Quantum Noise. X. Density-Matrix Treatment of Field and Population-Difference Fluctuations

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Starting with the quantum-noise-source model of a maser developed in the fourth paper in this series and specializing to the case in which all atomic parameters except the population difference vary rapidly compared with the field, an equation is developed for the density operator $\rho(b, b^\dagger, D, t)$ that depends on the electromagnetic field variables b, $b^\dagger$ and the population difference D. The corresponding equation obeyed by the density matrix $\rho_{mn}(D, t)$ in the "n representation" is obtained. The antinormal ordering correspondence $\rho(b, b^\dagger, D, t) = \mathcal{A}\bar{\rho}^{(a)}(\beta, \beta^*, D, t)$ [in which β is replaced by b and β^* by $b^\dagger$ with all $b^\dagger$'s to the right of all b's] defines an associated classical function $\bar{\rho}^{(a)}$. The equation for $\partial\rho/\partial t$ corresponds to a Fokker-Planck equation for $\partial\bar{\rho}^{(a)}/\partial t$. Under the Markoffian assumption, an associated classical random process is uniquely defined. The quantum-mechanical average of normal-ordered time-ordered operators is given by simple integrals over the corresponding classical distribution functions. For one-time operators this result is well known. For two-time operators, a proof is given using the quantum-regression theorem of the second paper in this series. Amplitude and phase fluctuations are discussed by rewriting the Fokker-Planck equation in the variables I, ϕ, D, where $\beta = I^{1/2}\exp(-i\phi)$. The Langevin description of the classical random process associated with $\bar{\rho}^{(a)}(\beta, \beta^*, D, t)$ is used to make a classical adiabatic elimination of D. The resulting equations for $\bar{\rho}^{(a)}(\beta, \beta^*, t)$ and $\rho(b, b^\dagger, t)$ are shown to agree with the corresponding results of the ninth paper in this series, obtained by a more difficult quantum adiabatic elimination of D. The equation for $\rho(b, b^\dagger, t)$, rewritten in the n representation agrees with corresponding results of Scully and Lamb.

1. INTRODUCTION

Summary of the Author's Previous Work

IN paper QIV of this series,[1] the active atoms plus the radiation field in a laser was treated as a system; reservoirs were eliminated and replaced by dissipation coefficients plus (noncommuting) random forces. This quantum Langevin description could be applied immediately, using quasilinear techniques, to a discussion of phase noise in[1] QV and correlated intensity and population fluctuation noise in QVII. A quantitatively correct description of laser noise is given by such quasilinear techniques away from threshold, and, by a judicious choice of variables, qualitatively correct results are obtained even in the threshold region. To obtain quantitatively correct results everywhere, including threshold, one must use distribution-function (Fokker-Planck) techniques in classical problems and density-operator techniques in quantum mechanics.

To obtain density operator equations, the author and Louisell[1] in QIX specialized to the case in which all atomic time constants are fast compared to the electromagnetic field time constants. In this case, the field alone should act in a Markoffian manner, and one could hope to obtain an equation for $\partial\rho/\partial t$ where $\rho = \rho(b, b^\dagger, t)$, the density operator, would be a function of the field variables alone. Instead of proceeding directly to this end, we found it more expedient to introduce an associated classical function $\bar{\rho}^{(a)}(\beta, \beta^\dagger, t)$ by means of the correspondence

$$\rho(b, b^\dagger, t) = \mathcal{A}\bar{\rho}^{(a)}(\beta, \beta^*, t), \tag{1.1}$$

where the antinormal ordering operation $\mathcal{A}$ means replace β by b and β^* by $b^\dagger$ placing all operators in antinormal order (all $b^\dagger$'s to the right of all b's). It was then easier to obtain an equation for $\partial\bar{\rho}^{(a)}/\partial t$ than for $\partial\rho/\partial t$. Moreover, this equation had Fokker-Planck form corresponding to a particular Markoff random process. Over a broad region including threshold, this classical random process was shown in QIX to reduce to that of a (classical) rotating wave Van der Pol oscillator. This permitted the quasilinear treatment of the Van der Pol oscillator[2] in V, and the exact numerical solutions for such an oscillator obtained by Hempstead and Lax in VI to be applied to the laser problem. Indeed, it is this work that permits us to assert with confidence that judicious quasilinear methods are valid away from the immediate vicinity of threshold.

From our point of view, the equation for $\partial\bar{\rho}^{(a)}/\partial t$ is more important than that for $\partial\rho/\partial t$ since we have shown

[1] The complete series by M. Lax is QI. Phys. Rev. **109**, 1921 (1958); QII. Phys. Rev. **129**, 2342 (1963); QIII. J. Phys. Chem. Solids **25**, 487 (1964); QIV. Phys. Rev. **145**, 110 (1966); QV. in *Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill Book Company, Inc., New York, 1966), p. 735; QVI. (with D. R. Fredkin, to be published); QVII. J. Quantum Electron. **QE3**, 37 (1967); QVIII. H. Cheng and M. Lax, in *Quantum Theory of the Solid State*, edited by Per-Olav Lowdin (Academic Press Inc., New York, 1966), p. 587; QIX. (with W. H. Louisell) J. Quantum Electron. **QE3**, 47 (1967). Note: Papers QVII and QIX were presented as part of one long talk at the Phoenix International Conference on Quantum Electronics, April 1966.

[2] V refers to the fifth paper in the author's series on classical noise: I. Rev. Mod. Phys. **32**, 25 (1960); II. J. Phys. Chem. Solids **14**, 248 (1960); III. Rev. Mod. Phys. **38**, 359 (1966); IV. Rev. Mod. Phys. **38**, 541 (1966); V. Bull. Am. Phys. Soc. **11**, 111 (1966) and (to be published); VI. (with R. D. Hempstead) Bull. Am. Phys. Soc. **11**, 111 (1966), and thesis by R. D. Hempstead, Department of Electrical Engineering, Massachusetts Institute of Technology, 1965 (unpublished) and Phys. Rev. (to be published). Some of the key curves of VI have already been presented in V, QVII, QIX, and Fig. 1 of this paper.

in QIX (at least for one- and two-time operators) that the quantum-mechanical average of a time-ordered normal-ordered operator is given by an ordinary integration over the appropriate distribution function of the associated classical process described by $\bar{\rho}^{(a)}$. Moreover, normal-ordered (with all b's to the right of all $b^\dagger$'s) time-ordered operators have been shown by Glauber[3] to be just those appropriate to detection schemes in which photons are absorbed. If we wished to describe stimulated emission counters, such as those discussed by Mandel,[4] we could be concerned with antinormally ordered operators, and would find it necessary, then, to introduce the normal ordering correspondence

$$\rho(b, b^\dagger, t) = \mathfrak{N}\bar{\rho}^{(n)}(\beta, \beta^*, t) \tag{1.2}$$

in which the normal ordering operation $\mathfrak{N}$ replaces β by b, β^* by $b^\dagger$, *and* places all operators in normal order. Our methods of calculation can, of course, be applied to $\bar{\rho}^{(n)}$ just as easily as to $\bar{\rho}^{(a)}$.

By using an ordering correspondence between ρ and a classical c-number function, we have eliminated quantum-mechanical difficulties, and reduced our problem to that of a classical (nonlinear) random process. Time-ordered, normal-ordered operator products have mean values that (we shall show in QXI) can be calculated by taking corresponding multitime averages in the associated classical random process. The proof of this result in QIX made use of the quantum regression theorem. The theorem was derived in QII. (See also QIV Sec. 2). Because of its central importance, a new brief derivation of the regression theorem will be given later in this section.

Contributions of the Present Paper

In the present paper we generalize QIX to permit the fractional population difference D defined by

$$D = N^{-1} \sum_M [(a_2^\dagger a_2)^M - (a_1^\dagger a_1)^M] \tag{1.3}$$

to vary at a rate that is comparable to the photon decay rates. All other atomic responses are assumed, as before, to be fast. The consideration of this problem has three advantages: (1) It is the simplest generalization that permits us to include the population dynamics of the laser in addition to that of the field. Moreover, the methods of this paper can be readily generalized to permit the explicit inclusion of several populations, and we shall do so in QXII. (2) The adiabatic elimination of D in QIX presented some ordering difficulties. In order to be sure that the previous calculations were correct (and indeed we detected an error in the first draft[5] of QIX), we shall obtain the Fokker-Planck equation for the variables b, $b^\dagger$, and D. The adiabatic elimination of D can then be performed entirely in the classical domain after the associated classical function has been set up. This avoids all difficulties of ordering. (3) In Paper QVII on the rate equations and amplitude noise in lasers, we set up a description in terms of the number of photons and the population difference and succeeded in obtaining a Fokker-Planck equation for these variables. However, intensity correlation experiments of the Hanbury Brown-Twiss variety[6] measure not the number correlation $\langle b^\dagger(t) b(t) b^\dagger(0) b(0)\rangle$ but the intensity correlation $\langle b^\dagger(0) b^\dagger(t) b(t) b(0)\rangle$. The discussion of such correlated intensity fluctuations in Sec. 9 of Paper QVII is based entirely on the associated classical random problem to be derived in the present paper. Thus one function of the present paper is to derive the starting equations QVII(9.5)–(9.9).

Operator Averages

If we define a density matrix that is a function of b, $b^\dagger$, and D, we can take the average of an arbitrary operator in the form

$$\langle M(t)\rangle = \langle M[b(t), b^\dagger(t), D]\rangle = \int dD \, \mathrm{Tr}[M(b, b^\dagger, D)\rho(b, b^\dagger, D, t)]. \tag{1.4}$$

Since D commutes with b and $b^\dagger$ it can be regarded as a c number in the subsequent discussion. Strictly speaking, the integral over D in (1.4) should be a sum over the possible discrete values of this population difference, but these are so closely spaced that we shall for simplicity replace such a sum by an integral. It is now possible to make the same antinormal correspondence between the density operator and its associated classical function as was used in QIX:

$$\rho(b, b^\dagger, D, t) = \mathfrak{A}\bar{\rho}^{(a)}(\beta, \beta^*, D, t). \tag{1.5}$$

In (1.5) the β and β^* are numbers that are to be replaced by the corresponding operators b and $b^\dagger$ following the antinormal ordering rule mentioned above. The population difference D can be placed in any position. Similarly, an arbitrary operator M can be associated with a classical function by means of a normally ordered correspondence

$$M(b, b^\dagger, D) = \mathfrak{N}\bar{M}^{(n)}(\beta, \beta^*, D), \tag{1.6}$$

[3] R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963); Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).

[4] L. Mandel, Phys. Rev. **152**, 438 (1966).

[5] In the first draft of QIX, the diffusion coefficient $D_{\beta\beta}$ was found to vanish, contradicting (9.10) in the present paper. This error arose because the commutator $[b, J]$ of QIX (3.14) was neglected. This term is of order $(\pi/\Gamma)bJ \sim 10^{-8}bJ$ relative to the drift terms. But as discussed in Sec. 10 of this paper, the diffusion terms are of order $\pi/\Gamma_2 \sim 10^{-8}$ times the drift terms. Hence the drift terms must be computed to an accuracy π/Γ_2 since the drift correction terms arising from commutators can take the form of diffusion terms, as shown in Sec. 10 of this paper.

[6] R. Hanbury Brown and R. W. Twiss, Proc. Roy. Soc. (London) **A242**, 300 (1957); **A243**, 291 (1958). See QVII Ref. 28 for additional references.

in which after the operator replacement all $b^\dagger$'s are to be shifted to the left and all b's are to be shifted to the right. We now take the well-known theorem that the trace of the product of a normally ordered and an antinormally ordered operator can be evaluated by an integral over the product of the associated functions and generalize the proof [see, for example, QIX(1.4)–(1.7)] merely by adding D to obtain

$$\langle M(t)\rangle=\int dD\int \bar{M}^{(n)}(\beta,\beta^*,D)\bar{\rho}^{(a)}(\beta,\beta^*,D,t)\,d^2\beta/\pi. \tag{1.7}$$

Thus the mean of a quantum-mechanical operator can be obtained in a nearly classical way by integrating a classical function against a "classical distribution function." If by means of the equation

$$\rho(b,b^\dagger,D,t)=\int P(\beta,D,t)\,|\,\beta\rangle d^2\beta\langle\beta\,| \tag{1.8}$$

we introduce a slight generalization $P(\beta, D, t)$ of the Glauber[7]-Sudarshan[8] diagonal weight function, then the usual proof (see, for example, QIX Sec. 1) leads to

$$\langle M(t)\rangle=\int dD\int M^{(n)}(\beta,\beta^*,D)P(\beta,D,t)\,d^2\beta. \tag{1.9}$$

Comparison with (1.7) then leads to the same relationship

$$\bar{\rho}^{(a)}(\beta,\beta^*,D,t)=\pi P(\beta,D,t) \tag{1.10}$$

as found in QIX in the absence of D. Since $P(\beta, D, t)\equiv P(\beta, \beta^*, D, t)$ obeys the same equation as $\bar{\rho}^{(a)}$, we shall use the simpler P notation in what follows and understand that a dependence on β implies a dependence on β^* as well. Moreover, Eq. (1.9) with $M=1$ tells us that $P(\beta, D, t)$ is normalized.

