Ultrasonic Attenuation in Dirty Superconductors*

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The attenuation coefficients of acoustic waves have been calculated for small-gap superconductors for the case where the electron mean free path l is much smaller than the pure superconductor coherence length ξ_0. The emphasis in this paper is placed on a study of longitudinal acoustic waves. It is shown that attenuation in this case arises from both electromagnetic absorption and the collision drag effect. In the transverse case the electromagnetic absorption is negligible in the low-frequency limit $|\Delta|/\omega \gg 1$. The results of the calculation have been applied to the Abrikosov mixed state and the superconducting surface sheath. Measurement of the attenuation in these regimes will determine the upper critical fields H_{c2} and H_{c3} as well as the averaged amplitude of the energy gap.

I. INTRODUCTION

THERE has been considerable interest recently in the possible use of ultrasound as a tool to investigate the nature of type-II superconductivity. In contrast to ordinary electromagnetic waves, which are damped out in the skin depth region, a sound wave penetrates into the bulk of a metallic sample. Thus, in principle the study of ultrasonic absorption should yield detailed information on the structure of the mixed state.

The problem of ultrasound absorption has been studied theoretically by several authors. Caroli and Matricon[1] treated the absorption due to a single vortex line in the field region $H_{c1}<H\ll H_{c2}$. Cooper, Lee, and one of the authors[2] considered the absorption of longitudinal waves in a model impurity-free type-II superconductor for $T\sim T_c$ and H near H_{c1}. They suggested the possibility of resonance absorption when the sound wavelength matches the spacing between the flux lines of the Abrikosov[3] mixed state. More recently, the attenuation of transverse waves in the dirty limit, that is, $l/\xi_0\ll 1$, where l is the electronic mean free path and ξ_0 is the coherence distance in a pure superconductor, has been discussed by Maki.[4] He concludes that measurement of the attenuation serves to predict the averaged amplitude of the energy-gap parameter as well as to determine the upper critical fields, H_{c2} for the mixed state and H_{c3} for the superconducting surface sheath.

In order to discuss the feasibility of experimental detection of the structure in the attenuation discussed by Cooper *et al.*,[2] it is important that a self-consistent calculation of the ultrasonic attenuation be carried out which includes the effects of impurities. In this paper we calculate the longitudinal attenuation coefficient in the dirty limit; a self-consistent generalization of the model calculation applicable for all l will be presented elsewhere. We have also calculated the transverse attenuation coefficient in the dirty limit and have obtained results which agree with those obtained by Maki.[4] We present these results here both for completeness and to clarify a point of formalism which occurs in the calculation of transverse transport coefficients in the presence of a self-consistent electromagnetic field.

The assumption that $l/\xi_0\ll 1$ allows considerable simplification in the calculation of the properties of the superconductor. First, the important electronic distance parameter is now l rather than ξ_0. Since the energy gap varies over distances of at least $(l\xi_0)^{1/2}$, it may be treated as essentially constant in integrals involving electronic Green's functions. Second, for magnetic fields of interest, $\omega_c\tau\ll 1$, and therefore the effects of the magnetic field on electronic Green's functions may be neglected. It is important, however, to consider the effect of the field in causing a spatial variation in the energy gap. Finally, as $l\ll\xi_0$ we have $T\tau\ll 1$, where in the superconducting state T is a temperature less than the transition temperature T_c and τ in the electron collision time.

In the next section we give a brief review of the general formalism of ultrasonic attenuation as developed by Tsuneto,[5] Kadanoff and Falko,[6] and Maki.[4] In Sec. III we indicate how this formalism may be applied to small-gap type-II superconductors. We expand the correlation functions and attenuation coefficients to second order in the energy gap, $\Delta(\mathbf{r})$, and include the effects of impurity scattering. In the next section we calculate the necessary correlation functions in the dirty limit using the technique of thermal Green's functions. Finally, in Secs. V and VI we give the longitudinal and transverse attenuation coefficients in this limit. These results are applied to the Abrikosov mixed state and the superconducting surface sheath.

II. REVIEW OF FORMALISM

As is well known, the attenuation coefficient is defined as

$$\alpha=Q/\tfrac{1}{2}\rho_{\rm ion}\,|u|^2\,v_s, \tag{1}$$

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[1] C. Caroli and J. Matricon, Physik Kondensielten Materie **3**, 380 (1965).

[2] L. N. Cooper, A. Houghton, and H. J. Lee, Phys. Rev. **148**, 198 (1966).

[3] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].

[4] K. Maki, Phys. Rev. **148**, 370 (1966).

[5] T. Tsuneto, Phys. Rev. **121**, 402 (1961).

[6] L. P. Kadanoff and I. I. Falko, Phys. Rev. **136**, A1170 (1964).

