K-Electron Ejection Accompanying Nuclear K Capture

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The momentum spectrum for the K electrons ejected during allowed K capture is calculated in the lowest order of perturbation theory using a nonrelativistic approximation for the electrons. The differences between this theory and the previously developed theory of Primakoff and Porter are pointed out. The new theory yields a spectrum which is virtually identical to the Primakoff-Porter spectrum over almost the entire momentum range of the ejected electrons. A discussion of relativistic effects and effects due to atomic screening and nuclear recoil is given. The discrepancies existing between the theoretical predictions and the experimental results are also discussed.

I. INTRODUCTION

IN nuclear decay by orbital electron capture the neutrino emitted in the decay is sometimes accompanied by another emitted particle, either a photon or an electron. The emission of the additional particle is a consequence of the fact that the electronic structure of the atom undergoes a rearrangement as a result of the nuclear decay, particle emission being one of the more drastic possible rearrangements. When two particles are emitted, the energy released in the decay process is shared statistically between them.

The case in which a photon is emitted, viz., radiative orbital electron capture, has been carefully studied by Glauber and Martin in two well-known papers.[1,2] These authors have calculated both the photon-energy spectrum and the total radiative capture probability per electron capture event. A complete mathematical analysis of this problem was made possible as a result of a crucial observation concerning the sum over intermediate electron states which appears in the calculation. Glauber and Martin noticed that this sum, which is simply the Green's function for the Dirac equation with a nuclear Coulomb potential, could be represented in closed form (provided one of the position coordinates was evaluated at the origin).

The most interesting case of electron emission is that of K-electron emission during K-electron capture. The relative transition rate per K-capture event is independent of all nuclear matrix elements and depends only on the atomic structure of the atom. In particular, the shape of the momentum spectrum of the ejected electrons is sensitive to screening and correlation effects between the two K electrons in the initial state. Hence, a study of the ejected-electron momentum spectrum provides a method for examining the validity of various assumptions concerning the structure of K-electron states in many electron atoms.

The problem has been studied theoretically by Primakoff and Porter.[3] Their treatment of the problem is a nonrelativistic one (except for the description of the neutrino) and is based on the use of the sudden-perturbation approximation. This approach necessitates the use of a two-electron wave function to describe the initial electronic configuration.[4] For this purpose Primakoff and Porter construct a two-parameter variational wave function, designed to take account of screening and correlation effects and adjusted to minimize the energy of the initial electronic configuration. They then calculate the momentum spectrum for the ejected electrons, the total ejection probability per K-capture event and the probability, per K-capture event, for the production of two holes in the K shell of the atom.

Primakoff and Porter published their theory in 1953. Since then a number of experimental observations have been reported[5–12] on various aspects of the problem. The experiments are rather difficult to perform, and as a result, the errors associated with many of the early attempts are so large that a detailed quantitative comparison with the Primakoff-Porter theory is not justified. However, during recent years experimental techniques have improved to the point where a quantitative comparison with the Primakoff-Porter theory is now meaningful. In particular, Lark and Perlman[11] have measured the probability for the production of two holes in the

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[2] P. C. Martin and R. J. Glauber, Phys. Rev. **109**, 1307 (1958).
[3] H. Primakoff and F. T. Porter, Phys. Rev. **89** 930 (1953).
[4] The presence of electrons in all shells above the K shell is ignored. See Sec. V for a discussion of effects due to these electrons.
[5] F. T. Porter and H. P. Holz, Phys. Rev. **89**, 938 (1953).
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[12] J. G. Pengra and B. Crasemann, Phys. Rev. **131**, 2642 (1963).

K shell during K capture in Cs^{131}. They reported an experimental value which was about a factor of two less than that predicted by the Primakoff-Porter theory. Most recently, Pengra and Crasemann[12] have measured the energy spectrum of the electrons ejected during K capture in Fe^{55}. For low ejection energies, their results differed from those predicted by the Primakoff-Porter theory by factors of from 2 to 5.

It has been suggested[13] that the apparent discrepancies between the experiments and the Primakoff-Porter theory reflect inadequacies in the structure of the two-electron variational wave function chosen by these authors to represent the initial state of the system. The use of such a wave function can be avoided by including the initial-state electron-electron interaction in the perturbed part of the Hamiltonian (rather than in the unperturbed part as Primakoff and Porter have done). This approach is equivalent to performing a perturbation expansion on the exact two-electron wave function where the perturbation is taken to be the electron-electron interaction.[14] The problem of K-electron ejection during K capture is then one in third-order perturbation theory and involves a sum over intermediate electron states. However, due to the work of Glauber and Martin it is now possible to represent this sum in closed form, and it is this simplification which makes a more exact analysis possible at the present time.

In Sec. II we develop the basic equations of the theory. It is shown that by restricting the analysis to a nonrelativistic description of the electrons, two significant simplifications result in the structure of the matrix elements. One is due to the vanishing of retardation effects while the other is due to a partial cancellation between the two matrix elements which contribute to the process. Section III is devoted to a review of the Primakoff-Porter theory and a discussion of the differences between that theory and the theory presented here. In Sec. IV, we carry out the detailed calculations of the theory and obtain the formula for the ejected-electron momentum spectrum. In Sec. V, the results of the theory are presented and comparison is made with the results of Primakoff and Porter and with recent experimental observations.

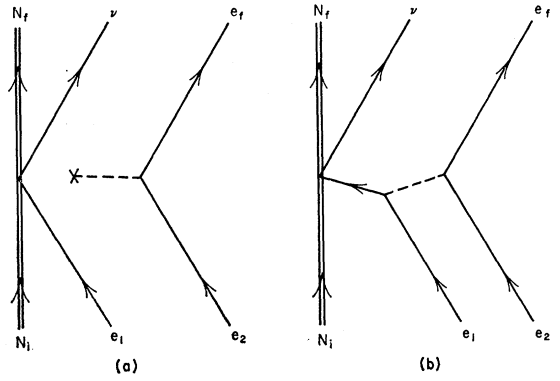

FIG. 1. Feynman diagrams for K-electron ejection during K capture.

II. MATRIX ELEMENTS AND TRANSITION RATE

The theory presented in this paper is based on treating the electron-electron interaction along with the beta interaction as perturbations on the nuclear Coulomb interactions of the system. Accordingly, the unperturbed electron field $\psi_e(x)$ satisfies a Dirac equation containing the Coulomb field of the nucleus,[15]

$$\left[\gamma_u\frac{\partial}{\partial x_u}+\gamma_4\frac{Z\alpha}{r}+1\right]\psi_e(x)=0. \tag{1}$$

In this representation, the interaction Hamiltonian density is

$$\mathcal{H}_I(x)=\mathcal{H}_\gamma(x)+\mathcal{H}_\beta(x), \tag{2}$$

where $\mathcal{H}_\gamma$ represents the interaction of the electron field with the radiation field and $\mathcal{H}_\beta$ represents the beta interaction, for which we use the usual $V-\lambda A$ coupling. The probability amplitude for the process may then be obtained from the appropriate terms of the expansion for the scattering matrix. In the lowest order, the two terms in the expansion which contribute to the process correspond to the diagrams shown in Fig. 1.