Associated Fokker-Planck Equation

One of the principal results of the present paper is a Fokker-Planck equation for the associated classical function. In order for the reader to understand the remarks that follow it may be helpful to have the form of this equation in mind, although at the moment the meaning of the parameters is not necessary. We find that the associated classical function obeys

$$\begin{aligned}\partial P(\beta,\beta^*,D,t)/\partial t=&-\partial(PA_{\beta*})/\partial\beta^*-\partial(A_\beta P)/\partial\beta-\partial\{[w_{20}(1-dD)-(\Gamma_2+\pi)D]P-\pi\beta DP\beta^*\}/\partial D\\&+(\gamma\bar{n}+\pi ND)\,\partial^2P/\partial\beta\partial\beta^*-\pi\partial^2(PD\beta^*)/\partial\beta^*\partial D-\pi\partial^2(\beta DP)/\partial\beta\partial D\\&+(2N)^{-1}\partial^2\{[w_{20}(1-dD)+\Gamma_2D+\pi D]P+\pi\beta P\beta^*\}/\partial D^2\\&+\pi\partial^3(DP)/\partial\beta\partial\beta^*\partial D-(\pi/2N)\,\partial^3(\beta DP)/\partial\beta(\partial D)^2,\end{aligned} \tag{1.11}$$

where

$$A_\beta=-\tfrac{1}{2}\gamma(1-i\alpha)\beta+\tfrac{1}{2}\pi(1-i\alpha)\beta ND;\qquad A_{\beta*}=A_\beta{}^*. \tag{1.12}$$

The parameter γ is a photon decay constant, Γ_2 is an atomic decay rate, w_{20} is the pump rate, π is the rate constant (3.4) such that the rate of emission of photons is π times the population difference times the number of photons plus one. The parameter d is to be set equal to unity. If one wishes to neglect depletion then this parameter can be set equal to zero. The parameter α is a measure of detuning. All of these parameters are defined precisely in Sec. 2 by the role they play in the Langevin Eqs. (2.1)–(2.8).

Quantum Regression Theorem

The density matrix, unfortunately, only supplies information concerning operators at one time, whereas the questions we wish to answer concerning noise in the electromagnetic field require means of products of operators at at least two times. To overcome this difficulty, we invoke the quantum regression theorem of QII. This theorem states [see QII(6.4)–(6.7)] that if the mean of an arbitrary system operator M at time t can be expressed in terms of the means of a set of system operators M_μ at an earlier time t' by means of

$$\langle M(t)\rangle=\sum_\mu O_\mu(t,t')\langle M_\mu(t')\rangle\qquad(t>t'), \tag{1.13}$$

then an arbitrary two-time correlation involving the operator M any other system operator N can be written in the form

$$\langle M(t)N(t')\rangle=\sum_\mu O_\mu(t,t')\langle M_\mu(t')N(t')\rangle, \tag{1.14}$$

where the *same* numerical function $O_\mu(t, t')$ enters the two-time correlation as was previously involved in the mean equation of motion. The theorem can easily be generalized to yield

$$\langle Q(t')M(t)N(t')\rangle=\sum_\mu O_\mu(t,t')\langle Q(t')M_\mu(t')N(t')\rangle, \tag{1.15}$$

where Q is another system operator. In Eqs. (1.13)–(1.15) the time dependence of the operators is induced by the full Hamiltonian including the system-reservoir interaction.

[7] R. J. Glauber, Ref. 3 and in *Quantum Optics and Electronics*, edited by C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, Science Publishers, Inc., New York, 1965), p. 63.

[8] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963); in *Proceedings of the Symposium on Optical Masers at the Polytechnic Institute of Brooklyn*, 1963 (John Wiley & Sons, Inc., New York, 1963), p. 45. See also C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965).

At first glance it may be thought that our regression theorem is either obvious or false. Indeed it is true that if (1.13) is valid as an operator equation (omitting the averaging brackets), then clearly it can be multiplied by $N(t')$ and a subsequent average taken. In this case, the theorem is obvious. If, for example, we had a complete system and no reservoir, the Heisenberg equation of motion for M could be valid as an operator equation, and (1.13) would represent the time integral of this equation, and would also be valid as an operator equation. A direct proof of our theorem for this pure system case was given in QII(3.14), (3.22), and (3.23).

If (1.13) is not an operator equation then (1.14) no longer follows directly from (1.13), but it is not necessarily false. Indeed QII showed that (1.14) is correct provided only that the density matrix of system and reservoir could be factored at the starting time t'. Although such a factorization is often performed in deriving Markoffian equations of motion, it is not equivalent to stating that the system is Markoffian. It is therefore desirable to restate our theorem as follows.

Quantum Regression Theorem. If M is a member (or a linear combination) of a complete set of system Markoffian variables M_μ then (1.14) *and* (1.15) *follow from* (1.13).

Our set is understood to be complete if the mean equations of motion

$$d\langle M_\mu\rangle/dt=\langle A_\mu\rangle \tag{1.16}$$

are closed by the A_μ being linearly expressible in terms of the same set $\{M_\mu\}$. In addition to the dissipative terms introduced into $\langle A_\mu\rangle$, the reservoir produces a random force acting on the system:

$$dM_\mu/dt=A_\mu+F_\mu(t). \tag{1.17}$$

In a Markoff system, the forces can have no memory and thus must obey QIV(2.3):

$$\langle F_\mu(t)N(t')\rangle=0 \qquad t>t' \tag{1.18}$$

for all μ and any system operator N. But this last equation can be rewritten as

$$d\langle M_\mu(t)N(t')\rangle/dt=\langle A_\mu(t)N(t')\rangle, \tag{1.19}$$

i.e., a differential form of the regression theorem. If the set is complete, this system of equations can be integrated up to produce a set of equations that includes (1.14). [This proof is closely related to using equations QIV(2.11)–(2.16) in reverse order.]

Further discussion of the regression theorem and quantum-classical correspondences will be given in QXI.

Density-Matrix Evolution

The solution of the Fokker-Planck equation (1.11) subject to a given initial condition can be written in the form

$$P(\beta, D, t)=\int P(\beta, D, t\mid\beta', D', t') \times P(\beta', D', t')\,d^2\beta'dD', \tag{1.20}$$

where the Green's function can be expanded in terms of eigenfunctions in the form

$$P(\beta, D, t\mid\beta', D', t')=\sum_l \exp[-\Lambda_l(t-t')]P_l(\beta, D)\phi_l(\beta', D')^*. \tag{1.21}$$

The discussion here precisely parallels that in QIX, with the simple addition of the parameter D. In particular we again set up the correspondences

$$P_l(b, b^\dagger, D)=\pi\mathfrak{A}[P_l(\beta, \beta^*, D)], \tag{1.22}$$

$$\phi_l(b, b^\dagger, D)=\mathfrak{N}[\phi_l(\beta, \beta^*, D)^*], \tag{1.23}$$

which permit us to write the density matrix in the form

$$\rho(b, b^\dagger, D, t)=\sum_l \exp[-\Lambda_l(t-t')] \times P_l(b, b^\dagger, D)\langle\phi_l(t')\rangle, \tag{1.24}$$

where

$$\langle\phi_l(t')\rangle\equiv\int \phi_l(\beta', D')^*P(\beta', D', t')\,d^2\beta'dD' =\int dD\ \mathrm{Tr}[\phi_l(b, b^\dagger, D)\rho(t')]. \tag{1.25}$$

Two-Time Averages

Making use of (1.9), the average of an arbitrary operator can then be written in the form

$$\langle M(t)\rangle=\sum_l \exp[-\Lambda_l(t-t')]\int \bar{M}^{(n)}(\beta, D)P_l(\beta, D)\,d^2\beta dD\langle\phi_l(t')\rangle. \tag{1.26}$$

Using our quantum-regression theorem (1.15), we can now write

$$\langle Q(t')M(t)N(t')\rangle=\sum_l \exp[-\Lambda_l(t-t')]\int \bar{M}^{(n)}(\beta, D)P_l(\beta, D)\,d^2\beta dD\langle Q(t')\phi_l(t')N(t')\rangle. \tag{1.27}$$

If the complete operator on the left-hand side of (1.27) were in normal order this would probably mean that Q involved only $b^\dagger$'s and N involved only b's. In this case, with the operator ϕ_l written in normal order, the operator

on the right-hand side of (1.27) is still in normal order. In this case, we can again invoke (1.9) and write the average on the right-hand side of (1.27) as a classical integration. These conditions are met by the two most important averages associated with the electromagnetic field. For these we can write

$$\langle b^\dagger(t)b(0)\rangle=\sum_l \exp(-\Lambda_l t)\int \beta^* P_l(\beta, D)\,d^2\beta dD\int \beta'\phi_l(\beta', D')\,{}^*P(\beta', D', 0)\,d^2\beta' dD' \tag{1.28}$$

and

$$\langle b^\dagger(0)b^\dagger(t)b(t)b(0)\rangle=\sum_l \exp(-\Lambda_l t)\int |\beta|^2 P_l(\beta, D)\,d^2\beta dD\int |\beta'|^2\phi_l(\beta', D')\,{}^*P(\beta', D', 0)\,d^2\beta' dD'. \tag{1.29}$$

These integrations are precisely the ones that would have been performed if one took the classical random problem literally. Indeed, we shall show in QXI that classically performed averages always yield corresponding quantum-mechanical averages, namely, those averages associated with appropriately time-ordered and normally ordered operators.

Relation between Our Work and That of Other Authors

The correspondence between a density matrix and a classical (c number) function is not new, nor unique. As examples we mention the work of Wigner, Husimi, Moyal, and Klauder.[9] The particular correspondence used here (β, $\beta^*\to b$, $b^\dagger$) for the electromagnetic field has been emphasized by Glauber[7] and by Sudershan,[8] and extensively reviewed and exploited by Mandel and Wolf[10] and others. In general, the above-mentioned work[9] deals with a closed physical system, rather than, as we do, with a system in dynamic interaction with a reservoir. For such closed systems, the first authors above[9] have concerned themselves with finding the classical dynamical equation from the quantum equation for $\partial\rho/\partial t$.

In the electromagnetic-field case, the free-field Hamiltonian is so simple that the second set of authors[7,8,10] have in general not sought to obtain $\partial P(\beta, t)/\partial t$, but instead have usually chosen to work in a Heisenberg representation in which $P(\beta)$ is independent of the time, and the field operators $b(t)$ and $b^\dagger(t)$ are given a free-field dependence. [Kelley and Kleiner[11] have criticized but continued to use free-field operators.] Plausible guesses for $P(\beta)$ have then been made by Glauber,[7] Morawitz,[12] and Lachs.[13]

Our own procedure exploits the antinormal correspondence[14] (1.1) to obtain an equation for $\partial P(\beta, t)/\partial t$. Our $P(\beta)$ is then the steady-state solution of this equation, and the solution of this equation subject to initial conditions $P(\beta t\,|\,\beta' t')$ is used to calculate the desired phase and amplitude fluctuations. [That $P(\beta t\,|\,\beta' t')$ is also the conditional probability depends on the field alone being Markoffian.]

To obtain the equation for $\partial P(\beta, t)/\partial t$ [or $\partial P(\beta, D)/\partial t$] we need a physical model for a laser: field+atoms+reservoir, rather than some plausible estimate of the field statistics. The quantum noise model of QIV provides us with the necessary starting point by eliminating the reservoirs and replacing them by dissipation coefficients plus random noncummuting noise sources. This model is essentially identical to that used by the Haken school.[15] We use the Einstein relations to determine the second moments of all our noise sources, whereas Haken *et al.*[15] give an explicit calculation in agreement with ours for atomic sources, and take over Senitzky's[16] sources for the field. Since our models are essentially equivalent they can be expected to yield the same results at the quasilinear level. There will, however, sometimes be differences between the results of our calculations based on a classical function that is *exactly* related to the density matrix and Haken's "semiclassical" procedure of taking over quantum-mechanically calculated diffusion coefficients and inserting them *ad hoc* into a Fokker-Planck equation. [See Risken, Schmid, and Weidlich.[15]]

Since Scully and Lamb[17] use a density-matrix ap-

[9] E. P. Wigner, Phys. Rev. **40**, 749 (1932); K. Husimi, Proc. Phys. Math. Soc. Japan **22**, 264 (1940); J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1949); J. R. Klauder, J. Math. Phys. **4**, 1055 (1963); **5**, 177 (1964); Phys. Rev. Letters **16**, 534 (1966); J. McKenna and J. R. Klauder, J. Math. Phys. **5**, 878 (1964); J. R. Klauder, J. McKenna, and D. G. Currie, *ibid.* **6**, 734 (1965).

[10] L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965).

[11] P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).

[12] H. Morawitz, Phys. Rev. **139**, A1072 (1965).

[13] G. Lachs, Phys. Rev. **138**, B1012 (1965).

[14] Most authors refer to the Glauber-Sudarshan weight function $P(\beta)$ of (1.8) as normally ordered since it is useful in obtaining the mean of normally ordered operators as in (1.9). This is a matter of taste. It is important, however, to recognize that although our antinormal ordering connection (1.1) is *equivalent* to the Glauber-Sudarshan resolution (1.8) of the density operator, the antinormal correspondence (1.1) is a particularly convenient way to do calculations. The calculations in QIX and QX were all done using (1.1) and ignoring (1.8) because the equivalence between these two viewpoints was not realized until the calculations were essentially complete.

[15] H. Haken, Z. Physik **190**, 327 (1966); H. Sauerman, *ibid.* **188**, 480 (1965); **189**, 312 (1966); H. Risken, C. Schmid, and W. Weidlich, Phys. Letters **20**, 489 (1966).

[16] I. R. Senitzky, Phys. Rev. **119**, 670 (1960); **124**, 642 (1961).

[17] M. Scully and W. E. Lamb, Jr., Phys. Rev. Letters **16**, 853 (1966). See also M. Scully, W. E. Lamb, Jr., and M. J. Stephen, in *Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill Book Company, Inc., New York, 1966).

proach and adiabatically eliminate the atoms, our results *should* be in agreement with theirs whenever a careful comparison is made. Such a nontrivial comparison is made in Sec. 10 [by translating our equation for $P(\beta, t)$ into one for the density matrix $\rho_{mn}(t)$] and agreement is indeed found.

Our procedure is more general than the Scully-Lamb[17] procedure insofar as we can handle situations in which some (or all) of the atomic rates are slow compared to the field. In this paper, we permit the population difference D to be slow, and in QXII we shall permit both upper- and lower-state populations to equilibrate at arbitrary rates compared to the field.