where $\rho_{\rm ion}$ is the ionic mass density, v_s the sound velocity, and $u(\mathbf{r}, t)$ is the velocity field of the ions. Q is the power dissipated by the sound wave per unit volume and is given by

$$Q=-\tfrac{1}{2}\,\mathrm{Re}[\mathbf{F}\cdot u^*], \tag{2}$$

where $\mathbf{F}(\mathbf{r}, t)$ is the force per unit volume on the ions

$$F_i(\mathbf{r}, t)=-i\omega\langle h_i(\mathbf{r}, t)\rangle. \tag{3}$$

The brackets $\langle\ \rangle$ denote a statistical average and also an average over the random placements of any impurities in the system. The operator $h_i(\mathbf{r}, t)$ is defined as

$$h_i(\mathbf{r}, t)=(q_j/\omega)\tau_{ij}(\mathbf{r}, t)-mj_i(\mathbf{r}, t), \tag{4}$$

where the current operator $j_i(\mathbf{r}, t)$ and the electronic stress field operator $\tau_{ij}(\mathbf{r}, t)$ are given by

$$j_i(\mathbf{r}, t)=\sum_\alpha[(\boldsymbol{\nabla}-\boldsymbol{\nabla}')_i/2im]n_\alpha(\mathbf{r}, t), \tag{5}$$

and

$$\tau_{ij}(\mathbf{r}, t)=\sum_\alpha[(\boldsymbol{\nabla}-\boldsymbol{\nabla}')_i/2im][(\boldsymbol{\nabla}-\boldsymbol{\nabla}')_j/2i]n_\alpha(\mathbf{r}, t), \tag{6}$$

where the density operator $n_\alpha(\mathbf{r}, t)$ is defined in terms of the Heisenberg field operators in the usual way:

$$n_\alpha(\mathbf{r}, t)=[\psi_\alpha^\dagger(\mathbf{r}', t)\psi_\alpha(\mathbf{r}, t)]_{\mathbf{r}'\to\mathbf{r}}. \tag{7}$$

Following Tsuneto,[5] the force $\mathbf{F}(\mathbf{r}, t)$ is calculated most conveniently in the reference frame fixed to the ions. The transition to the moving frame introduces an effective interaction between the impressed sound wave and the conduction electrons described by the Hamiltonian

$$H_I=-i\omega\int d^3r\,\phi_i(\mathbf{r}, t)h_i(\mathbf{r}, t). \tag{8}$$

In this expression the ion displacement field $\boldsymbol{\phi}(\mathbf{r}, t)$ is given by

$$\boldsymbol{\phi}(\mathbf{r}, t)=\boldsymbol{\phi}(\mathbf{q}, \omega)\,\exp[i(\mathbf{q}\cdot\mathbf{r}-\omega t)], \tag{9}$$

where ω is the frequency and $\mathbf{q}$ the wave vector of the sound wave which we take from now on to be in the z direction. The attenuation coefficients can now be calculated to first order in the ion displacement field and are given by

$$\alpha_L=\mathrm{Re}(\omega^2/i\omega\rho_{\rm ion}v_s)\langle[h_I^L, h_I^L]\rangle(\mathbf{q}, \omega) \tag{10}$$

for the longitudinal wave, and

$$\alpha_T=\mathrm{Re}(\omega^2/i\omega\rho_{\rm ion}v_s)\langle[h_I^T, h_I^T]\rangle(\mathbf{q}, \omega) \tag{11}$$

for the transverse wave. Here h_I^L and h_I^T are

$$h_I^L=(q/\omega)\tau_{zz}(\mathbf{r}, t)-(m\omega/q)n(\mathbf{r}, t) \tag{12}$$

and

$$h_I^T=(q/\omega)\tau_{xz}(\mathbf{r}, t)-mj_x(\mathbf{r}, t), \tag{13}$$

and the retarded product is defined as usual by

$$\langle[A, B]\rangle(\mathbf{q}, \omega)=-i\int_{-\infty}^t dt'\int d^3r\,\exp\{-i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]\}\langle[A(\mathbf{r}, t), B(\mathbf{r}', t')]\rangle. \tag{14}$$

The above retarded products are true electronic correlation functions and include the effects of electron-electron Coulomb interactions and current-current interactions. These interactions are treated in the random phase approximation, which amounts to treating the long-range electromagnetic fields generated by the sound wave in a self-consistent manner. For a longitudinal wave, the retarded products may be written[6]

$$\langle[A, B]\rangle(\mathbf{q}, \omega)=\langle[A, B]\rangle_0(\mathbf{q}, \omega)+\frac{\langle[A, n]\rangle_0(\mathbf{q}, \omega)\,(4\pi e^2/q^2)\,\langle[n, B]\rangle_0(\mathbf{q}, \omega)}{1-(4\pi e^2/q^2)\,\langle[n, n]\rangle_0(\mathbf{q}, \omega)}, \tag{15}$$

where the subscript zero denotes the fictitious system without electron-electron interactions. When the frequency $\omega\ll 10^{14}$ we have almost complete screening, that is

$$(4\pi e^2/q^2)\,\langle[n, n]\rangle_0(\mathbf{q}, \omega)\gg 1, \tag{16}$$

and the longitudinal attenuation coefficient takes the form

$$\alpha_L=\mathrm{Re}(q^2/i\omega\rho_{\rm ion}v_s)\left[\langle[\tau_{zz}, \tau_{zz}]\rangle_0(\mathbf{q}, \omega)-\frac{[\langle[\tau_{zz}, n]\rangle_0(\mathbf{q}, \omega)]^2}{\langle[n, n]\rangle_0(\mathbf{q}, \omega)}\right]. \tag{17}$$

Similarly, when the current-current interaction is taken into account, the attenuation in the transverse case, in the low-frequency limit $\omega \ll 10^9$, is given by

$$\alpha_T = \mathrm{Re}(q^2/i\omega\rho_{\mathrm{ion}}v_s)\left[\langle[\tau_{xz},\tau_{xz}]\rangle_0(\mathbf{q},\omega) - \frac{[\langle[\tau_{xz},j_x]\rangle_0(\mathbf{q},\omega)]^2}{(N/m)+\langle[j_x,j_x]\rangle_0(\mathbf{q},\omega)}\right]. \quad (18)$$

This expression differs from that given in Ref. (4) by replacement of $\langle[j_x,j_x]\rangle_0(\mathbf{q},\omega)$ by the total linear current response function in the presence of a transverse electromagnetic field. A discussion of this point is given in the Appendix. It should be noted that in the presence of a dc magnetic field such that $\omega_c\tau \gtrsim 1$ this simple procedure for treating the screening in the transverse case is no longer valid for arbitrary direction of propagation of the sound wave.