The first diagram represents the effects on the motion of the uncaptured K electron due to the sudden change in the nuclear charge as a result of the K-capture process. The matrix element for this diagram comes from the first-order term in the scattering matrix and is given, in configuration space, by[16]

$$M_a=G[1-P_{12}]\int d\mathbf{r}\bar{\phi}_f{}^N(\mathbf{r})\Gamma_\mu\phi_i{}^N(\mathbf{r})\bar{\phi}^\nu(\mathbf{r})\Lambda_\mu\phi_1{}^{(Z)}(\mathbf{r})$$
$$\times\int d\mathbf{r}'\phi_f{}^{\dagger(Z-1)}(\mathbf{r}')\phi_2{}^{(Z)}(\mathbf{r}'), \tag{3}$$

where $\Gamma_\mu=\gamma_\mu(1+\lambda\gamma_5)$, $\Lambda_\mu=\gamma_\mu(1+\gamma_5)$, and G is the vector coupling constant of the beta interaction. $\phi_i{}^N(\mathbf{r})$ and $\phi_f{}^N(\mathbf{r})$ represent the initial and final states of the nucleus, respectively.[17] Similarly, ϕ^ν represents the

[13] See, for instance, Ref. 12, p. 2648.

[14] This approach has also been suggested by Primakoff and Porter, Ref. 3.

[15] We employ units in which $m=c=\hbar=1$, (m is the electron mass), and $e^2=\alpha=1/137$. $x_\mu=(\mathbf{r}, it)$ and the Dirac matrices are defined as $\boldsymbol{\gamma}=-i\beta\boldsymbol{\alpha}$ and $\gamma_4=\beta$. Also $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$ and $\bar{\phi}=\phi^\dagger\gamma_4$.

[16] Although this term is formally of the first order, it is actually of order e^3 because of the near orthogonality of the initial and final states of the ejected electron; see Sec. III.

[17] In the nuclear wave functions, the coordinates of all the nucleons except the one involved in the capture process have been suppressed. It is understood that an integration over this coordinate implies an integration over the coordinates of the other nucleons as well. A summation over all the protons in the initial nucleus is also implied.

emitted neutrino and $\phi_1^{(Z)}$, $\phi_2^{(Z)}$, and $\phi_f^{(Z-1)}$ represent the two initial K electrons and the ejected electron, respectively; the electron wave functions $\phi^{(Z)}$ being Coulomb-field eigenfunctions corresponding to a nuclear charge Ze. P_{12} is the exchange operator which interchanges the two K electrons.

It will prove useful for our later work to rewrite the overlap integral appearing in (3) in a slightly different form. To this end, let us consider the functions $\phi_\mu^{(Z)}(\mathbf{r})$ and $\phi_\nu^{(Z-1)}(\mathbf{r})$ which satisfy

$$\left(H_0-\frac{Z\alpha}{r}\right)\phi_\mu^{(Z)}(\mathbf{r})=E_\mu^{(Z)}\phi_\mu^{(Z)}(\mathbf{r}), \tag{4a}$$

$$\left(H_0-\frac{Z\alpha}{r}+\frac{\alpha}{r}\right)\phi_\nu^{(Z-1)}(\mathbf{r})=E_\nu^{(Z-1)}\phi_\nu^{(Z-1)}(\mathbf{r}). \tag{4b}$$

If we treat the α/r term in (4b) as a perturbation, we can calculate the functions $\phi_\nu^{(Z-1)}(\mathbf{r})$ from the "unperturbed" functions $\phi_\mu^{(Z)}(\mathbf{r})$ by standard perturbation methods. Such a calculation gives

$$\phi_\nu^{(Z-1)}(\mathbf{r})=\phi_\nu^{(Z)}(\mathbf{r})+\alpha\sum_\lambda{}'\frac{\phi_\lambda^{(Z)}(\mathbf{r})}{(E_\nu^{(Z)}-E_\lambda^{(Z)})}\times\int\frac{d\mathbf{r}'}{r'}\phi_\lambda^{\dagger(Z)}(\mathbf{r}')\phi_\nu^{(Z)}(\mathbf{r}')+O(\alpha^2). \tag{5}$$

The prime on the summation sign signifies the omission of the term $\nu=\lambda$. If we insert this expansion into (3) and, in keeping with our approximation, retain only the first nonvanishing term, we can rewrite M_a as

$$M_a=\alpha G[1-P_{12}]\int d\mathbf{r}\,\bar\phi_f^N(\mathbf{r})\Gamma_\mu\phi_i^N(\mathbf{r})\bar\phi^\nu(\mathbf{r})\times\Lambda_\mu\phi_1(\mathbf{r})\frac{1}{(W-E_2)}\int\frac{d\mathbf{r}'}{r'}\phi_f^\dagger(\mathbf{r}')\phi_2(\mathbf{r}'), \tag{6}$$

where W is the energy of the ejected electron. We have suppressed the superscript (Z) since it is now the same for all the electron wave functions.

Let us now consider the second diagram appearing in Fig. 1. This diagram represents the effect on the motion of the uncaptured K electron due to the sudden vanishing of the electron-electron interaction between this electron and the K electron which has been captured. The matrix element corresponding to this diagram is

$$M_b=\frac{\alpha G}{2\pi^2}[1-P_{12}]\int d\mathbf{r}_1\int d\mathbf{r}_2\int d\mathbf{r}_3\int\frac{d\mathbf{K}\,e^{i\mathbf{K}\cdot(\mathbf{r}_1-\mathbf{r}_2)}}{[K^2-(W-E_2)^2]}\times\bar\phi_f(\mathbf{r}_2)\gamma_\nu\phi_2(\mathbf{r}_2)\bar\phi_f^N(\mathbf{r}_3)\Gamma_\mu\phi_i^N(\mathbf{r}_3)\times\bar\phi^\nu(\mathbf{r}_3)\Lambda_\mu G_E(\mathbf{r}_3,\mathbf{r}_1)\gamma_\nu\phi_1(\mathbf{r}_1), \tag{7}$$

where $E=E_1+E_2-W$. The electron Green's function, $G_E(\mathbf{r},\mathbf{r}')$, appearing in (7) is the Coulomb-Dirac Green's function. It satisfies the equation

$$[H_c(\mathbf{r})-E]G_E(\mathbf{r},\mathbf{r}')=-\gamma_4\delta(\mathbf{r}-\mathbf{r}'), \tag{8}$$

where H_c is the Coulomb-Dirac Hamiltonian.