When all atomic-rate constants are fast compared to the field, $P(\beta, t)$ and $\rho_{mn}(t)$ constitute two ways of viewing the same situation. Which is preferable depends on the question asked. Number fluctuations: $\langle b^\dagger(t)b(t)b^\dagger(0)b(0)\rangle$ are most simply described in terms of $\rho_{nn}(t)$, but intensity correlations $\langle b^\dagger(0)b^\dagger(t)b(t)b(0)\rangle$ (or other time-ordered, normal-ordered operators) are more simply described in terms of $P(\beta, t)$.

As a practical matter, $P(\beta, t)$ obeys a differential equation and $\rho_{mn}(t)$ obeys a difference equation. Over the broad region, including threshold, for which the Van der Pol description of QIX is valid, the differential equation for $P(\beta, t)$ can be scaled both in β and in t so that only one dimensionless parameter p remains (the dimensionless net pump rate). Thus numerical calculations need only explore a one-parameter space. On the other hand, since n is discrete, it can not be scaled except in an approximate sense, and numerical calculations for $\rho_{mn}(t)$ would, in principle, have to explore a two-parameter space.

Green's-function methods such as those used by Korenman[18] are necessarily moment methods with an approximate factorization valid above or below, but not near, threshold, as discussed in QVI and V.

The self-consistent approach of Picard and Willis[19] assumes a factorization of the density matrix into field and atom parts. This procedure (for a gas laser) is quite different, in principle, from allowing the atoms to follow the field. Whether Willis's[20] most recent treatment using the Bogoliubov kinetic-equation approach[21] is equivalent to ours is difficult to ascertain until Willis quotes more specific results, comparable to ours, including the effects of noise.

Recent work of Gordon[22] starts from our equation [QIV(A10)] for the density matrix of the system (field+atoms), uses the antinormal ordering correspondence, and seeks a solution which is a *superposition* of product density matrices. This procedure is, in principle, exact. Whenever the same results have been computed, Gordon has been in agreement with us and with Scully and Lamb.

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Modus Operandi

We start in Sec. 2 with our quantum Langevin model of a maser, and adiabatically eliminate the lower state population σ_{11} and the atomic polarization σ to obtain, in Sec. 3, a set of Langevin equations for $b^\dagger$, b, and D. From this, in Sec. 5, the motion of a general normally ordered operator $\partial\langle(b^\dagger)^r b^s f(D)\rangle/\partial t$ is computed. It is then easy to deduce in Sec. 6 the equation for the associated classical function $P(\beta, \beta^*, D, t)$ and from this the equation for the density operator $\rho(b, b^\dagger, D, t)$ which is our prime result.

From the Fokker-Planck equation for $P(\beta, \beta^*, D, t)$ we construct the associated classical random process for β, β^*, and D in Langevin form. In the latter form, it is easy in Sec. 9 to adiabatically eliminate D classically, resulting in an equation for $P(\beta, \beta^*, t)$. The conversion from the latter to an equation for $\rho(b, b^\dagger, t)$, a more difficult task, is accomplished in Sec. 10. The results for the density-matrix equation are then shown to agree with Scully and Lamb.

2. STARTING EQUATIONS

In Sec. 6 of QIV we developed a Markoffian model for a set of atoms and a radiation field interacting with reservoirs. In this "black-box" model, the reservoirs are replaced by a set of damping constants or transition probabilities and a set of quantum (noncummuting) noise generators. This model, when specialized to the homogeneously broadened case, as in QV, QVII, and QIX is described by the equations

$$db/dt = -\tfrac{1}{2}\gamma(1-i\alpha)b + N\mu\sigma + F(t), \tag{2.1}$$

$$d\sigma/dt = -\Gamma(1+i\alpha)\sigma + \mu bD + F_{12}(t), \tag{2.2}$$

$$d\sigma_{22}/dt = w_{20}\sigma_{00} + w_{21}\sigma_{11} - \Gamma_2\sigma_{22} - B + F_{22}, \tag{2.3}$$

$$d\sigma_{11}/dt = w_{10}\sigma_{00} + w_{12}\sigma_{22} - \Gamma_1\sigma_{11} + B + F_{11}, \tag{2.4}$$

where B, the radiative transition rate, and D, the population difference are given by

$$B \equiv \mu(b^\dagger\sigma + \sigma^\dagger b); \qquad D = \sigma_{22} - \sigma_{11}. \tag{2.5}$$

The (fractional) populations are defined by

$$\sigma_{ii} \equiv N^{-1}\sum(a^\dagger{}_i a_i)_M, \tag{2.6}$$

and (after removing the steady rapid time dependence)

$$\sigma \equiv N^{-1}\sum(a^\dagger{}_1 a_2)_M \exp(i\omega_0 t) \tag{2.7}$$

is proportional to the total electric dipole moment of the maser. The operating frequency of the maser was shown in QV to be

$$\omega_0 = (\tfrac{1}{2}\gamma\omega_a + \Gamma\omega_c)/(\tfrac{1}{2}\gamma + \Gamma), \tag{2.8}$$

and the parameter α is the detuning defined by

$$\alpha=(\omega_a-\omega_c)/(\tfrac{1}{2}\gamma+\Gamma), \tag{2.9}$$

where ω_a is the atomic frequency and ω_c is the cavity frequency.

3. ADIABATIC ELIMINATION OF σ AND σ_{11}

We now wish to specialize to the model adopted in QVII, in which the population difference is the only relevant atomic variable, and the population of the lower masing state is negligible. This can be accomplished by assuming

$$\Gamma_1\rightarrow\infty, \tag{3.1}$$

which means that the lower state empties so rapidly that its population can be neglected:

$$\sigma_{11}\rightarrow 0; \qquad \sigma_{22}=D. \tag{3.2}$$

As shown in (A27) of QIV the off-diagonal decay constant Γ depends on Γ_1 and also approaches infinity:

$$\Gamma\equiv\Gamma_{12}=\tfrac{1}{2}(\Gamma_1+\Gamma_2)+\Gamma_{12}^{\text{ph}}\rightarrow\infty. \tag{3.3}$$

In order not to lose the essential physics we must assume that the electric dipole parameter μ increases in such a way that the ratio

$$2\mu^2/[\Gamma(1+\alpha^2)]\equiv\pi\rightarrow\text{const} \tag{3.4}$$

approaches a constant. Since the parameter Γ is large, an adiabatic approximation should also be made to eliminate σ. This is accomplished by neglecting d/dt compared to Γ in (2.2) to obtain

$$\sigma=[\Gamma(1+i\alpha)]^{-1}(\mu bD+F_{12}). \tag{3.5}$$

The population of the supply state can be eliminated using particle conservation by means of

$$\sigma_{00}=1-d(\sigma_{11}+\sigma_{22})\approx 1-dD. \tag{3.6}$$

The parameter d is understood to be unity. We explicitly inserted it however because setting d equal to zero is equivalent to neglecting depletion of the supply state.

Working Equations

After these adiabatic eliminations have been made, we obtain the working equations

$$db/dt=A_b+\gamma H(t)/(1+i\alpha), \tag{3.7}$$

$$dD/dt=A_D+F_D, \tag{3.8}$$

where the drift vectors are defined by

$$A_b\equiv-\tfrac{1}{2}\gamma(1-i\alpha)b+\tfrac{1}{2}\pi(1-i\alpha)bND, \tag{3.9}$$

$$A_D\equiv w_{20}(1-dD)-\Gamma_2 D-\pi(b^\dagger b+1)D=A_D^{(n)}, \tag{3.10}$$

and the random forces are defined by

$$\gamma H\equiv(N\mu/\Gamma)F_{12}+(1+i\alpha)F, \tag{3.11}$$

$$F_D\equiv F_{22}-\frac{\mu}{\Gamma}\left[\frac{b^\dagger_c F_{12}}{1+i\alpha}+\frac{F^\dagger_{12}b_c}{1-i\alpha}\right], \tag{3.12}$$

where $b_c\equiv b(t-0)$ is b evaluated at a slightly earlier time as in QIX.

4. PRESERVATION OF COMMUTATION RULES

Before proceeding to the use of our working equations, we would like, in this section, to note that the adiabatic approximation has not interfered with our desired commutation rules. In the Appendix we show that to within the accuracy of the adiabatic approximations of (3.1) to (3.3),

$$d[b, D]/dt=[db/dt, D]+[b, dD/dt]\propto[b, D]. \tag{4.1}$$

In other words, if the commutator of b and D vanishes at any one time it will continue to vanish thereafter;

$$[b, D]=[b^\dagger, D]=0. \tag{4.2}$$

We also find that

$$d[b, b^\dagger]/dt=[db/dt, b^\dagger]+\text{H.a.}=0, \tag{4.3}$$

so that the commutator of b and $b^\dagger$ is constant in time and may as well be assigned the conventional value

$$[b, b^\dagger]=1. \tag{4.4}$$

In addition it is easy to see that

$$[db/dt, b]=0; \qquad [dD/dt, D]=0. \tag{4.5}$$

If we examine (3.7) we see that all terms commute with b except possibly $H(t)$. The methods of Appendix A reduce this to the question of whether or not the random force H commutes with itself at two different times. But as shown in QIV, forces such as F and F_{12}, of which H is composed, commute with themselves (although they do not commute with their Hermitian adjoints). In a similar way an examination of (3.8) shows that the second part of (4.5) reduces to the question of whether the random force F_D commutes with itself at different times. This can indeed be established by direct use of the definition (3.12) and the commutation rules associated with the random forces of which F_D is composed. It is simpler to note, however, that the Markoff property requires this commutator to be some operator times a δ function of the time difference, i.e., an even function of the time difference. On the other hand, the commutator of F_D at two different times is necessarily an odd function of the time difference. Thus this commutator necessarily vanishes and (4.5) is obeyed. The properties (4.5) will play an essential role in the equations developed in Sec. 5. It is to be noted, however, that

$$[dD/dt, b]\neq 0, \tag{4.6}$$

and it will not be assumed in the subsequent equations that dD/dt commutes with the operators b and $b^\dagger$.

5. MOTION OF A GENERAL OPERATOR

Introducing the notation

$$\Delta b\equiv b(t+\Delta t)-b(t) \tag{5.1}$$

to describe the change in an operator over a time interval Δt, the motion of a fairly general operator can be written in the form

$$\Delta[(b^\dagger)^r b^s f(D)]=r(b^\dagger)^{r-1}\Delta b^\dagger b^s f(D)+(b^\dagger)^r s b^{s-1}\Delta b f(D)+(b^\dagger)^r b^s \Delta D\partial f/\partial D+r(b^\dagger)^{r-1}\Delta b^\dagger \Delta b s b^{s-1} f(D)$$
$$+(b^\dagger)^r s b^{s-1}\Delta b\Delta D\partial f/\partial D+r(b^\dagger)^{r-1}\Delta b^\dagger b^s \Delta D\partial f/\partial D+(b^\dagger)^r b^s \tfrac{1}{2}(\Delta D)^2\partial^2 f/\partial D^2. \quad (5.2)$$

This procedure is similar to the corresponding classical procedure of III(5.5)–(5.9). We are permitted to stop at second-order terms because we have shown in QIV that our random forces are to a good approximation Gaussian, so that higher-order terms yield results of higher powers than the first in Δt. See, for example, the discussion[2] in V(5.17)–(5.19) or IV(3.8) ff. Note also that we have made use of (4.5) to position the pairs of differences as close to one another as possible. The fact that ΔD and $\Delta b^\dagger$ in the next-to-last term of (5.2) could not be brought completely together will lead to some interesting complications in the calculations that follow.

Drift and Diffusion Terms

We now wish to take the mean of (5.2) and divide by Δt. The terms involving only first differences lead to the drift terms

$$\langle \Delta b\rangle/\Delta t=-\tfrac{1}{2}\gamma(1-i\alpha)\langle b\rangle+\tfrac{1}{2}\pi(1-i\alpha)N\langle bD\rangle=\langle A_b\rangle, \quad (5.3)$$

$$\langle \Delta D\rangle/\Delta t=w_{20}(1-d\langle D\rangle)-\Gamma_2\langle D\rangle-\pi\langle bb^\dagger D\rangle=\langle A_D\rangle, \quad (5.4)$$

which are obtained directly from (3.9) and (3.10). The second-order or diffusion terms arise because of the random forces. From (3.7) and (3.9) we learn that

$$\langle \Delta b^\dagger \Delta b\rangle/\Delta t=(\Delta t)^{-1}\int_t^{t+\Delta t}\left\langle \gamma H^\dagger(s)\,ds\int_t^{t+\Delta t}\gamma H(s')\right\rangle ds'\Big/(1+\alpha^2)$$
$$=\gamma\bar{n}+N\pi D\equiv\gamma[\bar{n}+D/D_0]. \quad (5.5)$$

The second part of (5.5) is obtained using moments of the random forces available in Sec. 6 of QIV. The spectrum of H when no adiabatic approximation is made is given in QV (19) and (20). We see by comparison with the results quoted here that one must set $\omega=0$ in QV(20) to obtain the present result. This is equivalent to neglecting ω in comparison with Γ. Thus, as might be expected, our adiabatic approximation restricts us to frequencies small compared to those decay constants which have been assumed large. The parameter D_0 in (5.5) is defined by

$$D_0\equiv(\tfrac{1}{2}\gamma\Gamma/N\mu^2)(1+\alpha^2)=\gamma/(N\pi), \quad (5.6)$$

and has the meaning of the minimum population difference at which masing will occur according to QV(15). Using again the moments available from QIV, we obtain another diffusion constant in the form

$$\langle(\Delta D)^2\rangle/\Delta t=(\Delta t)^{-1}\int_t^{t+\Delta t}ds\int_t^{t+\Delta t}ds'\langle F_D(s)F_D(s')\rangle$$
$$=\langle w_{20}(1-dD)+\Gamma_2 D+\pi bb^\dagger D\rangle/N. \quad (5.7)$$