III. APPLICATION TO TYPE-II SUPERCONDUCTORS

In this section we apply the general formalism developed in Sec. II to calculate the attenuation in a type-II superconductor. Using the definition of the retarded two-particle Green's function

$$G_{\alpha\beta}^{\mathrm{II}R}(\mathbf{r},\mathbf{r}';t-t') = -i\Theta(t-t')\langle[n_\alpha(\mathbf{r},t),n_\beta(\mathbf{r}',t)]\rangle, \quad (19)$$

we rewrite the correlation functions as

$$\langle[A,B]\rangle(\mathbf{q},\omega) = \int_{-\infty}^{\infty}dt'\int d^3r' \exp\{-i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]\} \times V_A(\mathbf{r})V_B(\mathbf{r}')\sum_{\alpha\beta}G_{\alpha\beta}^{\mathrm{II}R}(\mathbf{r},\mathbf{r}';t-t'), \quad (20)$$

where $V_A(\mathbf{r})$ and $V_B(\mathbf{r}')$ are the vertex functions associated with the operators $A(\mathbf{r},t)$ and $B(\mathbf{r}',t)$, respectively. In Eqs. (19) and (20), $\langle\ \rangle$ indicates an average in the fictitious system without Coulomb or current-current interactions; the subscript zero will be dropped from now on. Following the usual convention we go to the temperature representation to evaluate correlation functions of the form $\langle[A,B]\rangle(\mathbf{q},\omega)$ using the fact that

$$G^{\mathrm{II}R}(i\omega_0) = G^{\mathrm{II}}(\omega_0), \quad (21)$$

where $\omega_0 = 2m\pi T(m=0,1,2,\cdots)$, and G^{II} denotes the thermal two-particle Green's function. To obtain $G^{\mathrm{II}R}(\omega)$, we must analytically continue off the points $\omega = i\omega_0$ in such a way that the resulting function is analytic in the upper half ω plane.

The Gor'kov[7] factorization procedure is used to expand $G^{\mathrm{II}}(\mathbf{r},\mathbf{r}';\tau-\tau')$, and the resulting one-particle Green's functions are Fourier analyzed in $(\tau-\tau')$. It is assumed that the energy-gap parameter is small,

$$\Delta(\mathbf{r})/\pi T_c \ll 1;$$

consequently, the superconducting Green's functions may be expanded to second order in the energy gap by making use of the Gor'kov[7] equations. The second-order contribution to the correlation functions $\langle[A,B]\rangle(\mathbf{q},\omega_0)$ is depicted graphically in Fig. 1.

Using the condition that the static magnetic field varies slowly in distances of the order l, and is sufficiently weak that $\omega_c\tau \ll 1$, we neglect the magnetic field dependence of the normal-metal Green's functions. Of course, we must include the effect of the field on the energy gap, as the momentum associated with the electron pairs is small. It should be noted that the effects of the dc magnetic field should also have been included in writing down the force density on the ions; but under the conditions given above, this effect is negligible.

The influence of impurities is included under the assumptions of elastic and isotropic scattering (in the reference frame moving with the ions), and it is assumed that no magnetic impurities are present. The effect of impurities on the normal-metal Green's functions[8] is incorporated simply by making the replacement

$$\omega \to \tilde{\omega} = \omega + (1/2\tau)(\omega/|\omega|).$$

Following Maki,[9] the energy-gap function is modified

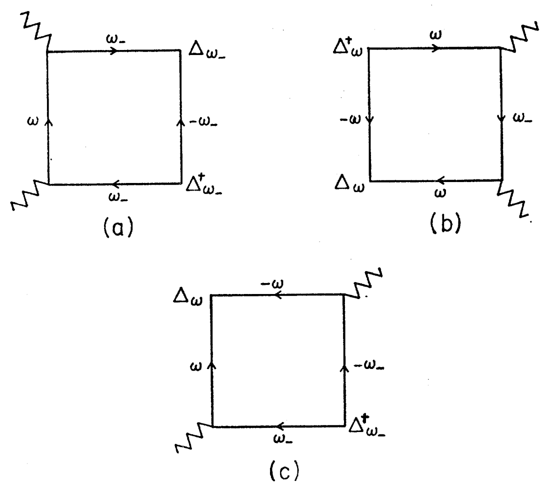

FIG. 1. The second-order diagrams in the expansion of the correlation functions in powers of Δ.

[7] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **34**, 735 (1958) [English transl.: Soviet Phys.—JETP **7**, 505 (1958)].

[8] See, for example, A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).

[9] K. Maki, Physics **1**, 21 (1964).

in the following manner:

$$\Delta(\mathbf{r})\rightarrow\tilde{\Delta}_\omega(\mathbf{r})=\eta_\omega\Delta(\mathbf{r}),\tag{22}$$

where

$$\eta_\omega=[1-(1/2\,|\,\tilde{\omega}\,|\,\tau)(1-2\alpha\tau)]^{-1}.\tag{23}$$

The Landau-Ginzburg eigenvalue $\alpha=\frac{1}{3}\tau v_F{}^2eH_0$, where H_0 is the external magnetic field. This renormalization includes all effects of the dc magnetic field to order l/ξ_0.