In this paper we restrict ourselves to ejected-electron energies for which the motion can be well described nonrelativistically. It is at these energies that Coulomb effects are most important and hence lead to the most interesting results.[18] In a nonrelativistic treatment, it is to be expected that the electron-electron interaction reduces to an instantaneous Coulomb interaction. This reduction and the subsequent simplification of the matrix element may be achieved most easily by first rewriting the integral over K, appearing in (7), in a more covariant form,

$$\int\frac{d\mathbf{K}\,\gamma_\nu\cdots\gamma_\nu}{[K^2-(W-E_2)^2]}=\int\frac{d^4k}{k^2}\delta(k_0-W+E_2)\gamma_\nu\cdots\gamma_\nu, \tag{9}$$

where the $\cdots$ represent intervening matrices and $k_\mu=(\mathbf{K},ik_0)$. The retardation effects can now be separated out by a technique due to Feynman[19] according to which

$$\frac{\gamma_\nu\cdots\gamma_\nu d^4k}{k^2}=\sum_{i=1}^{2}\frac{\boldsymbol{\gamma}\cdot\hat e_i\cdots\boldsymbol{\gamma}\cdot\hat e_i d^4k}{k^2}+\frac{\gamma_4\cdots\gamma_4 d^4k}{K^2}, \tag{10}$$

where the $\hat e_i$ are the two transverse polarization vectors. Equation (10) is an operator equation which is understood to hold true for the corresponding matrix elements. Feynman has derived this equation for the matrix elements of free-particle states. In Appendix A, we extend this proof to include Coulomb field states. The first two terms on the right represent the retardation effects due to the exchange of virtual transverse photons while the last term represents the instantaneous Coulomb interaction.

In a nonrelativistic treatment, only γ_4 can couple the final and initial wave functions of the ejected electron. Thus, for the corresponding matrix elements of (10), the first two terms on the right vanish and only the Coulomb interaction term survives. Incorporating this simplification into (9) and carrying out the k_0 integration, we obtain the result

$$\int\frac{d\mathbf{K}\gamma_\nu\cdots\gamma_\nu}{[K^2-(W-E_2)^2]}=\int\frac{d\mathbf{K}}{K^2}\gamma_4\cdots\gamma_4, \tag{11}$$

in the nonrelativistic limit. After substituting this result into (7) and carrying out the $\mathbf{K}$ integration, we obtain the following expression for M_b:

$$M_b=\alpha G[1-P_{12}]\int d\mathbf{r}_1\int d\mathbf{r}_2\int d\mathbf{r}_3\frac{1}{r_{12}}\phi_f^\dagger(\mathbf{r}_2)\phi_2(\mathbf{r}_2)\times\bar\phi_f^N(\mathbf{r}_3)\Gamma_\mu\phi_i^N(\mathbf{r}_3)\bar\phi^\nu(\mathbf{r}_3)\Lambda_\mu G_E(\mathbf{r}_3,\mathbf{r}_1)\phi_1(\mathbf{r}_1). \tag{12}$$

A further simplification of the matrix elements is motivated by the following considerations. While one K electron, with coordinate $\mathbf{r}_1$, is being captured by the

[18] Actually, since the energy released in the decay is usually quite small, relativistic effects can be neglected for all but the highest ejected-electron energies.

[19] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

nucleus, the second K electron, with coordinate $\mathbf{r}_2$, is being ejected. If electron one is much closer to the nucleus than is electron two when the capture process occurs, then the potential energy of electron two will undergo almost no change and thus electron two is not likely to be ejected. Therefore, when $r_2 \gg r_1$, we expect the corresponding amplitude for the process to be extremely small. Mathematically, this sharp decrease in the transition amplitude in the region $r_2 \gg r_1$, occurs through a mutual cancellation between the two matrix elements, M_a and M_b. In order to effect this cancellation explicitly, it is necessary that we rewrite the expression for M_a in a somewhat different form. To this end we consider the expression

$$(E_2-W)\int d\mathbf{r}_1 G_E(\mathbf{r}_3,\mathbf{r}_1)\gamma_4\phi_1(\mathbf{r}_1).$$

If we introduce the eigenfunction expansion for the Green's function,

$$G_E(\mathbf{r}_3,\mathbf{r}_1)=\sum_\alpha \frac{\phi_\alpha(\mathbf{r}_3)\bar{\phi}_\alpha(\mathbf{r}_1)}{(E-E_\alpha)},$$

and make use of the relation $E=E_1+E_2-W$, this expression can be reduced to

$$(E_2-W)\int d\mathbf{r}_1 G_E(\mathbf{r}_3,\mathbf{r}_1)\gamma_4\phi_1(\mathbf{r}_1)=\phi_1(\mathbf{r}_3). \quad (13)$$

Substitution of (13) into (6) then leads to

$$M_a=-\alpha G[1-P_{12}]\int d\mathbf{r}_3\bar{\phi}_f{}^N(\mathbf{r}_3)\Gamma_\mu\phi_i{}^N(\mathbf{r}_3)\bar{\phi}^\nu(\mathbf{r}_3)\Lambda_\mu$$
$$\times\int d\mathbf{r}_1 G_E(\mathbf{r}_3,\mathbf{r}_1)\gamma_4\phi_1(\mathbf{r}_1)\int\frac{d\mathbf{r}_2}{r_2}\phi_f{}^\dagger(\mathbf{r}_2)\phi_2(\mathbf{r}_2). \quad (14)$$

We can now conveniently combine (12) and (14) to get the total matrix element for the process

$$M\equiv M_a+M_b=-\alpha G[1-P_{12}]\int d\mathbf{r}_3\bar{\phi}_f{}^N(\mathbf{r}_3)\Gamma_\mu\phi_i{}^N(\mathbf{r}_3)$$
$$\times\bar{\phi}^\nu(\mathbf{r}_3)\Lambda_\mu\int d\mathbf{r}_1 G_E(\mathbf{r}_3,\mathbf{r}_1)\phi_1(\mathbf{r}_1)$$
$$\times\int d\mathbf{r}_2\phi_f{}^\dagger(\mathbf{r}_2)\phi_2(\mathbf{r}_2)\left(\frac{1}{r_2}-\frac{1}{r_{12}}\right). \quad (15)$$

For allowed K capture, the wave functions of the leptons involved in the beta interaction can be replaced by their values at the origin. Furthermore, in the nonrelativistic limit the nuclear matrix elements reduce to

$$(\phi_f{}^N,\Gamma_\mu\phi_i{}^N)\rightarrow\{i\lambda(\phi_f{}^N,\boldsymbol{\sigma}\phi_i{}^N),(\phi_f{}^N,\phi_i{}^N)\}\equiv\{\mathbf{B},B_4\}, \quad (16)$$

and (15) becomes

$$M=-\alpha G[1-P_{12}]\bar{\phi}^\nu(0)\mathfrak{B}\int d\mathbf{r}_1 G_E(0,\mathbf{r}_1)\phi_1(\mathbf{r}_1)$$
$$\times\int d\mathbf{r}_2\phi_f{}^\dagger(\mathbf{r}_2)\phi_2(\mathbf{r}_2)\left(\frac{1}{r_2}-\frac{1}{r_{12}}\right), \quad (17)$$

where $\mathfrak{B}=B_\mu\Lambda_\mu$.