This result is identical to the shot-noise result quoted in QVII(4.12). It is also easy to show that within the scope of our adiabatic approximation

$$\langle \Delta b\Delta D\rangle/\Delta t\approx 0. \quad (5.8)$$

The troublesome next-to-last term in (5.2) is evaluated explicitly in

$$\langle \Delta b^\dagger b^s \Delta D\rangle/\Delta t=(\Delta t)^{-1}\int_t^{t+\Delta t}\frac{N\mu}{\Gamma(1-i\alpha)}\left\langle F_{21}(u)\,du\,b^s\int_t^{t+\Delta t}du'\left[F_{22}(u')-\frac{\mu}{\Gamma}\left(\frac{b^\dagger F_{12}}{1+i\alpha}+\frac{F_{21}b}{1-i\alpha}\right)\right]\right\rangle$$
$$=[N\mu/\Gamma(1-i\alpha)]b^s(2D_{2122}/N)-\tfrac{1}{2}\pi b^s b^\dagger(2D_{2112}/\Gamma)$$
$$=-[\mu w_{21}/\Gamma(1-i\alpha)]b^s\sigma^\dagger-\pi b^s b^\dagger D\approx-\pi b^s b^\dagger D, \quad (5.9)$$

where the omission of the first term follows from the adiabatic approximation (3.3). The vanishing of the means

$$\langle F_{12}(s)F_{12}(s')\rangle=\langle F(s)F(s')\rangle=0 \quad (5.10)$$

is sufficient to ensure that the second moments

$$\langle(\Delta b)^2\rangle=\langle(\Delta b^\dagger)^2\rangle=0 \quad (5.11)$$

vanish. This explains the omission of these terms in (5.2). Our operator equation of motion can now be written in the form

$$\partial\langle(b^\dagger)^r b^s f(D)\rangle/\partial t=\langle r(b^\dagger)^{r-1}(A_b)^\dagger b^s f(D)\rangle+\langle(b^\dagger)^r s b^{s-1}A_b f(D)\rangle$$
$$+\langle(b^\dagger)^r b^s[w_{20}(1-dD)-\Gamma_2 D-\pi b b^\dagger D]\partial f/\partial D\rangle+\langle r(b^\dagger)^{r-1}s b^{s-1}[\gamma\bar{n}+\pi N D]f(D)\rangle$$
$$-\langle r(b^\dagger)^{r-1}b^s(\pi b^\dagger D)\partial f/\partial D\rangle+\langle(b^\dagger)^r b^s N^{-1}[w_{20}(1-dD)+\Gamma_2 D+\pi b b^\dagger D]\tfrac{1}{2}\partial^2 f/\partial D^2\rangle. \quad (5.12)$$

We shall now try to rearrange the operators in (5.12) so that they appear in normal order. If we set $M=b^{\dagger r}b^s$, then we can write

$$Mbb^\dagger=(b^\dagger)^{r+1}b^{s+1}+(b^\dagger)^r s b^s+(b^\dagger)^r b^s$$
$$=b^\dagger M b+\partial M/\partial b\; b+M. \quad (5.13)$$

Rearranging the terms in this way, (5.12) then takes the form

$$\partial\langle M(b,b^\dagger)f(D)\rangle/\partial t=\langle A_b{}^\dagger\partial M/\partial b^\dagger f\rangle+\langle\partial M/\partial b\; A_b f\rangle$$
$$+\langle\{M[w_{20}(1-dD)-\Gamma_2 D]-\pi[b^\dagger M b+\partial M/\partial b\; b+M]D\}\;\partial f/\partial D\rangle$$
$$+\langle\partial^2 M/\partial b\partial b^\dagger[\gamma\bar{n}+\pi N D]f(D)\rangle-\pi\langle(b^\dagger\partial M/\partial b^\dagger+\partial^2 M/\partial b\partial b^\dagger)D\partial f/\partial D\rangle$$
$$+\langle[w_{20}M(1-dD)+M\Gamma_2 D+\pi D(b^\dagger M b+\partial M/\partial b\; b+M)]N^{-1}\tfrac{1}{2}\partial^2 f/\partial D^2\rangle. \quad (5.14)$$

Since an arbitrary operator can be written as a linear combination of operators of the form $b^{\dagger r}b^s$, we may regard (5.14) as valid for an operator M that is an arbitrary function of b and $b^\dagger$.

6. DENSITY-MATRIX EQUATION

The equation of motion of an arbitrary operator is clearly equivalent to a knowledge of the equation of motion of the density matrix. Indeed, the relation between these equations can be obtained by comparing (D2) and (D3) of QIV. Since, however, we are also interested in obtaining an equation for the associated classical function we shall proceed by a method which leads directly to the latter objective. We may as well assume that the operator M has been written in normal order. In that case (1.5) can be rewritten in the form of an integral

$$\langle M^{(n)}(b,b^\dagger)f(D)\rangle=\int dD f(D)\;\mathrm{Tr}M^{(n)}(b,b^\dagger)\rho(b,b^\dagger,D,t)$$
$$=\int dD\int d^2\beta f(D)M^{(n)}(\beta,\beta^*)P(\beta,D,t). \quad (6.1)$$

Similarly, the time derivative of a general operator is expressible in terms of the time derivative of the density matrix so that

$$\partial\langle M^{(n)}(b,b^\dagger)f(D)\rangle/\partial t=\int dD\int d^2\beta f(D)M^{(n)}(\beta,\beta^*)\partial P(\beta,D,t)/\partial t. \quad (6.2)$$

We next replace the left-hand side of (6.2) by (5.14). The left-hand side is then written in integral form following the prescription (6.1). All derivatives acting on M are then integrated by parts so that they act instead on the density matrix. The arbitrary function $M^{(n)}(\beta,\beta^*)$ then appears as a common factor and its coefficient must necessarily vanish. In this way we obtain the equation for the associated classical distribution function quoted in the Introduction as (1.11). Since that equation was already written with the parameters β and β^* in an antinormal order, it can immediately be rewritten in the form of an operator equation;

$$\partial\rho^{(a)}\partial t=-\partial[\rho^{(a)}A_b{}^\dagger]/\partial b^\dagger-\partial[A_b\rho^{(a)}]/\partial b-\partial\{[w_{20}(1-dD)-\Gamma_2 D-\pi D]\rho^{(a)}-\pi b D\rho^{(a)}b^\dagger\}/\partial D$$
$$+(\gamma\bar{n}+\pi N D)\partial^2\rho^{(a)}/\partial b\partial b^\dagger-\pi\partial^2(\rho^{(a)}Db^\dagger)/\partial b^\dagger\partial D-\pi\partial^2(bD\rho^{(a)})/\partial b\partial D+(2N)^{-1}\partial^2\{[w_{20}(1-dD)$$
$$+\Gamma_2 D+\pi D]\rho^{(a)}+\pi b\rho^{(a)}b^\dagger\}/\partial D^2+\pi\partial^3(D\rho^{(a)})/\partial b\partial b^\dagger\partial D-(\pi/2N)\partial^3(bD\rho^{(a)})/\partial b(\partial D)^2; \quad (6.3a)$$

or setting $N_2=ND$, $R_2=Nw_{20}(1-dD)$, and omitting the unnecessary (a) superscript (since the above equation must be true whether ρ is in antinormal order or not), we have

$$\partial\rho/\partial t=-i(\omega_c-\omega_0)[1-(D/D_0)][b^\dagger b,\rho]+\tfrac{1}{2}\gamma[\partial(b\rho)/\partial b+\partial(\rho b^\dagger)/\partial b^\dagger]+\gamma\bar{n}\partial^2\rho/\partial b\partial b^\dagger+[-\partial/\partial N_2+\tfrac{1}{2}\partial^2/\partial N_2{}^2-\cdots]$$
$$\times(R_2\rho)+\Gamma_2[\partial/\partial N_2+\tfrac{1}{2}\partial^2/\partial N_2{}^2+\cdots](N_2\rho)-\tfrac{1}{2}\pi N_2[\partial(b\rho)/\partial b+\partial(\rho b^\dagger)/\partial b^\dagger-2\partial^2\rho/\partial b\partial b^\dagger]$$
$$+\pi[\partial/\partial N_2+\tfrac{1}{2}\partial^2/\partial N_2{}^2+\cdots]N_2[b\rho b^\dagger-\partial(b\rho)/\partial b-\partial(\rho b^\dagger)/\partial b^\dagger+\partial^2\rho/\partial b\partial b^\dagger+\rho]. \quad (6.3b)$$

The first form (6.3a) has some but not all third-derivative terms. The relation between first-, second-, and third-derivative terms (when available) is just that due to shot noise: "Rate in" contributions appear with the same sign in all orders, "rate out" contributions alternate in sign. [See, for example, IV(9.4), IV(9.73).] We can then immediately extrapolate to all orders. The shot noise merely tells us that $N_2 \equiv ND$ must take on integral values, and our sums of derivatives add up to $\exp[\pm\partial/\partial N_2]$, so that we get the difference equation

$$\partial\rho/\partial t = -i(\omega_c-\omega_0)[1-D/D_0][b^\dagger b,\rho]+\tfrac{1}{2}\gamma[2b\rho b^\dagger-(b^\dagger b\rho+\rho b^\dagger b)]+\gamma\bar{n}[b,[\rho,b^\dagger]]$$
$$+[\exp(-\partial/\partial N_2)-1](R_2\rho)+[\exp(\partial/\partial N_2)-1](\Gamma_2 N_2\rho)$$
$$-\tfrac{1}{2}\pi N_2[\rho bb^\dagger+bb^\dagger\rho-2b^\dagger\rho b]+\pi[\exp(\partial/\partial N_2)-1](N_2 b^\dagger\rho b). \quad (6.3c)$$

The series of terms $(\partial/\partial N_2)^r$ converges as $(\bar{N}_2)^{-r}$ where $\bar{N}_2$ is the mean number of atoms in the excited state, so that terms with $r>2$ can be neglected with impunity since $\bar{N}_2 \sim \gamma/\pi \gg\gg 1$. We retain these terms here, however, because of the simplicity of the difference equation that results when one uses

$$\exp[\pm\partial/\partial N_2]\rho(N_2)=\rho(N_2\pm 1).$$

Density-Matrix Equation in the n Representation

Equation (6.3) has such a simple form that by taking its mn matrix element one can immediately obtain the equation of motion for the density matrix in the n representation;

$$\partial\rho_{mn}/\partial t = -i(\omega_c-\omega_0)(1-D/D_0)(m-n)\rho_{mn}+\tfrac{1}{2}\gamma[2(m+1)^{1/2}(n+1)^{1/2}\rho_{m+1,n+1}-(m+n)\rho_{mn}]$$
$$+\gamma\bar{n}[(m+1)^{1/2}(n+1)^{1/2}\rho_{m+1,n+1}+(mn)^{1/2}\rho_{m-1,n-1}-(m+n+1)\rho_{mn}]+R_2[\exp(-\partial/\partial N_2)-1]\rho_{mn}$$
$$+\Gamma_2[\exp(\partial/\partial N_2)-1]N_2\rho_{mn}-\tfrac{1}{2}\pi N_2[(m+n+2)\rho_{mn}-2(mn)^{1/2}\rho_{m-1,n-1}]$$
$$+\pi[\exp(\partial/\partial N_2)-1]N_2(mn)^{1/2}\rho_{m-1,n-1}. \quad (6.4a)$$

The diagonal ($m=n$) form of (6.4a) agrees precisely with QVII (5.7), and is just the difference equation one would write down intuitively, with the given transition probabilities

$$\partial\rho_{nn}/\partial t=[\exp(-\partial/\partial N_2)-1](R_2\rho_{nn})+[\exp(\partial/\partial N_2)-1](\Gamma_2 N_2\rho_{nn})+\gamma(\bar{n}+1)[(n+1)\rho_{n+1,n+1}-n\rho_{n,n}]$$
$$+\gamma\bar{n}[n\rho_{n-1,n-1}-(n+1)\rho_{n,n}]+\pi[\exp(\partial/\partial N_2)N_2 n\rho_{n-1,n-1}-N_2(n+1)\rho_{nn}]. \quad (6.4b)$$

Associated Classical Langevin Equations for $b^\dagger$, b, D System

An alternative approach more closely related to the spirit of the present paper is to take the Fokker-Planck equation (1.11) obeyed by the associated distribution function and write the classical Langevin equations that yield this particular Fokker-Planck process. By comparing III(5.13) and IV(10.18)–(10.20), we see that the nonrandom drift vectors of the Langevin process can be read off as the coefficients that appear in the first-derivative terms of the Fokker-Planck equation, and the moments of the random force can be read off from the coefficients of the second derivatives in the Fokker-Planck equation. Thus we obtain the process

$$d\beta/dt=-\tfrac{1}{2}\gamma(1-i\alpha)\beta+\tfrac{1}{2}\pi(1-i\alpha)\beta ND+F_\beta(t), \quad (6.5)$$
$$d\beta^*/dt=-\tfrac{1}{2}\gamma(1+i\alpha)\beta^*+\tfrac{1}{2}\pi(1+i\alpha)\beta^* ND+F_{\beta^*}(t), \quad (6.6)$$
$$dD/dt=w_{20}(1-dD)-\Gamma_2 D-\pi(I+1)D+F_D(t), \quad (6.7)$$
$$I\equiv\beta^*\beta=|\beta|^2 \quad (6.8)$$
$$\langle F_{\beta^*}(t)F_\beta(u)\rangle=\langle F_\beta(t)F_{\beta^*}(u)\rangle=2D_{\beta\beta^*}\delta(t-u), \quad (6.9)$$
$$2D_{\beta\beta^*}=\gamma\bar{n}+\pi N_2=2D_{\beta^*\beta}; \qquad 2D_{\beta\beta}=2D_{\beta^*\beta^*}=0, \quad (6.10)$$
$$2D_{\beta D}=-\pi\beta D; \qquad 2D_{\beta^* D}=-\pi\beta^* D, \quad (6.11)$$
$$2D_{DD}=[w_{20}(1-dD)+\Gamma_2 D+\pi(1+I)D]/N. \quad (6.12)$$