We now include impurity corrections to the electromagnetic vertices V_A. To do this we must sum the set of ladder diagrams bridging the single vertex, or what is equivalent, solve the integral equation[8]

$$V_\omega(\mathbf{p})=V_0(\mathbf{p})+\frac{n}{(2\pi)^3}\int d^3p'\,|\,u(\mathbf{p}-\mathbf{p}')\,|^2\,G_\omega{}^0(\mathbf{p}')G_{\omega_-}{}^0(\mathbf{p}'-\mathbf{q})\,V_\omega(\mathbf{p}'),\tag{24}$$

where $u(\mathbf{q})$ is the Fourier component of the scattering potential, n is the impurity density, and $G_\omega{}^0(\mathbf{p})$ is the normal-metal Green's function. Here $V_0(\mathbf{p})$ is the vertex function in the absence of impurities, $\omega=(2n+1)\pi T$, where n is an integer and $\omega_-=\omega-\omega_0$. Solving Eq. (24) under the assumptions listed above, we write the results in the form

$$V_A=V_A{}^0+V_A{}^c\Lambda_-,\tag{25}$$

where the impurity corrections $V_A{}^c$ for the longitudinal vertex functions $[V_n{}^0=1$ and $V_{\tau_{zz}}{}^0=(p_0{}^2/m)\cos^2\theta]$ are found to be

$$V_n{}^c=\tan^{-1}x/(x-\tan^{-1}x)\tag{26}$$

and

$$V_{\tau_{zz}}{}^c=(p_0{}^2/m)(1/x^2),\tag{27}$$

where $x=q1$ and

$$\Lambda\pm=\tfrac{1}{2}\left(1\pm\frac{\omega}{|\,\omega\,|}\frac{\omega_-}{|\,\omega_-\,|}\right).$$

For isotropic scattering, all vertex corrections to the transverse correlation functions vanish.

In addition to impurity renormalization of the single vertex functions, V_A and $\tilde{\Delta}_\omega$, we must also take into account the series of ladder diagrams which bridge two vertices. In the graphs of Fig. 1, this corresponds to impurity lines running both horizontally and vertically across the boxes. After solving an integral equation similar to (24), we find that we can take into account impurity lines bridging two superconducting vertices [which occur only in Figs. 1(a) and 1(b)] simply by replacing the electron Green's function joining the two vertices by

$$G_{-\omega}{}^0(\mathbf{p})\rightarrow G_{-\omega}{}^0(\mathbf{p})-(i/4\tau\tilde{\omega}^2)(\omega/|\,\omega\,|).\tag{28}$$

In a similar fashion we find that impurity lines bridging an electromagnetic vertex V_A and a superconducting vertex $\tilde{\Delta}_\omega$ can be accounted for with the replacement

$$V_AG_\omega{}^0(\mathbf{p}-\mathbf{q})\rightarrow V_AG_\omega{}^0(\mathbf{p}-\mathbf{q})-\frac{\omega}{|\,\omega\,|}\frac{\omega_-}{|\,\omega_-\,|}\frac{i}{2\tilde{\omega}}V_A{}^c\tag{29}$$

if $G_\omega{}^0(\mathbf{p}-\mathbf{q})$ is the electron line joining the two vertices. This contribution enters in all three diagrams of Fig.1 for the longitudinal case. Again, if isotropic scattering is assumed, all impurity corrections to electromagnetic vertices vanish for transverse waves.

We now return to the correlation functions incorporating the above remarks. We use the condition that the energy gap $\Delta(\mathbf{r})$ varies slowly over distances of order l to remove it from the integrals. After Fourier expanding the Green's functions and making the replacement $d^3p\rightarrow mp_0d\Omega d\xi$ in the usual way, we find

$$\langle[A,B]\rangle(\mathbf{q},\omega_0)=\langle[A,B]\rangle_N(\mathbf{q},\omega_0)+\langle[A,B]\rangle_S(\mathbf{q},\omega_0),\tag{30}$$

where the normal-metal correlation function is

$$\langle[A,B]\rangle_N(\mathbf{q},\omega_0)=\frac{mp_0}{A\pi^3}T\sum_\omega\int d\Omega V_AV_B{}^0\int_{-\infty}^{\infty}d\xi\,G_\omega{}^0(\mathbf{p})G_{\omega_-}{}^0(\mathbf{p}-\mathbf{q})\tag{31}$$

and the lowest-order superconducting contribution is

$$\langle[A,B]\rangle_S(\mathbf{q},\omega_0)=-(mp_0/\pi^2)\,|\,\Delta(\mathbf{r})\,|^2\,T\sum_{\omega}\sum_{i=1}^{3}[R_i{}^0(\omega,\omega_0,\mathbf{q})+R_i{}^{s-s}(\omega,\omega_0,\mathbf{q})+R_i{}^{s-\mathrm{em}}(\omega,\omega_0,\mathbf{q})], \tag{32}$$

where

$$R_1{}^0(\omega,\omega_0,\mathbf{q})=\eta_{\omega_-}{}^2\int\frac{d\Omega}{4\pi}\,V_AV_B\int_{-\infty}^{\infty}d\xi\,G_\omega{}^0(\mathbf{p})G_{\omega_-}{}^0(\mathbf{p}-\mathbf{q})G_{-\omega_-}{}^0(\mathbf{p}-\mathbf{q})G_{\omega_-}(\mathbf{p}-\mathbf{q}), \tag{33}$$