Now, to bring about the cancellation in (17) which occurs when $r_2\gg r_1$, we expand $1/r_{12}$ in spherical harmonics in the usual way,

$$\frac{1}{r_{12}}=\frac{1}{r_>}+\frac{1}{r_>}\sum_{n=1}^{\infty}\left(\frac{r_<}{r_>}\right)^n P_n(\hat{r}_<\cdot\hat{r}_>),$$

and split up the integration in (17) into two regions according to whether $r_1\lesssim r_2$, thus obtaining

$$M=-\alpha G[1-P_{12}]\bar{\Phi}^\nu(0)\mathfrak{B}\int d\mathbf{r}_2\int d\Omega_1\left\{\int_0^{r_2} r_1^2 dr_1 G_E(0,\mathbf{r}_1)\phi_1(\mathbf{r}_1)\left[-\frac{1}{r_2}\sum_{n=1}^{\infty}\left(\frac{r_1}{r_2}\right)^n P_n(\hat{r}_1\cdot\hat{r}_2)\right]\right.$$
$$\left.+\int_{r_2}^{\infty} r_1^2 dr_1 G_E(0,\mathbf{r}_1)\phi_1(\mathbf{r}_1)\left[\frac{1}{r_2}-\frac{1}{r_1}-\frac{1}{r_1}\sum_{n=1}^{\infty}\left(\frac{r_2}{r_1}\right)^n P_n(\hat{r}_1\cdot\hat{r}_2)\right]\right\}\phi_f{}^\dagger(\mathbf{r}_2)\phi_2(\mathbf{r}_2). \quad (18)$$

Since we are treating the electrons nonrelativistically, $G_E(0,\mathbf{r}_1)$ and $\phi_1(\mathbf{r}_1)$ are spherically symmetric functions. Thus, with the aid of the addition formula for the Legendre functions, we see that all the terms involving the P_n's vanish when the integration over Ω_1 is performed and M thereby reduces to

$$M=-\alpha G[1-P_{12}]\bar{\phi}^\nu(0)\mathfrak{B}\int d\mathbf{r}_2\int d\mathbf{r}_1 G_E(0,r_1)\phi_1(r_1)$$
$$\times\phi_f{}^\dagger(\mathbf{r}_2)\phi_2(r_2)\theta(r_1-r_2)\left(\frac{1}{r_2}-\frac{1}{r_1}\right), \quad (19)$$

where we have introduced the step function $\theta(r)$. For further analysis it will be convenient to use the well-known integral representation for $\theta(r)$,

$$\theta(r)=\lim_{\epsilon\rightarrow 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dk\,e^{ikr}}{(k-i\epsilon)},$$

in which case M becomes

$$M=-\frac{\alpha G}{2\pi i}[1-P_{12}]\bar{\phi}^\nu(0)\mathfrak{B}\int\frac{dk}{(k-i\epsilon)}$$
$$\times\int d\mathbf{r}_2\phi_f{}^\dagger(\mathbf{r}_2)\phi_2(r_2)e^{-ikr_2}\int d\mathbf{r}_1 G_E(0,r_1)$$
$$\times\phi_1(r_1)e^{ikr_1}\left(\frac{1}{r_2}-\frac{1}{r_1}\right). \quad (20)$$

Section IV will be devoted to the evaluation of M. Here we simply note the connection between M and the ejected-electron momentum spectrum. The total transition rate for the ejection of a K electron into the momentum range dP is

$$dw=2\pi P^2dP\int d\Omega_P\int d\mathbf{P}_\nu\tfrac{1}{4}\sum_{S_f,S_\nu,S_1,S_2}|M|^2 \times\delta(W+E_\nu-\Delta E-E_1-E_2), \quad (21)$$

where the spin states of the two emitted particles have been summed over and the spin states of the two initial K electrons have been averaged over. A summation over all final nuclear spin states and an average over the initial nuclear spin states are also implied. $\mathbf{P}$ and $\mathbf{P}_\nu$ represent the momenta of the ejected electron and the neutrino, respectively. E_ν is the energy of the neutrino and ΔE is the energy released in the nuclear transformation. Anticipating the results of Sec. IV which show that $\sum|M|^2$ is independent of $\mathbf{P}_\nu$ and thus depends only on $|\mathbf{P}|$, we can easily perform the integration indicated in (21). The result is

$$dw=8\pi^3(\Delta E+E_1+E_2-W)^2P^2dP\sum|M|^2. \quad (22)$$

III. RELATIONSHIP TO PRIMAKOFF-PORTER THEORY

Primakoff and Porter treat K-electron ejection during K capture by means of first-order time-dependent perturbation theory. Since the beta interaction is assumed to take place instantaneously, the method is in fact a sudden perturbation approximation. These authors work in a representation in which the interaction Hamiltonian consists solely of the beta interaction; all correlation effects between the two initial K electrons are included in the initial unperturbed wave function of the system. In this representation, the matrix element for K-electron ejection during allowed K capture is

$$M=G[1-P_{12}]\int d\mathbf{r}\,\phi_f^{(Z-1)\dagger}(\mathbf{r})\bar{\phi}^\nu(0)\mathcal{O}\phi_{1,2}^{(Z)}(0,\mathbf{r}), \quad (23)$$

where $\phi_{1,2}(\mathbf{r}_1,\mathbf{r}_2)$ represents the space and spin dependence of the initial two-electron state including correlation effects between the two K electrons. To represent this two-electron state Primakoff and Porter construct an approximate wave function of the form

$$\phi_{1,2}(\mathbf{r}_1,\mathbf{r}_2)=N\phi_1(\mathbf{r}_1)\phi_2(\mathbf{r}_2)e^{\alpha_1(r_1+r_2)}e^{\alpha_2 r_{12}}, \quad (24)$$

in which ϕ_1 and ϕ_2 are ground-state hydrogenic wave functions for a nuclear change Ze and N is the appropriate normalization factor. The factor $e^{\alpha_2 r_{12}}$ accounts for the correlation due to the electron-electron interaction and the factor $e^{\alpha_1(r_1+r_2)}$ accounts for the partial screening of the nuclear Coulomb interaction. The values of the parameters α_1 and α_2 are chosen such as to make this wave function a reasonably good fit to Hylleraas's variational nonrelativistic wave function for the problem of two electrons in the ground state about a nucleus of charge Ze.[20]

Using the wave function (24) to represent the two initial electrons, the appropriate Coulomb eigenfunction for the ejected electron and a Dirac plane wave for the neutrino, these authors calculate M by means of (23). Here we simply wish to quote their final result for the relative probability per K-capture event for the ejection of the uncaptured K electron with a momentum in the range dP,

$$\frac{dw_r}{dP}=\frac{16\alpha^2a^4P\exp[-(4a/P)\tan^{-1}(P/a)]}{(a^2+P^2)^4(1-e^{-2\pi a/P})} \times\left[1-\frac{P^2}{2(\Delta E+1)}\right]^2, \quad (25)$$

where $a=Z\alpha$.