7. POPULATION AND PHOTON-NUMBER FOKKER-PLANCK EQUATION

If we are unconcerned with phase fluctuations we can seek a solution of the Fokker-Planck equation (1.11) in the form

$$P(\beta,\beta^*,D,t)\equiv P(I,D,t), \quad (7.1)$$

in other words a solution that is a function of β and β^* only in the combination I. Making use of the simplifications

$$\partial(\beta P)/\partial\beta=(\partial/\partial I)(IP)=(\partial/\partial\beta^*)(\beta^* P), \tag{7.2}$$

we can rewrite (1.9) in the form

$$\partial P(I,D,t)/\partial t=-\partial[(-\gamma I+IND)P]/\partial I-\partial\{[w_{20}(1-dD)-\Gamma_2 D-\pi(I+1)D]P\}/\partial D$$
$$+\partial^2[(-2\pi ID)P]/\partial I\partial D+(\gamma\bar{n}+N\pi D)(\partial/\partial I)[I(\partial P/\partial I)]+(2N)^{-1}\partial^2\{[w_{20}(1-dD)+\Gamma_2 D+\pi(I+1)D]P\}/\partial D^2$$
$$+\pi\frac{\partial}{\partial I}\left(I\frac{\partial}{\partial I}\frac{\partial}{\partial D}DP\right)-\frac{1}{2}\frac{\pi}{N}\frac{\partial^2}{\partial D^2}\frac{\partial}{\partial I}(IDP). \tag{7.3}$$

We next rearrange this equation so that all functions of I and D appear inside the derivatives, in order to obtain the Fokker-Planck form of this equation. Note that in so doing we have retained third-derivative terms, which, because of the rearrangement, affect the resulting second-derivative terms. Our final result is

$$\partial P/\partial t=-\partial\{[\gamma\bar{n}-\gamma I+\pi(I+1)ND]P\}/\partial I-\partial\{[w_{20}(1-dD)-\Gamma_2 D-\pi(I+1)D]P\}/\partial D$$
$$+(2N)^{-1}\partial^2\{[w_{20}(1-dD)+\Gamma_2 D+\pi(I+1)D]P\}/\partial D^2+\partial^2[(\gamma\bar{n}+\pi ND)IP]/\partial I^2$$
$$+\partial^2[-\pi(2I+1)DP]/\partial I\partial D+\pi(\partial/\partial I)(\partial/\partial D)\{[(\partial/\partial I)(ID)-(2N)^{-1}(\partial/\partial D)(ID)]P\}. \tag{7.4}$$

The ones which appear in combination with I in the first and next-to-last terms of (7.4) arise because of these rearrangements. Aside from this small effect, we can neglect third derivatives completely.

Equivalent Langevin Formulation

We now ignore the third-derivative term that appears in (7.4) and write the equivalent classical Langevin equations that give rise to the Fokker-Planck equation (7.4). Our results have the form

$$dI/dt=A_I+F_I, \tag{7.5}$$

$$dN_2/dt=A_N+F_N; \qquad N_2\equiv ND, \tag{7.6}$$

$$A_I=-\gamma(I-\bar{n})+\pi(I+1)N_2, \tag{7.7}$$

$$A_N=w_{20}(N-dN_2)-[\Gamma_2 N_2+\pi(I+1)N_2], \tag{7.8}$$

$$D_{II}=(\gamma\bar{n}+\pi N_2)I, \tag{7.9}$$

$$D_{NN}=\tfrac{1}{2}\{[Nw_{20}-dw_{20}N_2]+[\Gamma_2 N_2+\pi(I+1)N_2]\}, \tag{7.10}$$

$$D_{IN}=D_{NI}=-\tfrac{1}{2}\pi(2I+1)N_2. \tag{7.11}$$

These are the results quoted without proof[23] at the beginning of Sec. 9 of QVII. In that paper, a quasilinear analysis of these equations was used to obtain the total noise and the spectrum of intensity correlation noise. To save space these results will not be repeated here.

8. TRANSFORMATION TO I, ϕ, D REPRESENTATION

Under a transformation from one set of random variables a_m to another set a_i', the drift vectors were shown in IV(3.28) to transform according to

$$A_i'=\partial a_i'/\partial t+(\partial a_i'/\partial a_k)A_k+\partial^2 a_i'/\partial a_m\partial a_n D_{mn}. \tag{8.1}$$

[23] These results were discussed at the Phoenix International Conference on Quantum Electronics, April 1966 as part of our presentation of Papers QVII and QIX.

Using (8.1), we make an explicit calculation of the photon drift vector

$$A_I=(\partial I/\partial\beta)A_\beta+(\partial I/\partial\beta^*)A_{\beta^*}+2(\partial^2 I/\partial\beta\partial\beta^*)D_{\beta\beta^*}$$
$$=-\gamma I+\pi IND+(\gamma\bar{n}+\pi N_2)$$
$$=\gamma(\bar{n}-I)+\pi(I+1)ND. \tag{8.2}$$

In a similar way we find the remaining drift vectors

$$A_\phi=\tfrac{1}{2}\alpha(\pi ND-\gamma), \tag{8.3}$$

$$A_D=w_{20}(1-dD)-\Gamma_2 D-\pi(I+1)D. \tag{8.4}$$

Diffusion Coefficients in I, ϕ, D Representation

Equation IV(3.27) tells us that the corresponding transformation rule for diffusion coefficients is given by

$$D'_{ij}=\partial a'_i/\partial a_m(\partial a'_j/\partial a_n)D_{mn}. \tag{8.5}$$

Applying this rule yields the complete set of diffusion coefficients

$$D_{II}=2\beta\beta^* D_{\beta\beta^*}=I(\gamma\bar{n}+\pi N_2), \tag{8.6}$$

$$D_{DD}=\tfrac{1}{2}[w_{20}(1-dD)+\Gamma_2 D+\pi(I+1)D]/N,$$

unchanged (8.7)

$$D_{\phi\phi}=2(\partial\phi/\partial\beta)(\partial\phi/\partial\beta^*)D_{\beta\beta^*}=(4I)^{-1}(\gamma\bar{n}+\pi N_2), \tag{8.8}$$

$$D_{I\phi}=D_{\phi I}=D_{D\phi}=D_{\phi D}=0, \tag{8.9}$$

$$D_{ID}=\beta^* D_{\beta D}+\beta D_{\beta^* D}=-\pi ID. \tag{8.10}$$

These results could also have been obtained by making the transformation directly on the differential equation (1.9) with the omission of the third-derivative terms. The only change induced by retaining the third-

derivative terms is the replacement of the last result, (8.10), by

$$D_{ID}=-\pi(I+\tfrac{1}{2})D, \tag{8.11}$$

in agreement with the corresponding term in (7.4).

9. ADIABATIC ELIMINATION OF POPULATION DIFFERENCE D

As a check on the results of QIX, we wish to make an adiabatic elimination of the population difference D by assuming that Γ_2 is large compared to γ. We start from the Langevin equations (6.5)–(6.7) and obtain an adiabatic approximation for D by setting $dD/dt=0$ in the last equation. In this way we obtain the elimination

$$D=J(w_{20}+F_D); \quad J\equiv[\Gamma_2+dw_{20}+\pi(I+1)]^{-1}. \tag{9.1}$$

Inserting this result into (6.5) yields

$$d\beta/dt=-\tfrac{1}{2}\gamma(1-i\alpha)\beta+\tfrac{1}{2}N\pi(1-i\alpha)\beta Jw_{20}+F_\beta{}^{\rm ad}, \tag{9.2}$$

where the random force is defined by

$$F_\beta{}^{\rm ad}\equiv F_\beta+\tfrac{1}{2}N\pi(1-i\alpha)J\beta F_D. \tag{9.3}$$

We have calculated the mean of this new random force and found that

$$\langle F_\beta{}^{\rm ad}\rangle\approx(\pi/\Gamma_2)[\text{leading terms in } d\beta/dt]\approx 0, \tag{9.4}$$

so that the new drift vector takes the simple form

$$A_\beta=-\tfrac{1}{2}\gamma(1-i\alpha)\beta+\tfrac{1}{2}N\pi(1-i\alpha)\beta Jw_{20}; \quad A_{\beta*}=A_\beta{}^*. \tag{9.5}$$

Using (9.3) we can write the second moments of our noise sources in the form

$$\langle F_\beta{}^{\rm ad}(t)^*F_\beta{}^{\rm ad}(u)\rangle=\langle F_{\beta*}{}^{\rm ad}(t)F_\beta{}^{\rm ad}(u)\rangle =2D_{\beta*\beta}{}^{\rm ad}\delta(t-u), \text{ etc.,} \tag{9.6}$$

where

$$\begin{aligned}2D_{\beta*\beta}{}^{\rm ad}&=2D_{\beta\beta*}{}^{\rm ad}\\ &=\gamma\bar{n}+\pi N_2(1-\pi IJ)+\tfrac{1}{2}(1+\alpha^2)\pi N_2(\pi IJ)U\\ &\doteq\gamma\bar{n}+\pi N_2-\tfrac{1}{2}\pi N_2JI(1-\alpha^2),\end{aligned} \tag{9.7}$$

where in these expressions N_2 or D are merely abbreviations for

$$N_2\equiv ND\equiv Nw_{20}J=Nw_{20}[\Gamma_2+dw_{20}+\pi(I+1)]^{-1} \tag{9.8}$$

and

$$U=\{w_{20}(1-dD)+[\Gamma_2+(I+1)]D\}/(2w_{20})\doteq 1. \tag{9.9}$$

The dotted equal sign in (9.9) means that the parameter U reduces to unity if depletion is neglected. This conclusion can be seen by inserting the appropriate value (9.8) for D into (9.9). The second expression for $D_{\beta\beta*}$ results when we set $U=1$ into the first. The two remaining diffusion coefficients are given by

$$\begin{aligned}2D_{\beta\beta}{}^{\rm ad}&=-\pi N_2(1-i\alpha)\pi\beta^2J^2[1-\tfrac{1}{2}U(1-i\alpha)]\\ &\doteq-\tfrac{1}{2}\pi N_2(1+\alpha^2)\pi\beta^2J^2,\end{aligned} \tag{9.10a}$$

$$\begin{aligned}2D_{\beta*\beta*}{}^{\rm ad}&=-\pi N_2(1+i\alpha)\pi\beta^{*2}J^2[1-\tfrac{1}{2}U(1+i\alpha)]\\ &\doteq-\tfrac{1}{2}\pi N_2(1+\alpha^2)\pi\beta^{*2}J^2.\end{aligned} \tag{9.10b}$$

These diffusion coefficients are, of course, complex conjugates of one another. The drift vectors and diffusion coefficients given in (9.5)–(9.10) define a Fokker-Planck equation in accord with the usual formula III(5.13). We shall not write this equation until (10.1), but we shall remark that to obtain this equation directly from (1.11) would require the use of a Born-Oppenheimer approximation. The motion associated with D in the differential equation would have to be solved for regarding β and β^* as adiabatic parameters. The determination of this quasistatic solution and its elimination to obtain an effective Fokker-Planck equation for the variables β and β^* requires an amount of effort that is likely to be significantly greater than our rather trivial procedures using the Langevin equations.

I, ϕ Formulation

In order to discuss intensity and phase fluctuations it is convenient to transform from the variables β and β^* to the variables I and ϕ. Applying the transformation equation (8.1), we find our drift vectors to be

$$\begin{aligned}A_I&=\beta^*A_\beta+\beta A_{\beta*}+2D_{\beta\beta*}\\ &\approx\gamma(\bar{n}-I)+\pi N_2(I+1),\end{aligned} \tag{9.11}$$

$$\begin{aligned}A_\phi&=(\tfrac{1}{2}i/\beta)A_\beta-(\tfrac{1}{2}i/\beta^*)A_{\beta*}-(\tfrac{1}{2}i)D_{\beta\beta}/\beta^2+\tfrac{1}{2}iD_{\beta*\beta*}/\beta^{*2}\\ &=\tfrac{1}{2}\alpha[N_2w_{20}J-\gamma].\end{aligned} \tag{9.12}$$

Making use of the transformation equation (8.5) we find our diffusion coefficients to take the new forms

$$\begin{aligned}D_{II}&=\gamma\bar{n}I+\pi N_2I[1-\pi IJ(2-U)]\\ &\doteq\gamma\bar{n}I+\pi N_2I(1-\pi IJ),\end{aligned} \tag{9.13}$$

$$D_{\phi\phi}=(4I)^{-1}[\gamma\bar{n}+\pi N_2(1+\pi IUJ\alpha^2)], \tag{9.14}$$

$$D_{I\phi}\approx\tfrac{1}{2}\alpha\pi N_2I(\pi J)(U-1)\doteq 0. \tag{9.15}$$

Quasilinear Treatment of Amplitude Fluctuations

The operating point, which we shall call $\bar{I}$ to conform with the notation of QVII, can be determined by setting the drift vector of (9.11) equal to zero. When the expression (9.8) for N_2 is introduced, this yields precisely the same operating point as that found in Sec. 6 of QVII. The decay parameter of a quasilinear treatment (see[2] I) is given by the negative derivative of

the drift vector at the operating point;

$$\Lambda_a=-\partial A_I{}^{\rm ad}/\partial I|_{I=\bar I}$$
$$=\gamma\{1-(N\pi/\gamma)(Jw_{20})[1-\pi(I+1)J]\}|_{I=\bar I}$$
$$=\frac{\gamma\pi(\bar I-\bar n)}{\Gamma_2+dw_{20}+\pi(\bar I+1)}+\frac{\gamma(\bar n+1)}{\bar I+1}. \tag{9.16}$$

This result for the half-width Λ_a associated with intensity fluctuations is identical to the slow eigenvalue found in QVII(7.8). This is as it should be since only the slow eigenvalue remains in the latter problem after an adiabatic approximation has been made to eliminate the fast eigenvalue. Over a broad range of operating points that includes the threshold value $(\Gamma_2/2\pi)^{1/2}$, this eigenvalue Λ_a reduces to the form

$$\Lambda_a\approx\gamma[(\pi\bar I/\Gamma_2)+(\bar n+1)/\bar I] \quad \text{if} \quad 1\ll\bar I\ll\Gamma_2/\pi. \tag{9.17}$$

After taking account of the scaling performed in Sec. 5 of QIX, this result reduces identically to the last form in V(9.16). This last form is referred to in V as the intelligent quasilinear approximation because it has been expressed in terms of the operating point $\bar I$, and when compared with experiment one would presumably use not the quasilinear expression QVII(6.5) for this operating point, but simply the experimental number of photons. Figure 1 shows a comparison between the half-width Λ_a calculated in this intelligent quasilinear way and the exact value for this parameter obtained by finding the lowest eigenvalue of the Fokker-Planck equation associated with the drift vector A_I and the diffusion constant D_{II}. The numerical methods of evaluating Λ_a and also of making a direct evaluation of the spectrum of noise associated with noise intensity fluctuations are discussed in VI.