$$R_2{}^0(\omega,\omega_0,\mathbf{q})=\eta_\omega{}^2\int\frac{d\Omega}{4\pi}\,V_AV_B\int_{-\infty}^{\infty}d\zeta\,G_{\omega_-}{}^0(\mathbf{p})G_\omega{}^0(\mathbf{p}+\mathbf{q})G_{-\omega}{}^0(\mathbf{p}+\mathbf{q})G_\omega{}^0(\mathbf{p}+\mathbf{q}), \tag{34}$$

and

$$R_3{}^0(\omega,\omega_0,\mathbf{q})=\eta_\omega\eta_{\omega_-}\int\frac{d\Omega}{4\pi}\,V_AV_B\int_{-\infty}^{\infty}d\xi\,G_\omega{}^0(\mathbf{p})G_{-\omega}{}^0(\mathbf{p})G_{\omega_-}{}^0(\mathbf{p}-\mathbf{q})G_{-\omega_-}{}^0(\mathbf{p}-\mathbf{q}). \tag{35}$$

The functions $R_i{}^{s-s}$, which describe the corrections due to impurity lines bridging two superconducting vertices (for $i=1,2$), are obtained from $R_i{}^0$ by making the replacement of Eq. (28); similarly, the functions $R_i{}^{s-\mathrm{em}}$, arising from the ladder series bridging an electromagnetic vertex and a superconducting vertex, are obtained from $R_i{}^0$ by the replacement given by Eq. (29).

Since we have expanded $\langle[A,B]\rangle$ to second order in $\Delta(\mathbf{r})$, for consistency we must also expand the attenuation coefficients to order $|\,\Delta(\mathbf{r})\,|^2$. For the longitudinal wave we may write

$$\alpha_L=\alpha_L{}^N+\alpha_L{}^S, \tag{36}$$

$$\alpha_L{}^N=\mathrm{Re}(q^2/i\omega\rho_{\mathrm{ion}}v_s)\{[\tau_{zz},\tau_{zz}]_N-[\tau_{zz},n]_N{}^2/[n,n]_N\}, \tag{37}$$

and

$$\alpha_L{}^S=\mathrm{Re}\,\frac{q^2}{i\omega\rho_{\mathrm{ion}}v_s}\left\{[\tau_{zz},\tau_{zz}]_S-\frac{[\tau_{zz},n]_N}{[n,n]_N}\left[2[\tau_{zz},n]_S-\frac{[\tau_{zz},n]_N[n,n]_S}{[n,n]_N}\right]\right\}, \tag{38}$$

where the brackets $\langle\ \rangle(\mathbf{q},\omega_0)$ have been dropped for brevity. It is implicit in these equations that we spatially average at the end of the calculation to obtain the experimentally measurable coefficients. This just amounts to replacing $|\,\Delta(\mathbf{r})\,|^2$ by $\langle|\,\Delta(\mathbf{r})\,|^2\rangle$. In the next section we proceed to explicitly calculate the correlation functions occurring in Eqs. (37) and (38).

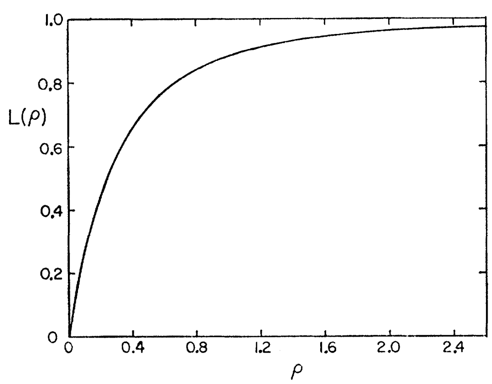

FIG. 2. The function $L(\rho)$ is plotted against $\rho=\alpha/2\pi T$.

IV. CALCULATION OF THE CORRELATION FUNCTIONS

The calculations proceed in a straightforward manner. It is important to remember that, in expressions such as Eq. (31), the sum over ω and the integral over ξ are only conditionally convergent. It turns out[8] that the sum must be done first, for only then are the integrals over ξ rapidly convergent near the Fermi surface. It is well known,[6] however, that it is still possible to do the integral first if we subtract from the correlation functions terms which account for the spurious high ω and ξ contribution. These high-energy factors can be safely estimated from the normal-metal contribution alone, and are given by

$$\langle[n,n]\rangle_\infty=mp_0/\pi^2, \tag{39}$$

$$\langle[\tau_{zz},n]\rangle_\infty=N, \tag{40}$$

and

$$\langle[\tau_{zz},\tau_{zz}]\rangle_\infty=3Np_0{}^2/5m, \tag{41}$$

where $N=p_0{}^3/3\pi^2$ is the electron density.

The normal-metal correlation functions are now easily obtained. Integrating over ξ we have

$$\langle[A,B]\rangle_N(\mathbf{q},\omega_0)=\frac{mp_0}{4\pi^3}T\sum_\omega\Lambda_-\int d\Omega\, V_AV_B{}^0\frac{2\pi}{\tilde{\omega}-\tilde{\omega}_-+i\mathbf{v}\cdot\mathbf{q}}-\langle[A,B]\rangle_\infty, \tag{42}$$

but in the dirty limit $\tilde{\omega}-\tilde{\omega}_-=\omega_0+\tau^{-1}\cong\tau^{-1}$ and consequently the ω sum is trivial.