The only difference between the theory of Primakoff and Porter and the theory developed in the present paper is in the method of construction of the initial two-electron wave function. Whereas Primakoff and Porter use a variational method, we calculate the wave function from perturbation theory. In particular, we treat the electron-electron interaction as a perturbation on the nuclear Coulomb interaction of the electrons. Since the exact wave function $\phi_{1,2}(\mathbf{r}_1,\mathbf{r}_2)$ satisfies

$$[H_c(\mathbf{r}_1)+H_c(\mathbf{r}_2)+e^2/r_{12}]\phi_{1,2}(\mathbf{r}_1,\mathbf{r}_2)=E_{1,2}\phi_{1,2}(\mathbf{r}_1,\mathbf{r}_2), \quad (26)$$

the unperturbed wave function is simply the product of two hydrogenic wave functions. A standard perturbation calculation then leads to the following expansion for the wave function,

$$\phi_{1,2}(\mathbf{r}_1,\mathbf{r}_2)=\phi_1(r_1)\phi_2(r_2)+\alpha\sum_{l,n}{}'\frac{\phi_l(\mathbf{r}_1)\phi_n(\mathbf{r}_2)}{(E_1+E_2-E_l-E_n)} \times\int d\mathbf{r}_1'\int d\mathbf{r}_2'\frac{1}{r_{12}'}\phi_l^\dagger(\mathbf{r}_1')\phi_1(r_1') \times\phi_n^\dagger(\mathbf{r}_2')\phi_2(r_2')+O(\alpha^2). \quad (27)$$

The prime on the summation implies that the term for which the energy denominator vanishes is omitted from the sum. Let us now insert (27), rather than (24), into (23). An integration over $\mathbf{r}$ and the introduction of the Green's function by means of the eigenfunction expansion then leads to the result

$$M=G[1-P_{12}]\bar{\phi}^\nu(0)\mathcal{O}\phi_1^{(Z)}(0)\int d\mathbf{r}\phi_f^{(Z-1)\dagger}(\mathbf{r})\phi_2^{(Z)}(r) +G\alpha[1-P_{12}]\bar{\phi}^\nu(0)\mathcal{O}\int d\mathbf{r}_1\int d\mathbf{r}_2\frac{1}{r_{12}} \times G_E(0,r_1)\phi_1(r_1)\phi_f^\dagger(\mathbf{r}_2)\phi_2(r_2). \quad (28)$$

[20] See H. A. Bethe and E. E. Salpeter, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 35, p. 232.

With the introduction of (5) and (13) into the first term of (28), it is then easily seen that (28) becomes identical with (17).

The advantages of the perturbation method are well known. Generally it provides us with more accurate wave functions than does the variational method. Most important, however, it allows us to make a reliable estimate of our error; this, of course, is not possible with the variational method.

IV. DETAILED CALCULATIONS AND FINAL RESULTS

In this section we wish to evaluate M in detail and obtain a final formula for the relative differential transition rate. We begin by introducing the appropriate representations for the wave functions and Green's function appearing in the matrix element. For the initial and final electron states these are the familiar Coulomb eigenfunctions,

$$\phi_i(r)=\left(\frac{a^3}{\pi}\right)^{1/2}e^{-ar}\chi_i, \tag{29a}$$

$$\phi_f(\mathbf{r})=\frac{1}{(2\pi)^{3/2}}e^{\pi a/2P}\Gamma(1+\frac{ia}{P})e^{i\mathbf{P}\cdot\mathbf{r}} \times F(-ia/P,\,1,\,-i\mathbf{P}\cdot\mathbf{r}-iPr)\chi_f, \tag{29b}$$

in which the Pauli spinor χ determines the spin dependence of the state and $F(c,d,z)$ is the confluent hypergeometric function.

In considering the form of the Green's function, we observe that since we are working in the nonrelativistic limit ($P\ll 1$), the condition $|E|<1$ is always satisfied. In this case the Green's function does not represent a freely propagating wave but rather is a function which decreases very rapidly away from the nucleus. The Green's function with these particular properties has already been studied by Glauber and Martin[21] in connection with the problem of radiative K capture. These authors have shown that this Green's function is simply related to the Whittaker function, $\mathcal{W}_{\eta,1/2}(2\mu r)$, and, in the nonrelativistic limit, is given by

$$G_E(0,r)=\frac{-\Gamma(1-\eta)}{2\pi}\frac{\mathcal{W}_{\eta,1/2}(2\mu r)}{r}, \tag{30}$$

with $\mu=[2(1-E)]^{1/2}$ and $\eta=a/\mu$. From the defining equation for E we see that $|E|\leq 1-a^2$ and consequently $0<\eta\leq 1/\sqrt{2}$. In this case we can make use of a real integral representation for the Whittaker function, which is valid for $0<\eta<1$, to rewrite the Green's function as

$$G_E(0,r)=-\frac{\mu}{\pi}e^{-\mu r}\int_0^\infty e^{-2\mu rs}\left(\frac{1+s}{s}\right)^\eta ds. \tag{31}$$

[21] Glauber and Martin, Ref. 1, especially Sec. 6.

This representation proves to be a convenient one for carrying out the spatial integration.

We begin our evaluation of M by performing the two spatial integrations appearing in (20). This involves the evaluation of the following four integrals:

$$I_1=\int\frac{d\mathbf{r}}{r}\phi_f^\dagger(\mathbf{r})\phi_2(r)e^{-ikr}, \tag{32a}$$

$$I_2=\int d\mathbf{r}\,\phi_f^\dagger(\mathbf{r})\phi_2(r)e^{-ikr}=i\frac{\partial I_1}{\partial k}, \tag{32b}$$

$$I_3=\int\frac{d\mathbf{r}}{r}G_E(0,r)\phi_1(r)e^{ikr}, \tag{32c}$$

$$I_4=\int d\mathbf{r}G_E(0,r)\phi_1(r)e^{ikr}=-i\frac{\partial I_3}{\partial k}. \tag{32d}$$