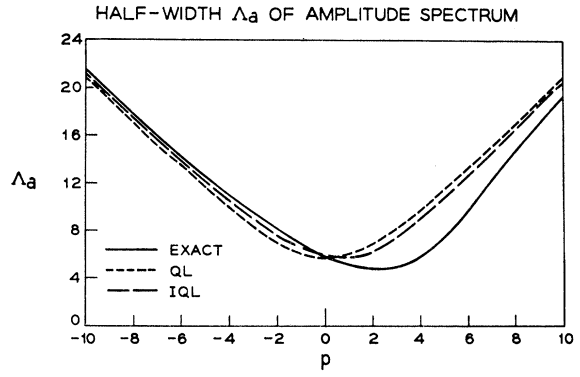

FIG. 1. The dimensionless amplitude decay eigenvalue $(\Lambda_a)^{\rm dim}=\Lambda_a T$ is plotted against the pump parameter

$$p=\tfrac{1}{2}\gamma T[D_w/D_0-1],$$

where $D_w=w_{20}/(\Gamma_2+dw_{20})$ is the population inversion that would be produced by the pump if no radiation could occur, and $D_0=\gamma/(N\pi)$ of (5.6) is the minimum inversion needed for masing. The unit of time T is defined by

$$\tfrac{1}{2}\gamma T=(\Gamma_2/\pi)^{1/2}[2/(\bar n+1)]^{1/2},$$

where π is the photon rate constant of (3.4). If one scales the photon number by

$$\bar I=\xi^2\bar p; \qquad \xi^2\equiv[(\bar n+1)\Gamma_2/(2\pi)]^{1/2},$$

the quasilinear approximation (9.17) reduces in dimensionless form to the result V (9.16)

$$(\Lambda_a)^{\rm dim}=2\bar p+(4/\bar p)$$

appropriate to the rotating-wave Van der Pol oscillator.

The quasilinear approximation $(\Lambda_a)^{\rm QL}=2(p^2+8)^{1/2}$ results when $\bar p=\rho_0$ is given its quasilinear value $\frac{1}{2}[p+(p^2+8)^{1/2}]$ of V (11.12); the intelligent quasilinear value $(\Lambda_a)^{\rm IQL}=2\langle\rho\rangle+4/\langle\rho\rangle$, where $\langle\rho\rangle$ is the exact mean value of ρ in the ground state. [See QIX (5.29) and Fig. 4 of V, or Fig. 2 of QVII, or QIX.] The exact half-width $(\Lambda_a)^{\rm dim}$ is the eigenvalue associated with the first nonvanishing eigenvalue Λ of the Fokker-Planck equation appropriate to amplitude fluctuations, i.e., eigenfunction independent of phase ϕ;

$$d^2R/d\rho^2+[\rho^{-1}-\tfrac{1}{2}(p-\rho)](dR/d\rho)+[1+(\Lambda-2p)/4\rho]R=0.$$

See QIX (5.27), V (Fig. 1), and VI.

A quasilinear approximation for the total noise associated with intensity fluctuations is given by the Einstein relation I(5.20)

$$\langle(\Delta I)^2\rangle\approx D_{II}{}^{\rm ad}/\Lambda_a. \tag{9.18a}$$

Over the same broad intermediate range, this second moment can be simplified to the form

$$\langle(\Delta I)^2\rangle\approx\frac{(\bar n+1)\bar I^2}{(\bar n+1)+(\pi\bar I^2)/\Gamma_2}; \qquad 1\ll\bar I\ll\Gamma_2/\pi, \tag{9.18b}$$

which corresponds precisely to the form V(10.27). At very high operating levels (9.18a) reduces to

$$\langle(\Delta I)^2\rangle\to\bar n I+(\Gamma_2/\pi) \qquad \bar I\gg\Gamma_2/\pi, \qquad d=0 \tag{9.18c}$$

in agreement with QVII(9.11) and QVII(8.10) when we assume $\Gamma_2\gg\gamma$ in the latter. It should be noted that when $\bar n$ can be neglected the shot noise that is present in photon-number fluctuations is not present in the total noise associated with I. Compare with QVII(8.10).

Phase Fluctuations above Threshold

By introducing the definitions

$$I=\exp 2u; \qquad \cos\beta'=(1+\alpha^2)^{-1/2};$$
$$w=\phi\cos\beta'-u\sin\beta', \tag{9.19}$$

the Langevin equations for amplitude and phase fluctuations can be transformed to the form

$$du/dt\approx\tfrac{1}{2}(\pi N_2-\gamma)+(2I)^{-1}F_I, \tag{9.20}$$
$$d\phi/dt=\tfrac{1}{2}\alpha(\pi N_2-\gamma)+F_\phi. \tag{9.21}$$

The particular linear combination w has been so chosen that the drift associated with this phase vanishes;

$$dw/dt=F_\phi\cos\beta'-F_I(2I)^{-1}\sin\beta'=F_w. \tag{9.22}$$

As a result, the phase w executes a simple Brownian motion described by

$$\langle[w(t)-w(0)]^2\rangle/t=t^{-1}\int_0^t ds\int_0^t ds'\langle F_w(s)F_w(s')\rangle$$
$$=\cos^2\beta'(2D_{\phi\phi})+\sin^2\beta'(2D_{II})/(4I^2). \tag{9.23}$$

Linewidth due to Phase Fluctuations

As discussed in V and QV, in the region well above threshold amplitude fluctuations are suppressed, but the line has a residual width associated with phase fluctuations. The full width at half-power of the Lorentzian describing the spectrum of these fluctuations [see, for example, QV(32) and (33)] is given by

$$2\Lambda_p = W = \langle [\phi(t)-\phi(0)]^2\rangle/t = (\cos\beta')^{-2}\langle [w(t)-w(0)]^2\rangle/t, \quad (9.24)$$

where the mean-square growth in w in accord with (9.23) is given by

$$\langle [w(t)-w(0)]^2\rangle/t = (\gamma\bar{n}+\pi N_2)/(2I)+\pi^2 N_2 J(U-1)\alpha^2/(1+\alpha^2) \doteq (\gamma\bar{n}+\pi\bar{N}_2)/(2\bar{I}). \quad (9.25)$$

The last form in (9.25) results when the average over I is performed by replacing the expression on the right-hand side of (9.25) by its quasilinear value. Our final result for the full width at half-power can be written

$$2\Lambda_p = W = (1+\alpha^2)(\gamma\bar{n}+\pi\bar{N}_2)/(2\langle b^\dagger b\rangle). \quad (9.26)$$

Comparison with Previous Calculations

The same width, as calculated in QV(34), is given by

$$W_{\mathrm{QV}} = (1+\alpha^2)(\Delta\omega)^2 S/(2\gamma\langle b^\dagger b\rangle), \quad (9.27)$$

where the half-width $\Delta\omega$ is defined by

$$\Delta\omega = \gamma\Gamma/(\tfrac{1}{2}\gamma+\Gamma)\to\gamma, \quad (9.28)$$

and takes the limiting value γ in the adiabatic limit. The parameter S is defined by

$$S=\tfrac{1}{2}(S_-+S_+); \qquad S_-=\bar{n}+(\tilde{\sigma}_{22}/D_0); \qquad S_+=\bar{n}+1+\sigma_{11}/D_0. \quad (9.29)$$

Since

$$\gamma\bar{n}+\pi\bar{N}_2=\gamma S_-, \quad (9.30)$$

our present result indicates that the appropriate linewidth above threshold is given by

$$W_{\text{above threshold}} = (1+\alpha^2)(\Delta\omega)^2 S_-/(2\gamma\langle b^\dagger b\rangle), \quad (9.31)$$

which differs from the earlier result (9.27) merely by the replacement of S by S_-. This correction, which is small above threshold, applies not only to our previous work in QV but also to the corresponding calculations of Haken and co-workers.[15] Although the replacement of S by S_- constitutes a minor correction when the number of photons is large, it is symptomatic of an important difference when the number of photons is small. In particular we see by examining the Fokker-Planck equation (1.11) that the relevant diffusion term takes the form

$$\gamma S_-\partial^2 P/\partial\beta\partial\beta^*, \quad (9.32)$$

whereas a quasiclassical analysis would have taken the form

$$\tfrac{1}{2}\gamma S_-\partial^2 P/\partial\beta^*\partial\beta+\tfrac{1}{2}\gamma S_+\partial^2 P/\partial\beta\partial\beta^*. \quad (9.33)$$

The difference between these two diffusion coefficients in the Fokker-Planck equation, γS_- and γS, is large below threshold. To see this difference we note that at absolute zero, if the pump is turned off, S_- approaches zero whereas S approaches the value unity. It is just these differences that arise from the commutation rules that make it risky to follow the procedure suggested by Risken, Schmid, and Weidlich.[15] Their procedure is to take a set of quantum-mechanical variables a_1, a_2, a_3, $\cdots$, and define a set of diffusion coefficients in the usual way QIV(1.6);

$$D_{ij} = \langle \Delta a_i \Delta a_j\rangle/(2\Delta t). \quad (9.34)$$

This diffusion coefficient is computed in a quantum-mechanical way and then it is inserted by Risken, Schmid, and Weidlich[15] in *ad hoc* fashion into a classical Fokker-Planck equation. Such a Fokker-Planck equation automatically makes use only of the symmetric part of D_{ij}, and thus discards the quantum-mechanical information contained in the antisymmetric part of D_{ij}. Of course, this procedure should be adequate above threshold where the number of photons is large and an essentially classical treatment is valid. This procedure is equivalent to replacing the original problem by a classical problem with c-number noise sources $F_i(t)$ whose correlations $\langle F_i(t)F_j(u)\rangle$ are given by $(D_{ij}+D_{ji})\delta(t-u)$. For many questions, such a semiclassical procedure is adequate. Indeed, this procedure was the basis of our early calculation of the phase linewidth.[24] These calculations were not published because it was felt that the proper task of a quantum theory of noise is to ascertain when such semiclassical procedures are valid. The Fokker-Planck equation derived in the present paper for the function $P(\beta, \beta^*, D, t)$ associated in a definite way with the density matrix $\rho(b, b^\dagger, D, t)$ is *not* precisely equivalent to the semiclassical Fokker-Planck equation for the variables b, $b^\dagger$, and D. The procedure of the present paper permits one to calculate the intensity correlation $\langle b^\dagger(0)b^\dagger(t)b(t)b(0)\rangle$ and to distinguish it from the number correlation $\langle b^\dagger(t)b(t)b^\dagger(0)b(0)\rangle$. Even in the region well above threshold, when the statistics are nearly classical, there is a significant difference between the spectra associated with these two correlations. [Compare QVII(8.15) and QVII(9.14).]

10. EQUATION FOR ADIABATIC DENSITY OPERATOR

Associated Classical Fokker-Planck Equation

In Sec. 6 we obtained the equation of motion for the density matrix in the variables $b^\dagger$, b, and D. In addition we obtained the equation of motion for the associated

[24] Presented at the 1964 Durham Conference on the Quantum Electrodynamics of High Intensity Photon Beams (unpublished).

classical distribution function in those variables. Replacing this classical Fokker-Planck problem by an equivalent Langevin problem, we were able in Sec. 9 to make an adiabatic elimination of the population difference D. The new drift and diffusion vectors after D had been adiabatically eliminated were given in (9.1)–(9.10). From these drift and diffusion vectors we can immediately write down the Fokker-Planck description of this same process in which D has been adiabatically eliminated.

$$\begin{aligned}\partial P/\partial t=&-\partial[(1-i\alpha)(-\tfrac{1}{2}\gamma+\tfrac{1}{2}N\pi w_{20}J)\beta P]/\partial\beta\\&+\tfrac{1}{2}\partial^2[(\gamma\bar{n}+N\pi w_{20}J)P]/\beta\partial\beta^*\\&-(\pi^2Nw_{20}/4)[(1-\alpha^2)\partial^2(\beta\beta^*J^2P)/\partial\beta\partial\beta^*\\&+(1+\alpha^2)\partial^2(\beta^2J^2P)/\partial\beta^2]\\&+\text{complex conjugate.}\end{aligned}\tag{10.1}$$

The classical function

$$P(\beta,\beta^*,t)\equiv\bar{\rho}^{(a)}(\beta,\beta^*,t)/\pi\tag{10.2}$$

is simply the classical function associated with the density matrix of the electromagnetic field. The symbol J in (10.1) is defined by (9.1) or in our present notation by

$$J=J(\beta\beta^*)=[\Gamma_2+dw_{20}+\pi+\pi\beta\beta^*]^{-1}.\tag{10.3}$$

In QIX Sec. 5 we showed that (10.1) reduces in the vicinity of threshold to the rotating wave Van der Pol oscillator. A detailed discussion of the rotating wave Van der Pol oscillator was provided in V, and exact numerical solutions were obtained in VI. From these numerical solutions we learned that the approximate quasilinear methods of V were adequate in a region well below or well above threshold. By "well above or well below," we mean roughly the regions in which there are at least ten times as many photons as at threshold or at most $\frac{1}{10}$ as many photons as at threshold. Since the number of photons at threshold has been shown in QIX to be of order $(\Gamma_2/\pi)^{1/2}\sim10^4$, and the deviations of (10.1) from a rotating wave Van der Pol oscillator do not become important until the number of photons is of order $\Gamma_2/\pi\sim10^8$, we can conclude that at these high operating levels a quasilinear analysis of (10.1) is adequate. Such a quasilinear analysis of the amplitude and phase fluctuations was already provided in the previous section. Conversely, in the region where a careful solution of the differential equation (10.1) is necessary, it is possible to reduce (10.1) to the rotating-wave Van der Pol oscillator. And the solutions found in VI are already adequate to cover the latter region.