After doing the angular integral and performing the analytical continuation, we obtain

$$\langle[n,n]\rangle_N(\mathbf{q},\omega)=(mp_0/\pi^2)\{i\omega\tau[\tan^{-1}x/(x-\tan^{-1}x)]-1\}, \tag{43}$$

$$\langle[\tau_{zz},n]\rangle_N(\mathbf{q},\omega)=(p_0{}^3/\pi^2)[(i\omega\tau/x^2)-\tfrac{1}{3}], \tag{44}$$

and

$$\langle[\tau_{zz},\tau_{zz}]\rangle_N(\mathbf{q},\omega)=(p_0{}^5/3\pi^2m)[(i\omega\tau/x^2)-\tfrac{3}{5}]. \tag{45}$$

The evaluation of the superconducting contributions is conveniently divided into three steps: (1) the ξ integration, that is, calculation of the $R_i(\omega,\omega_0,\mathbf{q})$; (2) the angular integrals; and (3) the frequency sums. The functions $R_i{}^0(\omega,\omega_0,\mathbf{q})$ are found to be

$$R_1{}^0=\frac{\pi\tau}{2(|\omega_-|+\alpha)^2}\int\frac{d\Omega}{4\pi}V_AV_B\left\{\frac{-\Lambda_+}{1+ixu}+\Lambda_-\left[\frac{1}{1+ixu}+\frac{1}{(1+ixu)^2}\right]\right\}, \tag{46}$$

$$R_2{}^0=\frac{\pi\tau}{2(|\omega|+\alpha)^2}\int\frac{d\Omega}{4\pi}V_AV_B\left\{\frac{-\Lambda_+}{1-ixu}+\Lambda_-\left[\frac{1}{1+ixu}+\frac{1}{(1+ixu)^2}\right]\right\}, \tag{47}$$

and

$$R_3{}^0=\frac{\pi\tau}{(|\omega|+\alpha)(|\omega_-|+\alpha)}\int\frac{d\Omega}{4\pi}\frac{V_AV_B}{(1+x^2u^2)}, \tag{48}$$

where again we have made the approximation $T\tau\ll1$. When we add the corrections arising from impurity lines bridging two superconducting vertices,[10] $R_i{}^{s-s}$, we find that they cancel the terms $(1+ixu)^{-2}$ in $R_1{}^0$ and $R_2{}^0$. Finally, adding $R_i{}^{s-\text{em}}$, we find the results

$$R_1(\omega,\omega_0,\mathbf{q})=\frac{\pi\tau}{2(|\omega_-|+\alpha)^2}\frac{1}{4\pi}\int\frac{d\Omega}{1+x^2u^2}\{-\Lambda_+[V_A{}^0V_B{}^0+V_A{}^0V_B{}^c]+\Lambda_-[V_AV_B-V_AV_B{}^c]\}, \tag{49}$$

$$R_2(\omega,\omega_0,\mathbf{q})=\frac{\pi\tau}{2(|\omega|+\alpha)^2}\frac{1}{4\pi}\int\frac{d\Omega}{1+x^2u^2}\{-\Lambda_+[V_A{}^0V_B{}^0+V_A{}^0V_B{}^c]+\Lambda_-[V_AV_B-V_AV_B{}^c]\}, \tag{50}$$

and

$$R_3(\omega,\omega_0,\mathbf{q})=\frac{\pi\tau}{(|\omega|+\alpha)(|\omega_-|+\alpha)}\frac{1}{4\pi}\int\frac{d\Omega}{1+x^2u^2}\{\Lambda_+[V_A{}^0V_B{}^0+V_A{}^0V_B{}^c]+\Lambda_-[V_AV_B-V_AV_B{}^c]\}. \tag{51}$$

After doing the angular integrals and summing over ω, we find that the correlation functions involve products of functions of $x=ql$ with one of the following sums:

$$S_1=2T\sum_\omega\Lambda_+\left[\frac{1}{(|\omega|+\alpha)^2}+\frac{1}{(|\omega_-|+\alpha)^2}-\frac{2}{(|\omega_-|+\alpha)(|\omega|+\alpha)}\right], \tag{52}$$

$$S_2=2T\sum_\omega\Lambda_-\left[\frac{1}{(|\omega|+\alpha)^2}+\frac{1}{(|\omega_-|+\alpha)^2}+\frac{2}{(|\omega_-|+\alpha)(|\omega|+\alpha)}\right]. \tag{53}$$

It is possible to express S_i in terms of poly γ functions; we can then perform the analytical continuation simply by replacing ω_0 by $i\omega$. Further, since the usual experimental conditions are such that $\omega/\pi T_c\ll1$, we expand S_i to first order in ω. The results are

$$S_1=0, \tag{54}$$

$$S_2=(2/\pi)[i\omega/(2\pi T)^2][\rho^{-1}\psi'(\rho+\tfrac{1}{2})-\psi''(\rho+\tfrac{1}{2})], \tag{55}$$

where ψ' and ψ'' are the trigamma and tetragamma functions, respectively, and $\rho=\alpha/2\pi T$.