To evaluate I_1, we introduce the representations (29) for the wave functions along with the well-known integral representation for the confluent hypergeometric function,

$$F(ib,1,z)=\frac{1}{2\pi i}\oint_C dt\,t^{ib-1}(t-1)^{-ib}e^{tz}, \tag{33}$$

in which the path of integration encircles the branch cut extending from 0 to 1 along the real axis in the positive sense. The spatial integration in I_1 may then be carried out by elementary methods. The result is

$$I_1=\frac{e^{\pi a/2P}\Gamma(1-ia/P)}{(2\pi)^{3/2}2iP^2}\left(\frac{a^3}{\pi}\right)^{\frac{1}{2}}\chi_f^\dagger\chi_2 \times\oint_C\frac{dt\,t^{ia/P-1}(t-1)^{-ia/P-1}}{(t-t_0)}, \tag{34}$$

with $t_0=(k+P-ia)/2P$. In order to ensure convergence of the radial integral, it is necessary that the contour C satisfy the requirement, $\mathrm{Im}t>-a/2P$. To evaluate I_3, we introduce the representations (29a) and (31) into (33c). This leads, after completing the spatial integration, to the result

$$I_3=-4\mu\left(\frac{a^3}{\pi}\right)^{1/2}\chi_1\int_0^\infty ds\left(\frac{1+s}{s}\right)^\eta\frac{1}{(a+\mu+2\mu s-ik)^2}. \tag{35}$$

By inserting (32b), (32d), (34), and (35) into (20), we then obtain the following expression for the matrix element:

$$M=\frac{-\alpha G\mu a^3e^{\pi a/2P}}{2^{3/2}P^2\pi^{7/2}}\Gamma(1-\frac{ia}{P})[1-P_{12}] \times\bar{\phi}^\nu(0)\mathcal{B}\chi_1\chi_f^\dagger\chi_2 I_a, \tag{36}$$

where

$$I_a=\frac{1}{2iP}\int_{-\infty}^{\infty}\frac{dk}{(k-i\epsilon)}\oint_C\frac{dt\,t^{ia/P-1}(t-1)^{-ia/P-1}}{(t-t_0)}\times\int_0^{\infty}ds\left(\frac{1+s}{s}\right)^{\eta}\frac{1}{(a+\mu+2\mu s-ik)^2}\times\left[\frac{4iP}{(a+\mu+2\mu s-ik)}+\frac{1}{(t-t_0)}\right]. \quad (37)$$

The squared absolute value of the matrix element must now be summed over the spin states of the two emitted leptons as well as over the spin states of the two initial K electrons. These calculations are carried out in the usual way by introducing the positive-energy projection operators $(1+\gamma_4)/2$ and $(\boldsymbol{\alpha}\cdot\mathbf{P}_\nu+E_\nu)/2E_\nu$ for the electrons and neutrino, respectively, and extending the summations over all states. The result of the calculation is

$$\sum_{s_f,s_\nu,s_1,s_2}|M|^2=\frac{\alpha^2G^2\mu^2a^7}{8P^5\pi^9}\frac{(\mathbf{B}\cdot\mathbf{B}^*+B_4B_4^*)}{(1-e^{-2\pi a/P})}|I_a|^2. \quad (38)$$

An inspection of (37) and (38) confirms our earlier statement that $\sum|M|^2$ depends only on P.

In order to obtain the relative rate for K-electron ejection, we also need to know the transition rate for ordinary allowed K capture. This is easily computed by the same method with the result,

$$w_K=\frac{G^2a^3}{\pi^2}(\Delta E+E_1)^2(\mathbf{B}\cdot\mathbf{B}^*+B_4B_4^*). \quad (39)$$

Combining (22), (38), and (39) and neglecting terms of order $(Z\alpha)^2$, we then obtain for the relative differential transition rate per K-capture event,

$$dw_r\equiv\frac{dw}{w_K}=\frac{\alpha^2a^4(P^2+2a^2)}{P^3\pi^4(1-e^{-2\pi a/P})}\times\left[1-\frac{P^2}{2(\Delta E+1)}\right]^2|I_a|^2dP. \quad (40)$$

To complete the analysis, we must still evaluate the integral I_a. This is done in Appendix B and leads to the final results given by (B14), (B15), and (B16). With the aid of these results we then obtain for the differential transition rate per K-capture event the formula,

$$\frac{dw_r}{dP}=\frac{64\alpha^2a^4Pe^{-2\pi a/P}\exp\{(4a/P)\tan^{-1}[(2a+\mu)/P]\}}{(\mu+a)^4[(2a+\mu)^2+P^2]^2(1-e^{-2\pi a/P})}\times\left[1-\frac{P^2}{2(\Delta E+1)}\right]^2I^2, \quad (41)$$

where I is defined by (B13) and (B14).

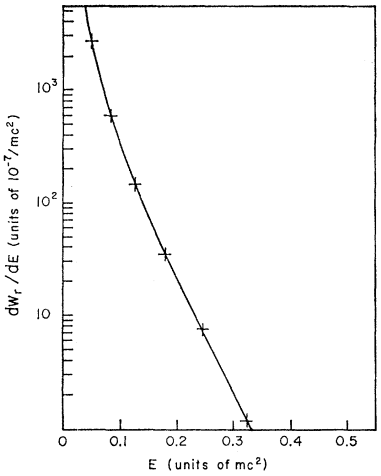

FIG. 2. Theoretical energy spectrum of K electrons ejected during K capture by Fe^{55}. The solid curve represents the result of the present calculation. The crossed points were computed from the Primakoff-Porter theory. The theoretical end point is at $E=0.425$.

As pointed out in Appendix B, the series representation for I is expected to converge quite rapidly. The error introduced by terminating this series after a finite number of terms is of the order of the next successive term in the expansion. In particular, detailed numerical evaluation for the case $Z=26$ shows that retaining only the first two terms of the expansion leads to an error of order $Z\alpha$, whereas retaining the first three terms reduces the error to order $(Z\alpha)^2$.

The theoretical forms of the ejected K-electron spectrum are plotted in Fig. 2 for Fe^{55}, which has a maximum ejected electron kinetic energy of 218 keV. The solid curve and crossed points represent plots, converted to the energy scale, of (41) and (25), respectively.

V. DISCUSSION OF RESULTS AND CONCLUSIONS

It is clear from Fig. 2 that the theory developed in the previous sections predicts an ejected-electron spectrum which is virtually identical with the Primakoff-Porter spectrum for almost the entire energy range of the ejected electrons. This is rather surprising in view of Primakoff and Porter's simplified description of the initial two-electron state. In Sec. III it was shown that, for the nonrelativistic two-electron problem, the perturbation treatment neglects terms of order Z^{-1}. That Primakoff and Porter's variational wave function should yield a result which is this accurate is more than one would normally expect.