It may be remarked that (10.1) was already presented in QIX(4.19). In QIX this equation was obtained by the elimination in one quantum-mechanical step of all atomic variables. In the present paper all variables but D were eliminated at first, and then the population difference D was eliminated in a second stage. A desirable feature of the present procedure in that the population difference was eliminated in a classical portion of the problem, so that possible ambiguities of order could not arise. The agreement obtained between (10.1) and QIX(4.19) is a check on the arithmetic and assumptions of both calculations.

Our work is complete in the sense that QIX(4.19) and (10.1) provide all the information necessary to calculate the desired statistical properties of the electromagnetic field. Scully and Lamb[17] have presented an equation for the density matrix in the number representation that should be comparable to our Eq. (10.1). Since their methods are not yet presented in detail, it is desirable to compare answers. The purpose of the present section will then be to convert (10.1) to an operator equation and thence to an equation for the density matrix.

Ordered-Density-Operator Equation

Equation (10.1) can at least formally be written as an operator equation

$$\begin{aligned}\partial\rho/\partial t=&\tfrac{1}{2}\gamma(1-i\alpha)\partial(b\rho)/\partial b-\tfrac{1}{2}(1-i\alpha)N\pi w_{20}\partial(b\hat{\mu})/\partial b\\&+\tfrac{1}{2}\gamma\bar{n}\partial^2\rho/\partial b\partial b^\dagger+\tfrac{1}{2}N\pi w_{20}\partial^2\hat{\mu}/\partial b\partial b^\dagger-(\pi^2Nw_{20}/4)\\&\times[(1-\alpha^2)\partial^2(b\hat{\nu}b^\dagger)/\partial b\partial b^\dagger+(1+\alpha^2)\partial^2(b^2\hat{\nu})/\partial b^2]\\&+\text{Hermitian adjoint.}\end{aligned}\tag{10.4}$$

I say formally because the symbols $\hat{\nu}$ and $\hat{\mu}$, although well defined by

$$\hat{\mu}\equiv\mathfrak{A}[J(\beta\beta^*)\bar{\rho}^{(a)}(\beta,\beta^*,t)],\tag{10.5}$$

$$\hat{\nu}\equiv\mathfrak{A}[J^2(\beta\beta^*)\bar{\rho}^{(a)}(\beta,\beta^*,t)],\tag{10.6}$$

have not yet been presented in a form that is easy to compute. The antinormal ordering symbol $\mathfrak{A}$ in (10.5) is the same as that defined in (1.1).

Feynman Ordering Notation

In order to facilitate the computation of $\hat{\mu}$ we shall introduce the Feynman ordering notation[25] in place of the antinormal ordering notation. In Feynman's notation, for example, (10.5) could be rewritten in the form

$$\hat{\mu}\equiv J(b_1b^\dagger{}_5)\rho_3.\tag{10.7}$$

In Feynman notation, it is understood that the operators appear in an order determined by their indexing symbol and not by the order in which they happen to be written. Thus if A, B, C, D, and E are any five possibly noncommuting operators, then

$$A_2B_4C_5D_1E_3\equiv DAEBC\tag{10.8}$$

is the manner in which the symbol on the left-hand side of (10.8) is to be interpreted.

To properly interpret (10.5), it would be necessary to make a complete expansion of (10.3) in powers of $\beta\beta^*$. Such an expansion would unfortunately not converge when the number of photons is large. Neverthe-

[25] R. P. Feynman, Phys. Rev. **84**, 108 (1951); F. Dyson, *ibid.* **75**, 486 (1949); **75**, 1736 (1949).

less, the first term in such an expansion, taken in the n representation, has the form

$$\begin{aligned}\langle m \mid b_1 b^\dagger{}_5 \rho_3 \mid n\rangle &= \langle m \mid b\rho b^\dagger \mid n\rangle \\ &= (m+1)^{1/2}(n+1)^{1/2}\langle m+1 \mid \rho \mid n+1\rangle \\ &\approx \tfrac{1}{2}[(m+1)+(n+1)]\langle m \mid \rho \mid n\rangle \\ &= \langle m \mid \tfrac{1}{2}(b_2 b^\dagger{}_2 + b_4 b^\dagger{}_4)\rho_3 \mid n\rangle. \end{aligned} \tag{10.9}$$

The approximation on the right-hand side of (10.9) makes use of the fact that the geometric mean of two large numbers with a small difference can be approximated by their arithmetic mean. It also makes use of the fact that the density matrix is a slowly-varying function of its indices when these indices are large. Equation (10.9) suggests that $b_1 b^\dagger{}_5$ be approximated by $\frac{1}{2}(b_2 b^\dagger{}_2 + b_4 b^\dagger{}_4)$. We note, however, that small frequency shifts lead to matrix elements of the form

$$\tfrac{1}{2}i\alpha'(m-n)\langle m \mid \rho \mid n\rangle = \langle m \mid \tfrac{1}{2}i\alpha'(b_2 b_2{}^\dagger - b_4 b_4{}^\dagger)\rho_3 \mid n\rangle. \tag{10.10}$$

To allow for such frequency shifts as well, we shall make the replacement

$$b_1 b^\dagger{}_5 \equiv \tfrac{1}{2}(1+i\alpha')b_2 b^\dagger{}_2 + \tfrac{1}{2}(1-i\alpha')b_4 b^\dagger{}_4 + \epsilon \equiv q+\epsilon. \tag{10.11}$$

The sum of the first two terms in (10.11), which is denoted briefly by q, represents a good approximation to $b_1 b^\dagger{}_5$. The symbol ϵ represents the small difference between $b_1 b^\dagger{}_5$ and its approximation q.

Drift-Correction Terms

The advantage of (10.11) is that it splits the operator $b_1 b^\dagger{}_5$ into two operators the first of which, q, is of the order of a typical number of photons and the second of which, ϵ, is the order unity. Thus we can expand J not about zero but about q;

$$J(q+\epsilon) \approx J(q) - \pi\epsilon J^2(q). \tag{10.12}$$

Inserting this approximation into (10.7) we obtain

$$\begin{aligned}\hat{\mu} &\approx J(q)\rho_3 - \pi J^2(q)[b_1 b^\dagger{}_5 - \tfrac{1}{2}(1+i\alpha')b_2 b^\dagger{}_2 \\ &\qquad - \tfrac{1}{2}(1-i\alpha')b_4 b^\dagger{}_4] \\ &= \mu - \pi b\nu b^\dagger + \tfrac{1}{2}\pi(1+i\alpha')bb^\dagger\nu + \tfrac{1}{2}\pi(1-i\alpha')\nu bb^\dagger, \end{aligned} \tag{10.13}$$

where μ and ν are defined by[26]

$$\begin{aligned}\mu &\equiv J(q)\rho_3 \\ &= J(\tfrac{1}{2}(1+i\alpha')b_2 b^\dagger{}_2 + \tfrac{1}{2}(1-i\alpha')b_4 b^\dagger{}_4)\rho_3, \end{aligned} \tag{10.14}$$

$$\begin{aligned}\nu &\equiv J^2(q)\rho_3 \\ &= [J(\tfrac{1}{2}(1+i\alpha')b_2 b^\dagger{}_2 + \tfrac{1}{2}(1-i\alpha')b_4 b^\dagger{}_4)]^2\rho_3. \end{aligned} \tag{10.15}$$

The expansion we have been using, (10.12), is a rapidly converging one since it is an expansion of powers of π/Γ_2. This justifies the neglect of all correction terms beyond the first in (10.12). Indeed, one may wonder why even the first correction term is necessary. The reason for this is that the diffusion terms in (10.4) are smaller than the drift terms by just this factor π/Γ_2. Thus the drift terms in (10.4) must be evaluated to this additional accuracy. However the diffusion terms do not require this accuracy and it is sufficient to introduce the approximation $\hat{\nu} \approx \nu$.

The last three terms in (10.13), the drift correction terms, can be rewritten in derivative form.

$$\begin{aligned}&2b\nu b^\dagger - (1+i\alpha')bb^\dagger\nu - (1-i\alpha')\nu bb^\dagger \\ &= (1+i\alpha')\partial(b\nu)/\partial b + (1-i\alpha')\partial(\nu b^\dagger)/\partial b^\dagger - 2\nu. \end{aligned} \tag{10.16}$$

When (10.16) is used in the second term of (10.4), the drift correction terms take the form

$$\begin{aligned}&(N\pi^2 w_{20}/4)(1-i\alpha)\times\partial[(1+i\alpha')\partial(b^2\nu)/\partial b \\ &\qquad + (1-i\alpha')\partial(b\nu b^\dagger)/\partial b^\dagger - (3+i\alpha')b\nu]/\partial b. \end{aligned} \tag{10.17}$$

The last term in (10.17) is a pure drift term. When it is compared in size with the dominant drift term, the first term in (10.4), we find the ratio

$$N\pi^2 w_{20} J^2/\gamma = (N\pi/\gamma)(w_{20}J)(\pi J) < \pi/\Gamma_2. \tag{10.18}$$

Thus we shall neglect the last term in (10.17).

If in (10.17) we set α' equal to α, we see that the first term cancels the last diffusion term in (10.4), and the Hermitian part of the second term in (10.17) cancels the next-to-last diffusion term in (10.4).

Density-Operator Equation

By the use of the cancellations mentioned above, (10.4) can be simplified to the form

$$\begin{aligned}\partial\rho/\partial t &= \tfrac{1}{2}(1-i\alpha)[\gamma\partial(b\rho)/\partial b - N\pi w_{20}\partial(b\mu)/\partial b] + \tfrac{1}{2}(1+i\alpha)[\gamma\partial(\rho b^\dagger)/\partial b^\dagger - N\pi w_{20}\partial(\mu b^\dagger)/\partial b^\dagger] \\ &\qquad + \gamma\bar{n}\partial^2\rho/\partial b\partial b^\dagger + N\pi w_{20}\partial^2\mu/\partial b\partial b^\dagger. \end{aligned} \tag{10.19}$$

Making use of the identities

$$-\partial(b\mu)/\partial b + \partial^2\mu/\partial b\partial b^\dagger = -\partial(\mu b)/\partial b, \tag{10.20}$$

$$\partial(\rho b^\dagger)/\partial b^\dagger - \partial(b\rho)/\partial b = b^\dagger b\rho - \rho b^\dagger b, \tag{10.21}$$

our density-operator equation can be reduced to the simple form

$$\begin{aligned}\partial\rho/\partial t &= \tfrac{1}{2}i\alpha\gamma[b^\dagger b, \rho] + \tfrac{1}{2}\gamma[\partial(b\rho)/\partial b + \partial(\rho b^\dagger)/\partial b^\dagger] - \tfrac{1}{2}N\pi w_{20}[(1+i\alpha)\partial(b^\dagger\mu)/\partial b^\dagger + (1-i\alpha)\partial(\mu b)/\partial b] \\ &\qquad + \gamma\bar{n}\partial^2\rho/\partial b\partial b^\dagger. \end{aligned} \tag{10.22}$$

[26] The temporary use of μ here as a kind of density operator should not cause confusion with our previous use of μ (see Sec. 2) as a parameter proportional to the electric dipole matrix element.

Density-Matrix Equation

Using (10.14), with α' set equal to α, the matrix elements of μ in the n representation are given simply by

$$\mu_{mn}=J_{mn}\rho_{mn}, \tag{10.23}$$

where J_{mn} is defined by

$$\begin{aligned}J_{mn}&=J\left(\tfrac{1}{2}(1+i\alpha)\,(m+1)+\tfrac{1}{2}(1-i\alpha)\,(n+1)\right)\\&=\left[\Gamma_2+dw_{20}+\pi+\tfrac{1}{2}\pi(m+1+n+1)+\tfrac{1}{2}i\alpha(m-n)\right]^{-1}.\end{aligned} \tag{10.24}$$

Our density-operator equation (10.22) can now be transformed immediately to a density-matrix equation. If all the terms proportional to $i\alpha$ are combined into one term, this density matrix equation takes the form

$$\begin{aligned}\partial\rho_{mn}/\partial t=&-i(m-n)\rho_{mn}(\omega_c-\omega_0)\left[1-(w_{20}J_{mn}/D_0)\right]-\tfrac{1}{2}N\pi w_{20}\left[(m+1+n+1)J_{mn}\rho_{mn}-2(mn)^{1/2}J_{m-1,n-1}\rho_{m-1,n-1}\right]\\&+\gamma\bar{n}\left[(m+1)^{1/2}(n+1)^{1/2}\rho_{m+1,n+1}+(mn)^{1/2}\rho_{m-1,n-1}-(m+n+1)\rho_{mn}\right].\end{aligned} \tag{10.25}$$

The first term in (10.25) was simplified by making use of the relation

$$\tfrac{1}{2}\gamma\alpha=\omega_0-\omega_c, \tag{10.26}$$

obtained from QV(13). Since our starting equations used ω_0 as the origin of frequency, the replacement of J_{mn} by a mean value in (10.25) leads to an approximate operating frequency of the form

$$\omega_0+\Delta\omega\approx\omega_0+(\omega_c-\omega_0)\,(1-\bar{D}/D_0), \tag{10.27}$$

which reduces to ω_c well below threshold and ω_0 well above threshold. The frequency shift found in (10.27) is in agreement with that found in QV(38)–(40). See also V(10.3).