It is now but a simple matter to combine the results of this section to obtain the leading order superconducting

[10] We wish to acknowledge an informative discussion with Professor Maki on the importance of this contribution.

contributions to the correlation functions. We list the results below:

$$\langle[n,n]\rangle_S(\mathbf{q},\omega)=-|\Delta(\mathbf{r})|^2(mp_0\tau/4\pi)[S_2\tan^{-1}x/(x-\tan^{-1}x)],\tag{56}$$

$$\langle[\tau_{zz},n]\rangle_S(\mathbf{q},\omega)=-|\Delta(\mathbf{r})|^2(p_0^3\tau/4\pi)[S_2(1/x^2)],\tag{57}$$

$$\langle[\tau_{zz},\tau_{zz}]\rangle_S(\mathbf{q},\omega)=-|\Delta(\mathbf{r})|^2(p_0^5\tau/4\pi m)[S_2(1/3x^2)].\tag{58}$$

V. LONGITUDINAL ATTENUATION

From Eq. (37) we find the usual Pippard[11] result for the normal-metal attenuation

$$\alpha_N{}^L=(Nm/\rho_{\rm ion}v_s\tau)[h(x)-1],\tag{59}$$

where $x=ql$ and

$$h(x)=x^2\tan^{-1}x/[3(x-\tan^{-1}x)].\tag{60}$$

The superconducting contribution is

$$\alpha_S{}^L=-|\Delta(\mathbf{r})|^2\frac{q^2}{i\omega\rho_{\rm ion}v_s}\frac{p_0^5\tau}{4\pi m}\left[S_2\frac{h(x)-1}{3x^2}\right].\tag{61}$$

Here, terms of relative order $(v_s/v_F)^2$ have been dropped. Substituting for the sum S_2 and taking the ratio of the attenuation in the superconducting state to the attenuation in the normal state, we obtain

$$\alpha_L/\alpha_L{}^N=1-\langle|\Delta(\mathbf{r})|^2\rangle/2(2\pi T)^2\{\rho^{-1}\psi'(\rho+\tfrac{1}{2})-\psi''(\rho+\tfrac{1}{2})\}.\tag{62}$$

The result of Eq. (62) is valid in the small gap region for arbitrary magnetic field. In this form, comparison with experiment will determine the average value of the energy gap. Using Maki's[9] results for $\langle|\Delta(\mathbf{r})|^2\rangle$ in the high field region, we may express the attenuation in terms of the upper critical fields. Following Maki[9] we consider two cases, Abrikosov's mixed state and the superconducting surface sheath.

(a) The mixed state $H_0\lesssim H_{c2}$. For this case Maki finds the average value of $|\Delta(\mathbf{r})|^2$ to be

$$\langle|\Delta(\mathbf{r})|^2\rangle=(eT/\sigma)[(H_{c2}-H_0)/(2\kappa_2{}^2(T)-1)\beta][\psi'(\rho+\tfrac{1}{2})]^{-1},\tag{63}$$

where $\beta=1.16$ and $\sigma=\mathrm{N}e^2\tau_{\rm tr}/m$ is the normal-metal dc conductivity. The parameter $\kappa_2(T)$ is defined in Ref. (9) and as pointed out by Caroli *et al.*[12] its temperature dependence is almost identical to $\kappa_1(T)$, the usual Landau-Ginzburg parameter. Thus the attenuation can be rewritten

$$\alpha_L/\alpha_L{}^N=1-(4\pi)^{-1}(e/\sigma\alpha)[(H_{c2}-H_0)/\beta(2\kappa_2{}^2(T)-1)][1+L(\rho)],\tag{64}$$

where

$$L(\rho)=-\rho(\partial/\partial\rho)\ln\psi'(\rho+\tfrac{1}{2}).\tag{65}$$

In Fig. 2, we plot $L(\rho)$ for $\rho=\alpha/2\pi T$ ranging between 0 and 2; in this case $\alpha=\frac{1}{3}\tau_{\rm tr}v^2eH_{c2}$.

(b) Superconducting surface sheath $(H_0\lesssim H_{c3})$. In this case we consider a film of thickness $d>(l\xi_0)^{1/2}$ in an external field applied parallel to the surface. Maki[9] finds

$$\langle|\Delta(\mathbf{r})|^2\rangle=\frac{2}{d}\left(\frac{\pi}{1.18eH_{c3}}\right)^{1/2}\frac{eT}{\sigma}\frac{H_{c3}-H_0}{[2\kappa_2{}^2(T)-3.12]}[\psi'(\rho+\tfrac{1}{2})]^{-1}.\tag{66}$$

Therefore, the attenuation coefficient in this case becomes

$$\frac{\alpha_L}{\alpha_L{}^N}=1-(\alpha d\sigma)^{-1}\left(\frac{e}{1.18\pi H_{c3}}\right)^{1/2}\frac{H_{c3}-H_0}{(2\kappa_2{}^2(T)-3.12)}[1+L(\rho)],\tag{67}$$

where α was found by Saint-James and De Gennes[13] to be given by

$$\alpha=\frac{0.59}{3}\tau_{\rm tr}v^2eH_{c3}.$$

The above calculations show that the attenuation in the longitudinal case arises from both electromagnetic absorption and the collision drag effect. As in the case of the transverse wave,[4] measurement of the attenuation coefficient at low frequencies in the dirty limit serves to determine both the average amplitude of the energy gap and the upper critical field H_{c2} and/or H_{c3}. *Note added in proof*: Since this paper was submitted for publication the longitudinal attenuation coefficient has been calculated in the dirty limit by Maki and Fulde [See K. Maki

[11] A. B. Pippard, Phil. Mag. **46,** 1104 (1955).
[12] C. Caroli, M. Cyrot, and P. G. DeGennes, Solid State Commun. **4,** 17 (1966).
[13] D. Saint-James and P. G. DeGennes, Phys. Letters **7,** (1963) 306.

and P. Fulde, Solid State Commun. **5**, 21 (1967)] who exploit the analogy with a thin-film superconductor in a parallel magnetic field. Equation (62), obtained by direct calculation, agrees with their result.