The most recent and extensive experimental study on the spectral distribution of the ejected electrons has been that of Pengra and Crasemann[13] on Fe^{55}. These measurements covered the range of ejected-electron energies from 30 to 190 keV. and showed serious disagreements with the predictions of the Primakoff-Porter theory for energies below about 60 keV. The only comparable previous experiment, that of Daniel, Schupp, and Jensen,[10] also showed large deviations from the theoretical curve at low energies. Several experimenters have also measured the total relative K-electron

ejection rate obtaining approximate agreement with the Primakoff-Porter theory. However, such measurements do not provide a sensitive test of the theory. A summary of all experimental work done prior to that of Pengra and Crasemann, including their own observations, may be found in the paper of Lark and Perlman.[11]

It has been frequently suggested that the apparent discrepancy between the Primakoff-Porter theory and the experimental results is due to the inadequate treatment of screening and correlation effects in the description of the initial two-electron state. On the contrary, the present calculation has shown that, in the nonrelativistic limit such corrections are only of the order Z^{-1}, which amounts to about 4% for Fe^{55}. The greatest source of error in both the Primakoff-Porter theory and the present one is the neglect of relativistic effects. Even at very low ejection energies there is a relative error of order $Z\alpha$ due to the neglect of relativistic effects in the initial and intermediate electron states. For Fe^{55} this error is about 20%. As has been shown by Levinger,[22] nuclear recoil effects are quite negligible in β-decay type processes, contributing corrections of much less than 1%. One final source of error which should be considered results from the modification of the ejected-electron wave function as a result of screening by electrons in shells above the K shell. A rough estimate of this error may be made by taking, as the correction factor to the transition rate, the ratio

$$F_s=|\phi_s(0)|^2/|\phi_c(0)|^2,$$

in which $\phi_s(\mathbf{r})$ and $\phi_c(\mathbf{r})$ are the screened and unscreened one-electron wave functions for the ejected electron, respectively. An approximate analytic expression for $\phi_s(\mathbf{r})$, valid for $r\ll 1/a$, has been derived by Good[23] for a Thomas-Fermi potential. With the aid of Good's results, the screening correction factor may be written as

$$F_s=\frac{(W-\Delta)}{W}\frac{(1-e^{-2\pi a/P})}{(1-e^{-2\pi a/\bar{P}})},$$

in which $\Delta=1.795\alpha^2Z^{1/3}$ and $\bar{P}=[(W-\Delta)^2-1]^{1/2}$. Numerical evaluation of F_s for $Z=26$ shows that the screening correction factor is never more than about 5%.

From the above discussion, it seems clear that the predictions of the theory presented here, as well as those of the Primakoff-Porter theory, are correct to within a relative error of order $Z\alpha$. However, such an error is far too small to account for the existing differences between theory and experiment, where the error is of the order of hundreds of percent (at low energies). The present authors know of no other factors which might seriously influence the theoretical results. It may therefore be necessary to re-examine the experimental situations to be sure that all possible processes leading to electron production have been properly accounted for.

[22] J. S. Levinger, Phys. Rev. **90**, 11 (1953).
[23] R. H. Good, Phys. Rev. **94**, 931 (1954).

APPENDIX A

In a well-known paper[19] on quantum electrodynamics, Feynman has given a convenient method for separating retardation effects from the instantaneous Coulomb interaction when applying perturbation theory to two interacting electrons whose unperturbed states are momentum eigenstates. For our purposes we wish to generalize the result to include Coulomb field eigenstates as the unperturbed states.

In configuration space the lowest-order two-electron interaction has the form

$$\int\frac{d^4k}{k^2}\int d^4x_1\bar{\phi}_b(x_1)\gamma_\mu\phi_a(x_1)e^{ik\cdot x_1}\cdots \times\int d^4x_2\bar{\phi}_d(x_2)\gamma_\mu\phi_c(x_2)e^{-ik\cdot x_2},$$

where the ϕ's represent the stationary states of a one-electron Dirac Hamiltonian containing an arbitrary external field. We expand the stationary states in momentum eigenstates

$$\phi_i(x)=\int\mathbf{dP}\,a_i(\mathbf{P})e^{ip\cdot x}U(\mathbf{P}),$$

where $U(\mathbf{P})$ is the usual free-particle Dirac spinor. It is now a simple matter to prove that the matrix elements of $\mathcal{K}=\gamma_\mu k_\mu$ analogous to those of the γ_μ's in the interaction term vanish. For example, using the expansion in momentum eigenstates and the fact that the Dirac spinors satisfy $\mathcal{P}U(\mathbf{P})=iU(\mathbf{P})$ and $\bar{U}(\mathbf{P})\mathcal{P}=i\bar{U}(\mathbf{P})$, where $\mathcal{P}=\gamma_\mu P_\mu$, one easily proves that

$$\int d^4x_1\bar{\phi}_b(x_1)\mathcal{K}\phi_a(x_1)e^{ik\cdot x_1}=0.$$

Thus, for matrix elements of the type appearing in the interaction term, the operator equation $\mathcal{K}=0$ is satisfied. It now follows immediately by Feynman's original argument that for such matrix elements the operators may be separated into the form (10).

APPENDIX B

Here we complete the evaluation of the integral I_a defined by (37). The calculation is most conveniently carried out by performing the integration over k first. For this purpose, we rewrite (37) in the form

$$I_a=2iP\oint_C dt\,t^{ia/P-1}(t-1)^{-ia/P-1}\int_0^\infty ds\left(\frac{1+s}{s}\right)^\eta \times\int_{-\infty}^{\infty}\frac{dk}{(k-i\epsilon)(k-k_0)(k-k_1)^2} \times\left[\frac{2}{(k-k_1)}+\frac{1}{(k-k_0)}\right],\quad\text{(B1)}$$

with $k_0=2Pt-P+ia$ and $k_1=-i(a+\mu+2\mu s)$. For large values of k, the integrand is proportional to k^{-5}. Thus, if we go into the complex k plane, we can close the contour of integration with a semicircular contour at infinity without affecting the value of the integral. Accordingly, we close the contour in the upper half-plane and evaluate the integral by the method of residues with the result that

$$I_a=-4\pi P\oint_C dt\, t^{ia/P-1}(t-1)^{-ia/P-1}\int_0^\infty ds\left(\frac{1+s}{s}\right)^\eta \times\left[\frac{2}{k_0k_1^3}+\frac{1}{k_0^2k_1^2}-\frac{1}{k_0^2(k_0-k_1)^2}\right]. \quad \text{(B2)}$$

In order to proceed with the t integration, we make use of the definitions of k_0 and k_1 to rewrite I_a as

$$I_a=\frac{\pi}{P}(J_1+J_2+J_3), \quad \text{(B3)}$$

in which

$$J_1=4iP\oint_C \frac{dt\, t^{ia/P-1}(t-1)^{-ia/P-1}}{(t-t_0)}\int_0^\infty ds\left(\frac{1+s}{s}\right)^\eta \times\frac{1}{(a+\mu+2\mu s)^3}, \quad \text{(B4a)}$$

$$J_2=\oint_C \frac{dt\, t^{ia/P-1}(t-1)^{-ia/P-1}}{(t-t_0)^2}\int_0^\infty ds\left(\frac{1+s}{s}\right)^\eta \times\frac{1}{(a+\mu+2\mu s)^2}, \quad \text{(B4b)}$$

$$J_3=\oint_C dt\, t^{ia/P-1}(t-1)^{-ia/P-1}\int_0^\infty ds\left(\frac{1+s}{s}\right)^\eta \times\frac{1}{(2Pt+2ia+i\mu+2i\mu s-P)^2}. \quad \text{(B4c)}$$