If (10.22) is rewritten in matrix form without rearranging the terms, we obtain

$$\begin{aligned}\partial\rho_{mn}/\partial t=&-i(m-n)\rho_{mn}(\omega_c-\omega_0)-\left[(m+1)R_{mn}+(n+1)R_{nm}{}^*\right]\rho_{mn}+\left[R_{m-1,n-1}+R^*{}_{n-1,m-1}\right](mn)^{1/2}\rho_{m-1,n-1}\\&+\gamma\bar{n}\left[(m+1)^{1/2}(n+1)^{1/2}\rho_{m+1,n+1}+(mn)^{1/2}\rho_{m-1,n-1}-(m+n+1)\rho_{mn}\right],\end{aligned} \tag{10.28}$$

where

$$R_{mn}=\tfrac{1}{2}N\pi w_{20}(1+i\alpha)J_{mn}. \tag{10.29}$$

We have rewritten (10.26) in this alternative form (10.28) because the latter is directly comparable with Scully and Lamb.[17] The only difference is that their matrix R_{mn} is given in our notation by

$$R_{mn}=\tfrac{1}{2}N\pi w_{20}\left[(1+i\alpha)+(\tfrac{1}{2}\pi/\Gamma_1)\,(1+i\alpha^2)\,(m-n)\right]/\left[\Gamma_2+\tfrac{1}{2}\pi(m+n+2)\,(\Gamma_1+\Gamma_2)/\Gamma_1+\tfrac{1}{2}i\alpha\pi(m-n)\,(\Gamma_1-\Gamma_2)/\Gamma_1\right]. \tag{10.30}$$

In (10.30) a term in the denominator,

$$\tfrac{1}{4}(\pi^2/\Gamma_1)\,(m-n)^2\sim(\pi^2/\Gamma_1\Gamma_2)\times\Gamma_2\sim10^{-16}\Gamma_2, \tag{10.31}$$

was omitted because it is small, of the order of the terms neglected in Scully-Lamb analysis and in our analysis. Table I permits a comparison between the Scully-Lamb notation and our notation.

In the present paper, we have chosen a model in which the lower state is essentially empty, namely in which the decay rate Γ_1 is large. Thus we can simplify the Scully-Lamb results by making the replacements

$$(\Gamma_1+\Gamma_2)/\Gamma_1\to1, \tag{10.32}$$

$$(\Gamma_1-\Gamma_2)/\Gamma_1\to1. \tag{10.33}$$

In this case the Scully-Lamb matrix reduces to

$$R_{mn}\to\tfrac{1}{2}N\pi w_{20}(1+i\alpha)J_{mn}, \tag{10.34}$$

our result, provided that we neglect depletion in our expression for J_{mn}. A comparison of (10.28) with the corresponding Scully-Lamb equation indicates complete agreement in the region of overlap of the two theories. The Scully-Lamb theory must be specialized by letting Γ_1 approach infinity to agree with ours and our theory must be specialized by letting $\bar{n}=d=0$ to agree with theirs.

Our result (10.28) is, of course, a special case of our more general equation (6.4) for the density matrix when the population difference is not assumed to equilibrate fast compared to the field equilibration rates. A straightforward application of the methods discussed here has permitted Louisell and the author (in QXII) to generalize (6.4) to an equation for the

TABLE I. Comparison of notations.

Scully and Lamb	Lax
ω, ν, Δ	ω_a, ω_0, $\omega_a-\omega_0$
γ_a, γ_b, γ_{ab}	Γ_2, Γ_1, $\Gamma=\Gamma_{12}$
r_a, g	Nw_{20}, μ
Δ/γ_{ab}	$(\omega_a-\omega_0)/\Gamma=\alpha$
$C=\nu/Q$	γ
$2g^2/\gamma_{ab}$	$2\mu^2/\Gamma\equiv\pi(1+\alpha^2)$
A	$(Nw_{20}/\Gamma_2)\;(2\mu^2/\Gamma)$
B	$(2Nw_{20}/\Gamma_2^2)\;(2\mu^2/\Gamma)^2$
B/A	$4\mu^2/(\Gamma\Gamma_2)=2\pi(1+\alpha^2)/\Gamma_2$

variables b, $b^\dagger$, σ_{11}, and σ_{22}. Adiabatic elimination of σ_{11} and σ_{22} has also been accomplished. When the results are translated into the n representation, a direct comparison with the Scully-Lamb result (10.30) should be possible.

11. SUMMARY

Using the model for a maser developed in QIV and specializing to the case in which all atomic response times are fast compared to those of the field with the exception of that associated with the population difference, we have succeeded in deriving Eq. (6.3) for the density matrix $\rho(b,b^\dagger,D,t)$, where b and $b^\dagger$ are field variables and D is the normalized population difference. This density-matrix equation has been written in operator form and also in the n representation as (6.4). In addition we have shown that the operator density matrix can be obtained from an associated classical function $\bar{\rho}^{(a)}(\beta,\beta^*,D,t)$ by replacing β by b and β^* by $b^\dagger$, subject to the rule of placing the results in antinormal order. This correspondence between the density operator and its associated classical function is a dynamical one because the field operators are not assumed to propagate freely in time but rather are variables that obey a quantum-mechanical Markoff process. The equation of motion of $\bar{\rho}^{(a)}(\beta,\beta^*,D,t)$ is given by (1.11).

Although the density matrix is descriptive only of fluctuations at one time, we showed in Sec. 1 how to use a theorem developed in QII to evaluate the means of two-time operators and to show that these can also be obtained from appropriate solutions of a corresponding "classical" random process.

For the purpose of discussing amplitude and phase fluctuations we transformed the Fokker-Planck equation obeyed by our associated function to the new variables I and ϕ defined by $\beta=I^{1/2}\exp(-i\phi)$. As a result of this transformation we obtain the differential equation obeyed by $\bar{\rho}^{(a)}(I,\phi,D,t)$. [See Sec. 8.] For a discussion of amplitude fluctuations alone we can seek a solution in which this associated density function is independent of ϕ. This leads to the differential equation (7.4) for $\bar{\rho}^{(a)}(I,D,t)$. From the Fokker-Planck equation for these two variables we constructed an associated classical Langevin process. It is this classical Langevin process that was used in Sec. 9 of QVII to discuss the total noise associated with intensity correlations and the spectrum of that noise.

As a check on our work in QIX we made an adiabatic approximation to eliminate D and in this way obtained Fokker-Planck equations for $\bar{\rho}^{(a)}(\beta,\beta^*,t)$ as well as for $\bar{\rho}^{(a)}(I,\phi,t)$. A quasilinear analysis of the last process gave amplitude noise which could be compared with corresponding results in QVII and a phase linewidth could be compared with earlier results in QV. In addition, by the use of ordered operator techniques, we translated $\bar{\rho}^{(a)}(\beta,\beta^*,t)$ into an operator equation for $\rho(b,b^\dagger,t)$ and finally into an equation for the density matrix ρ_{mn} of the electromagnetic field in the photon representation. The latter equation (10.28) was found to be in precise agreement with corresponding results of Scully and Lamb.[17]

APPENDIX A: PRESERVATION OF COMMUTATION RULES

Making use of our working equations (3.7)–(3.12), we can express the time derivative of a commutation rule in the form

$$\begin{aligned}(d/dt)[b,D]&=[db/dt,D]+[b,dD/dt]\\&=-\tfrac{1}{2}\gamma(1-i\alpha)[b,D]+\tfrac{1}{2}N\pi(1-i\alpha)[b,D]D\\&\quad-(\Gamma_2+w_{20}d+\pi bb^\dagger)[b,D]-\pi bD\\&\quad+[\gamma H,D]/(1+i\alpha)+[b,F_D].\end{aligned}\tag{A1}$$

By writing

$$D(s)=D(t)+\int_t^s[dD(s')/ds']ds',\tag{A2}$$

and using the fact that a random force commutes with an arbitrary operator at an earlier time, we can simplify the next-to-last term in (A1) to the form

$$\begin{aligned}[\gamma H,D]&=(\Delta t)^{-1}\int_t^{t+\Delta t}ds[\gamma H(s),D(s)]\\&=(\Delta t)^{-1}\int_t^{t+\Delta t}ds\int_t^s ds'[\gamma H(s),F_D(s')].\end{aligned}\tag{A3}$$

Making use of (3.11) and (3.12), and retaining only those parts of γH and F_D that do not commute, we obtain

$$\frac{[\gamma H,D]}{1+i\alpha}=-\frac{1}{2}\frac{N\pi}{\Gamma}(\Delta t)^{-1}\int_t^{t+\Delta t}ds\int_t^s ds'[F_{12}(s),F^\dagger{}_{12}(s')]b.\tag{A4}$$

Using QIV(6.25) we can write the commutator in (A4) in the form

$$\langle[F_{12}(s),F_{21}(s')]\rangle=[\delta(s-s')/N]\langle 2D_{1221}-2D_{2112}\rangle.\tag{A5}$$

The diffusion constants that appear in (A5) can be written in the forms

$$2D_{2112}=(\Gamma_1+2\Gamma_{12}^{\rm ph})\sigma_{22}+\sum_{p\neq 2} w_{2p}\sigma_{pp}\rightarrow\Gamma_1\sigma_{22}=2\Gamma D, \tag{A6}$$

$$2D_{1221}=(\Gamma_2+2\Gamma_{12}^{\rm ph})\sigma_{11}+\sum_{q\neq 1} w_{1q}\sigma_{qq}\ll\Gamma D. \tag{A7}$$

To obtain these forms we make use of QIV(6.28) and then keep the dominant terms appropriate to the requirements (3.1) and (3.3) of our model. In this way we obtain the result for the first commutator

$$\langle[\gamma H, D]\rangle\rightarrow\langle\tfrac{1}{2}\pi Db\rangle. \tag{A8}$$

The last term in (A1) can be treated in a similar fashion, and we find it can be reduced to a form identical to (A3) so that it has precisely the same value;

$$\langle[b, F_D]\rangle=(\Delta t)^{-1}\int_t^{t+\Delta t} ds\left\langle\left[\int_t^s \frac{db(s')}{ds'}\,ds', F_D(s)\right]\right\rangle$$

$$=(\Delta t)^{-1}\int_t^{t+\Delta t} ds\int_t^s ds'\left\langle\frac{[\gamma H(s), F_D(s)]}{(1+i\alpha)}\right\rangle=\langle\tfrac{1}{2}\pi Db\rangle. \tag{A9}$$

The last three terms in (A1) can be combined to form a commutator. Thus (A1) tells us that

$$\langle d[b, D]/dt\rangle\propto\langle[b, D]\rangle=0. \tag{A10}$$

In other words, the right-hand side is completely expressible in terms of commutators, so that if the commutator of b and D vanishes at an initial time, it continues to vanish thereafter.

By using (3.7) we can write

$$d[b, b^\dagger]/dt=[db/dt, b^\dagger]+\text{Hermitian adjoint}$$
$$=\{\tfrac{1}{2}(\pi D-\gamma)(1-i\alpha)+[\gamma H, b^\dagger]\}+\text{H.a.} \tag{A11}$$

Following the procedures used above, the commutator can be expressed in the form

$$[\gamma H, b^\dagger]=(\Delta t)^{-1}\int_t^{t+t} ds\int_t^s ds'[\gamma H(s), \gamma H^\dagger(s')] \tag{A12}$$

in terms of the random forces. Using QIV(6.22), (6.23), and (6.25), the commutator of the random forces can be written in the form

$$\langle[\gamma H(s), \gamma H^\dagger(s')]\rangle=(\gamma-\pi D)\delta(s-s'). \tag{A13}$$

The numerical integration in (A13) produces a factor of $\frac{1}{2}$. Thus the commutator in the braces of (A12) cancels the real part of the first term of that equation, leaving the content of the braces to be pure imaginary. Thus the braces cancels against its Hermitian adjoint and we arrive at

$$\langle d[b, b^\dagger]/dt=0\rangle, \tag{A14}$$

which tells us that the commutation rule relating b and $b^\dagger$ is preserved in time.

Strictly speaking, we have only shown that $\langle[b, D]\rangle$ remains zero for all time, if it is assumed to vanish at some initial time. More generally, we would like to show that the same is true of $[b, D]$ itself. More precisely, we shall attempt to show that $d\langle M[b, D]\rangle/dt=0$ if $[b, D]=0$ at time t for any $M\equiv M(b, b^\dagger, D)$. Using QIV(2.7)–(2.9) we can write

$$d\langle M[b, D]\rangle/dt=\langle M\{d[b, D]/dt\}\rangle$$
$$+\langle\{dM/dt\}[b, D]\rangle+2D_{M,[b,D]}, \tag{A15}$$

where the quantities in braces are the system operators that remain if a reservoir average is taken of the operator in the braces, i.e., $\{dM/dt\}=A_M$ is the drift vector associated with M. [See QIV(2.10).]

Our proof, through equation (A11), essentially consisted in showing that

$$\{d[b, D]/dt\}\propto[b, D]. \tag{A16}$$

Thus the first two terms are already of the desired form, and vanish at the initial time t. The diffusion coefficient $2D_{MN}$ with $N=[b, D]$ can be evaluated by setting $N=[b, D]$ in QIV(B22), and it is then found to be a sum of terms each of which contains $[b, D]$ and vanishes if the latter is assumed to vanish. