). The second term of M_{jk} has been evaluated in Ref. 7 under the assumption that the variations in $g_{\mu j}g_{\mu k}g_{\nu j}^*g_{\nu k}^*$ are uncorrelated with the magnitude of $(\mathcal{E}_\mu-\mathcal{E}_\nu)$. This assumption is valid only for the diagonal elements M_{jj} in the

[40] P. A. Moldauer, Rev. Mod. Phys. 36, 1079 (1964).

case of our model (for $j=k$ the product of the four g factors is independent of μ and ν), and we get from Ref. 7

$$M_{jk}=\frac{2\pi^2}{D^2}\left\{|\langle g_{\mu j}g_{\mu k}\rangle_\mu|^2 -\left\langle g_{\mu j}g_{\mu k}g_{\nu j}^*g_{\nu k}^*\Phi_0\left(\frac{\Gamma_\mu+\Gamma_\nu}{2D}\right)\right\rangle_{\mu\neq\nu}\right\},\ \text{for}\ j=k \quad (85)$$

where

$$\Phi_0\left(\frac{\Gamma_\mu+\Gamma_\nu}{2D}\right)=-2i\frac{D}{\pi}\int_{-\infty}^{\infty}d\epsilon\frac{R_2(\epsilon)}{\epsilon-i(\Gamma_\mu+\Gamma_\nu)}, \quad (86)$$

where the averages are with respect to the resonance level labels μ, ν. The two-level correlation function $R_2(\epsilon)$ gives the probability of finding another resonance level at a distance ϵ from any given level. For the picket-fence (P.F.) model we have clearly

$$R_2(\epsilon)=\sum_{n=-\infty}^{\infty}\delta(nD-\epsilon), \quad (87)$$

and from this

$$\Phi_0^{\text{P.F.}}(\Gamma/D)=\coth\frac{\pi\Gamma}{D}-\frac{D}{\pi\Gamma}. \quad (88)$$

Substituting this into Eq. (85) we find that in our model

$$M_{11}=\frac{2\pi}{D}\frac{|\bar{g}_1|^4}{\Gamma}-\frac{2\pi^2}{D^2}|g_1|^4\left(\coth\frac{\pi\Gamma}{D}-1\right). \quad (89)$$

In evaluating the second term of Eq. (9b) for the off-diagonal term M_{12}, we note that the sign of $g_{\mu1}g_{\mu2}g_{\nu1}^*g_{\nu2}^*$ depends on whether $\mathcal{E}_\mu$ and $\mathcal{E}_\nu$ are separated by an even number of spacings (+sign) or by an odd number of spacings (−sign). This level-level correlation of the resonance amplitudes is easily taken into account by evaluating M_{12} just as in Ref. 7, but with the substitution

$$R_2(\epsilon)\rightarrow\sum_{n=-\infty}^{\infty}(-1)^n\delta(nD-\epsilon). \quad (90)$$

The first term of M_{12} vanishes and so we find

$$M_{12}=\frac{2\pi}{D}\frac{|g_1|^2|g_2|^2}{\Gamma}-\frac{2\pi^2}{D^2}|g_1|^2|g_2|^2\operatorname{csch}\frac{\pi\Gamma}{D}. \quad (91)$$

From Eqs. (65) and (68) we have for $R^0=0$

$$\frac{2\pi\Gamma}{D}=4\ \text{tcth}^{-1}t_1+4\ \text{tcth}^{-1}t_2, \quad (92)$$

$$\frac{2\pi g_j^2}{D}=\frac{4t_j}{1-t_j^2}. \quad (93)$$

Putting (92) and (93) into Eqs. (84), (89), and (91) we find that the method of statistical theory yields exactly the same result as our direct integration (81) and (82).

$$\sigma_{11}^{\text{fl}}=\frac{8t_1^2}{(1-t_1^2)^2}\left(\coth\frac{\pi\Gamma}{D}-1\right) =\frac{4t_1^2}{(t_1+t_2)(1+t_1t_2)}\left(\frac{1-t_2}{1+t_1}\right)^2, \quad (81)$$

$$\sigma_{12}^{\text{fl}}=\frac{8t_1t_2}{(1-t_1^2)(1-t_2^2)}\operatorname{csch}\frac{\pi\Gamma}{D}=\frac{4t_1t_2}{(t_1+t_2)(1+t_1t_2)}. \quad (82)$$

It is now appropriate to compare Eqs. (81) and (82) with the predictions of the Hauser-Feshbach[2–5] (H.F.) relations which might be thought to apply here in their simplest form because partial widths do not fluctuate and so there is no width fluctuation correction.[31] We have then in units of $\pi\lambda\!\!\!^{-2}$

$$\sigma_{11}^{\text{H.F.}}=\frac{T_1^2}{T_1+T_2}=\frac{4t_1^2}{(t_1+t_2)(1+t_1t_2)+4t_1t_2}\left(\frac{1+t_2}{1+t_1}\right)^2, \quad (94)$$

$$\sigma_{12}^{\text{H.F.}}=\frac{T_1T_2}{T_1+T_2}=\frac{4t_1t_2}{(t_1+t_2)(1+t_1t_2)+4t_1t_2}, \quad (95)$$

$$\sigma_{12}^{\text{fl}}=\frac{\sigma_{12}^{\text{H.F.}}}{1-\sigma_{12}^{\text{H.F.}}}.$$

It is clear that only in the limit of small values of T_1 and T_2 are the Hauser-Feshbach cross sections good approximations to the fluctuation cross sections (81) and (82). Particularly striking is the fact that whenever T_2 is unity, the average compound-elastic cross section σ_{11}^{fl} vanishes, while $\sigma_{11}^{\text{H.F.}}$ becomes equal to $T_1^2/(1+T_1)$. Similarly when $T_1=T_2=1$ then $\sigma_{12}^{\text{fl}}=\pi\lambda\!\!\!^{-2}$, while $\sigma_{12}^{\text{F.H.}}=\frac{1}{2}\pi\lambda\!\!\!^{-2}$. On the other hand, our relations (74) between the widths and the transmission coefficients say that $\Gamma\gg D$ whenever either T_1 or T_2 are close to unity. Therefore we conclude that our model does not satisfy the Hauser-Feshbach relations when $\Gamma\gg D$.

This conclusion is to be contrasted with the derivation of the Hauser-Feshbach formula on the basis of statistical theory in Ref. 7. This derivation claims that the Hauser-Feshbach relations should hold in the limit of large values of Γ/D. The reason for the apparent contradiction between our model (79), (80) and the derivation of Ref. 7 is that in the latter it is explicitly assumed that the S-matrix poles amplitudes $g_{\mu j}$ are not only uncorrelated with respect to channel indices j, as ours are (i.e., $\langle g_{\mu1}g_{\mu2}\rangle=0$), but also that the values of $g_{\mu j}$ are uncorrelated with respect to the ordered resonance index μ. This condition is definitely not satisfied in our model, since the products $g_{\mu1}g_{\mu2}$ alternate in sign from resonance to resonance. Under the assumptions used to derive the

Hauser-Feshbach formula in Ref. 7, the expression (91) always vanishes.

We see therefore that the derivation of the Hauser-Feshbach formula is not applicable to our model. The next appropriate question is whether this circumstance is due to the special nature of our very simple model or whether it arises from the general requirements of unitarity. For this we have one clue. By adding the fluctuation cross sections computed from an S matrix with *uncorrelated* resonance amplitudes $g_{\mu j}$, one obtains transmission coefficients T_c which may be *greater* than $2\pi\bar{\Gamma}_c/D$ [see Eqs. (67) and (84) of Ref. 7] and which impose the limit $\pi\bar{\Gamma}_c/D\leq 1$ on the average channel width to spacing ratio [see Eq. (83) of Ref. 7]. However, on the basis of the present work we are strongly inclined to reject the notion of an upper limit on $\bar{\Gamma}_c/D$ as a requirement for unitarity and we are equally strongly inclined to believe that T_c is always *less* than $2\pi\bar{\Gamma}_c/D$. In fact we believe that T_c is given by Eq. (74) which is the one and only relation we have found to be common to all the models we have investigated analytically. For, $\bar{S}_{cc}$ and therefore T_c is expected to be relatively insensitive to the details of the distributions and correlations of the S-matrix resonance parameters, while $\langle|S_{cc'}|^2\rangle$ and therefore $\bar{\sigma}_{cc'}$ is expected to be sensitive to such correlations as can be seen from the structure of Eq. (9b) (averages of higher powers of $S_{cc'}$ which determine cross section fluctuations are even more sensitive to correlations).

We are therefore forced to conclude that unitarity imposes level-level correlations among the S-matrix resonance-pole amplitudes $g_{\mu j}$ of overlapping resonances. This supposition is certainly not unreasonable since unitarity certainly imposes *some* conditions on the values of the S-matrix elements (but not on those of the R matrix) and it is in fact not easy to see how such conditions can affect a pole expansion of the S matrix, except through conditions on the residues.

We also have numerical support for this supposition in some previous calculations[41]. There R matrices with random uncorrelated channel amplitudes $\gamma_{\mu c}$, were inverted numerically by means of the level-matrix formalism and the resulting $g_{\mu c}$ of the S-matrix pole expansion were found to exhibit both channel-channel and level-level correlations.

Not only the Hauser-Feshbach formula, but also the various formulas used to analyze cross-section fluctuations depend for their validity on the absence of correlations. Thus for our model (79), (80) the mean-square deviation of the cross section (or the autocorrelation) is given by

$$\frac{\langle\sigma_{12}^2\rangle-\bar{\sigma}_{12}^2}{\bar{\sigma}_{12}^2}=2\frac{(t_1+t_2)^2+(1+t_1t_2)^2}{(t_1+t_2)(1+t_1t_2)}-1\,, \qquad (96)$$

TABLE II. Average cross sections and cross-section fluctuations for three two-channel models: (1) Uncorrelated resonance pole amplitudes $g_{\mu c}$; (2) the cosecant model of Eq. (53); (3) the c^+ model, with c^+ replacing the cosecants in Eq. (53). All values are for $T_1=T_2=1$, and no direct coupling.

Model:	Uncorrelated	Cosecant	c^+
$\bar{\sigma}_{12}/\pi\lambda\!\!\!^{-}{}_1^2$	$\frac{1}{2}$	1	$\frac{2}{3}$
$[\langle\sigma_{12}^2\rangle-\bar{\sigma}_{12}^2]/\bar{\sigma}_{12}^2$	1	3	$\frac{1}{9}$

which diverges, as expected, in the region of isolated resonances where $t_1, t_2\ll 1$, and reaches the value 3 in the limit of very large Γ/D when $t_1\sim t_2\sim 1$. This is in contrast to the value of unity expected when Γ/D is large and there are no correlations.[8]

It should be emphasized that the results (81), (82), and (96) are valid *only* for the specific models embodied in Eqs. (79) and (80). In order to explore the effects of different level-level correlations, the values of $\sigma_{12}^{\mathrm{fl}}$ and the mean-square deviation from the average have also been calculated for the two-channel model in which the cosecants in Eq. (53) are replaced by the function c^+ defined in Eq. (76). The results are summarized together with those of the uncorrelated model and the cosecant model in Table II for the case where $T_1=T_2=1$. The indications are that while the average cross section does depend on the details of level-level correlations, the mean-square fluctuation of the cross section depends even more strongly on such details.

Small values of the mean-square cross-section fluctuation that have been observed in low-energy (p,α) reactions on medium weight nuclei,[42,43] and that can otherwise be explained only by assuming unusually large direct-reaction cross sections at backward angles could be due to unitarity conditioned correlations in the resonance amplitudes. Also the range of any level-level amplitude correlations will affect the observed cross-section correlation width, and since amplitude correlations will be expected to differ for different competing reaction processes, it is not impossible to observe different correlation widths in different reactions involving the same compound nucleus.