To perform the t integration we observe that, since all three integrals vanish at least as rapidly as t^{-3} as t approaches infinity, the original contour C can be replaced by the contour C' consisting of the original contour C plus a circular contour at infinity taken in the clockwise direction. The integration over C' can then be carried out using residue theory. For the integrals J_1 and J_2, the only residue comes from the pole at t_0 and its contribution leads to the results,

$$J_1=\frac{-32\pi P^3}{(P^2+a^2)}e^{-\pi a/P}\exp\left[\left(\frac{2a}{P}\right)\tan^{-1}\left(\frac{a}{P}\right)\right]\times\int_0^\infty ds\left(\frac{1+s}{s}\right)^\eta\frac{1}{(a+\mu+2\mu s)^3}, \quad \text{(B5a)}$$

$$J_2=0. \quad \text{(B5b)}$$

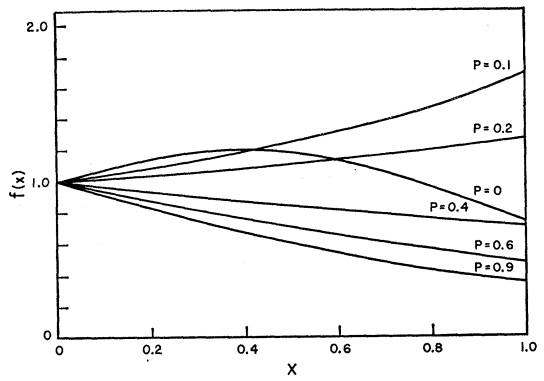

FIG. 3. A plot of the function $f(x)$ defined by (B14), for several values of the momentum (in mc units) with $Z=26$.

To complete the evaluation of J_1, it is advantageous to change the integration variable to $x=s/(1+s)$ and introduce $\lambda=(\mu-a)/(\mu+a)$ in which case J_1 becomes

$$J_1=\frac{-32\pi P^3e^{-\pi a/P}}{(P^2+a^2)(\mu+a)^3}\exp\left[\left(\frac{2a}{P}\right)\tan^{-1}\left(\frac{a}{P}\right)\right]\times\int_0^1\frac{dx\,x^{-\eta}(1-x)}{(1+\lambda x)^3}. \quad \text{(B6)}$$

Glauber and Martin[24] have shown how to evaluate integrals of this type. The integrand may be expanded in partial fractions and each of the resulting integrals expressed in terms of the function,

$$K(\lambda)=\lambda\int_0^1\frac{dx\,x^{-\eta}}{(1+\lambda x)}, \quad \text{(B7)}$$

and its derivatives by means of the identity,

$$\int_0^1\frac{dx\,x^{-\eta}}{(1+\lambda x)^{n+1}}=\frac{1}{n!\lambda^{n+1}}\left(\lambda^2\frac{d}{d\lambda}\right)^nK(\lambda). \quad \text{(B8)}$$

In this way, after algebraic reduction, we obtain

$$J_1=\frac{-16\pi P^3e^{-\pi a/P}}{(P^2+a^2)(\mu+a)^3}\exp\left[\left(\frac{2a}{P}\right)\tan^{-1}\left(\frac{a}{P}\right)\right]\times\left[\frac{1}{(1+\lambda)}+\frac{\eta}{\lambda}\right]. \quad \text{(B9)}$$

To evaluate J_3 we again change the integration variable to $x=s/(1+s)$ and introduce the definitions,

$$t_1=[P-i(2a+\mu)]/2P;\quad t_2=[P-i(2a-\mu)]/2P;\quad \rho=(t_2-t)/(t-t_1),$$

in which case J_3 can be written as

$$J_3=\frac{1}{4P^2}\oint_C\frac{dt\,t^{ia/P-1}(t-1)^{-ia/P-1}}{(t-t_0)^2(t-t_1)^2}\int_0^1\frac{dx\,x^{-\eta}}{(1+\rho x)^2}. \quad \text{(B10)}$$

[24] Glauber and Martin, Ref. 1, Sec. 8.

With the aid of (B7) and (B8) this expression can be reduced to

$$J_3=\frac{1}{4P^2(t_2-t_1)}\oint_C \frac{dt\, t^{ia/P-1}(t-1)^{-ia/P-1}}{(t-t_0)^2(t-t_1)}+\frac{\eta}{4P^2}\int_0^1 \frac{dx\, x^{-\eta}}{(1-x)}\oint_C \frac{dt\, t^{ia/P-1}(t-1)^{-ia/P-1}}{(t-t_0)^2(t-t_1)(t-t_3)}, \tag{B11}$$

where we have introduced $t_3=(t_1-t_2x)/(1-x)$. The t integration can now be carried out as before. The result, after algebraic reduction with (B7) and (B8) and the use of (B5), is

$$J_3=-J_1\frac{-8\pi P^3 e^{-\pi a/P}\exp\{(2a/P)\tan^{-1}[(2a+\mu)/P]\}I}{\mu(\mu+a)^2[(2a+\mu)^2+P^2]}, \tag{B12}$$

where

$$I=1+\eta\int_0^1 dx\, x^{-\eta-1}[1-(1-x)^4 f(x)], \tag{B13}$$

$$f(x)=\frac{\exp\{-(2a/P)\tan^{-1}[(2a+\mu)/P]\}\exp\{(2a/P)\tan^{-1}[(2a/P)+\mu(1+x)/P(1-x)]\}}{(1+\lambda x)^2(1+\sigma x)(1+\sigma^* x)}, \tag{B14}$$

and $\sigma=(\mu-2a-iP)/(\mu+2a+iP)$. Combining (B4), (B5b), and (B12), we obtain for I_a the formula

$$I_a=\frac{-8\pi^2 P^2 e^{-\pi a/P}\exp\{(2a/P)\tan^{-1}[(2a+\mu)/P]\}I}{\mu(\mu+a)^2[(2a+\mu)^2+P^2]}. \tag{B15}$$

The integral I defined by (B13) and (B14) cannot be evaluated in closed form. A rapidly converging series expansion is however obtainable. The appropriate expansion becomes apparent when we consider a plot of $f(x)$ for typical values of a and P. Such a plot is shown in Fig. 3. It indicates that $f(x)$ is a slowly varying function over the interval $0\leq x\leq 1$. An examination of (B13) reveals that most of the contribution to the integral comes from the region in the neighborhood of $x=0$. This suggests that, if $f(x)$ is expanded in a Maclaurin series, then the resulting series for I will converge quite rapidly. Performing such an expansion in (B13) then leads to the following series for I:

$$I=\eta\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left[-\frac{1}{(n-\eta)}+\frac{4}{(n+1-\eta)}-\frac{6}{(n+2-\eta)}+\frac{4}{(n+3-\eta)}-\frac{1}{(n+4-\eta)}\right]. \tag{B16}$$