VI. TRANSVERSE WAVE ATTENUATION

As mentioned in the Introduction, the attenuation in this case has been calculated previously by Maki.[4] For completeness, we briefly review the results here. Using Eq. (18), one finds that the normal-metal attenuation for low frequencies $\omega \lesssim 10^9$ is given by the familiar Pippard[11] result

$$\alpha_N{}^T = (Nm/\rho_{\text{ion}} v_s \tau)[(1-g(x))/g(x)], \tag{68}$$

where

$$g(x) = (3/2x^3)[(1+x^2)\tan^{-1}x - x]. \tag{69}$$

We note that the diamagnetic contribution to the current response function exactly cancels the high-energy factor arising from interchanging order of summation and integration.

In order to discuss the attenuation in the superconducting state it is instructive to write down the current response function

$$(N/m) + \langle[j_x, j_x]\rangle(\mathbf{q}, \omega) = (p_0{}^3/3\pi^2 m)(i\omega\tau) g(x) \times \{1 + [|\Delta(\mathbf{r})|^2/2(2\pi T)^2][\rho^{-1}\psi'(\rho+\tfrac{1}{2}) + 3\psi''(\rho+\tfrac{1}{2})] - [i|\Delta(\mathbf{r})|^2/\pi T\omega]\psi'(\rho+\tfrac{1}{2})\}. \tag{70}$$

For very small energy gap or sufficiently high frequency such that $|\Delta(\mathbf{r})|/\omega \ll 1$, we can consistently expand Eq. (18) to second order in Δ as in the longitudinal case; we obtain for the ratio of the attenuation in the superconducting and normal states

$$\alpha^T/\alpha_N{}^T = 1 - [\langle|\Delta(\mathbf{r})|^2\rangle/2(2\pi T)^2]\{g(x)[\rho^{-1}\psi'(\rho+\tfrac{1}{2}) - \psi''(\rho+\tfrac{1}{2})] + [1-g(x)][\rho^{-1}\psi'(\rho+\tfrac{1}{2}) + 3\psi''(\rho+\tfrac{1}{2})]\}. \tag{71}$$

For low frequencies such that $\Delta/\omega \gg 1$, the Meissner screening becomes very strong and the result found by Maki[4] is obtained:

$$\alpha^T/\alpha_N{}^T = g(x)\{1 - [\langle|\Delta(\mathbf{r})|^2\rangle/2(2\pi T)^2][\rho^{-1}\psi'(\rho+\tfrac{1}{2}) - \psi''(\rho+\tfrac{1}{2})]\} + [1-g(x)]\langle\{1 + [|\Delta(\mathbf{r})|^4/(\pi T)^2\omega^2][\psi'(\rho+\tfrac{1}{2})]^2\}^{-1}\rangle_{\text{av}}. \tag{72}$$

Finally, in the high-frequency limit, $\omega > 10^9$ and $|\Delta|/\omega \ll 1$, there is incomplete screening, and one obtains simply

$$\alpha^T/\alpha_N{}^T = 1 - [\langle|\Delta(\mathbf{r})|^2\rangle/2(2\pi T)^2][\rho^{-1}\psi'(\rho+\tfrac{1}{2}) - \psi''(\rho+\tfrac{1}{2})]. \tag{73}$$

We note that in this case the normal-metal attenuation goes like $1-g(x)$.

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APPENDIX

The purpose of this discussion is to note a correction to the expression of Ref. 4 when the effect of current-current interaction on the transverse response functions is treated. We consider the expectation value of an arbitrary operator $B(\mathbf{r}, t)$ in the presence of a transverse electromagnetic field described by the vector potential $\mathbf{A}(\mathbf{r}, t)$. If the operator $B(\mathbf{r}, t)$ has any transverse components, then in general it may be written in the form

$$B(\mathbf{r}, t) = B_0(\mathbf{r}, t) + B_1(\mathbf{r}, t)\mathbf{A}(\mathbf{r}, t) + O(A^2), \tag{A1}$$

where $B_0(\mathbf{r}, t)$ is the operator in the absence of the field. Then linear response theory gives

$$\frac{\delta\langle B\rangle(\mathbf{q}, \omega)}{\delta\langle\mathbf{A}\rangle} = \langle B_1\rangle + e\langle[B_0, \mathbf{j}]\rangle(\mathbf{q}, \omega), \tag{A2}$$

where $\mathbf{j}(\mathbf{r}, t)$ is defined in Eq. (5).

The effect of the current-current interaction is treated within the random phase approximation. The results are essentially those listed by Maki[4] except that one must use the total linear response functions as given by (A2). In the problem of transverse ultrasound attenuation, the only correction is the replacement of the retarded product $\langle[j_x, j_x]\rangle(\mathbf{q}, \omega)$ by the current response function

$$N/m + \langle[j_x, j_x]\rangle(\mathbf{q}, \omega). \tag{A3}$$

The constant factor in (A3) is just the diamagnetic contribution to the response. As noted in the text this constant term exactly cancels the high-energy factor arising when the orders of summation and integration are interchanged.