The average cross sections and cross-section fluctuations obtained in our three-channel model, Eq. (75) are more complicated functions of the resonance parameters. For example in the case where $T_1=T_3$ and no R^0 is present, the three-channel model gives

$$\sigma_{12}^{\mathrm{fl}}=\frac{4t_1t_2(1+t_3^2)}{(t_1+t_2+t_3+t_1t_2t_3)(1+t_1t_2+t_2t_3+t_3t_1)}\,, \qquad (97)$$

which again differs in general from the Hauser-Feshbach expression expected for uncorrelated amplitudes.

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The study of the statistical properties of slow-neutron resonances (see, for example, Garg *et al.*[44]) and the theoretical studies of complex spectra[22] have yielded much information on the actual distributions and correlations of R-matrix parameters, which are of course, much more complicated than those of any of the models discussed above. Work now in progress employs further generalizations of these models, as well as numerical methods to investigate the implications of unitarity and its effect on cross sections and their fluctuations for more realistic distributions of resonance parameters and for larger numbers of competing channels.

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Proton-Capture Gamma Rays from Be^8, C^{12}, Mg^{24}, and Ca^{40} in the Giant-Resonance Region*

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The 90° yield for gamma rays from Be^8, C^{12}, Mg^{24}, and Ca^{40} was determined for (p,γ) reactions using 10.4- to 14.5-MeV protons from the Columbia University variable energy cyclotron. For $B^{11}(p,\gamma)C^{12}$ the yields to both the first excited state and the ground state of the residual nucleus are presented. In the case of $K^{39}(p,\gamma)Ca^{40}$, only the ground-state yield was determined. Because the ground-state yields are very small, we report only the yield due to transitions to the first excited state for the $Li^7(p,\gamma)Be^8$ reaction and the combined first excited and ground-state yields for the $Na^{23}(p,\gamma)Mg^{24}$ reaction. In the region investigated, the yield curves exhibit a considerable amount of fine structure in all cases except $Li^7(p,\gamma)Be^8$. Fine-structure peaks were observed for the following excitation energies: 21.9, 22.4, 22.7, 23.0, 23.3, 24.1, 24.7, and 25.4 MeV for $Na^{23}(p,\gamma_0+\gamma_1)Mg^{24}$; at 18.8, 19.2, 19.5, 20.0, 21.0, and 21.7 MeV for $K^{39}(p,\gamma_0)Ca^{40}$; at 25.5, 26.9, 28.0, and 28.45 MeV for $B^{11}(p,\gamma_1)C^{12}$; and at 25.5, 27.45, 28.0, and 28.9 MeV for $B^{11}(p,\gamma_0)C^{12}$. A comparison with other experimental results shows that some of these peaks have not been previously observed.

I. INTRODUCTION

GIANT-RESONANCE phenomena have been extensively investigated by photonuclear reactions.[1] The source of the incident photons has been bremsstrahlung radiation in most of the photonuclear work, but more recently some experiments have been performed using monochromatic γ rays. Since the advent of variable energy cyclotrons and tandem accelerators, however, (p,γ) reactions have been used to investigate the giant-resonance region of nuclear excitation by the inverse process. The (p,γ) reactions have several distinct advantages, namely, (1) continuously variable, monochromatic beams are more readily attainable for protons than gamma rays; (2) nuclei with unstable ground states can be studied by the inverse reaction and not by the direct photonuclear reaction; (3) transitions resulting from de-excitation to low-lying excited states can be investigated by the inverse reaction provided the states are sufficiently well separated. The (p,γ) reactions provide only the proton widths of the giant resonance, whereas all the particle widths are required to obtain the total cross section for photonuclear reactions.

At the time these experiments were undertaken, the work that was reported employing these (p,γ) reactions was confined to proton energies below 10 to 11 MeV.[2–6] In some instances, this corresponded to energies below the peak of the giant resonance. Since the energy of the Columbia University 36-in. cyclotron had not been varied previously, the energy variation having been accomplished in conjunction with these experiments, as required, it was decided to extend the earlier (p,γ)

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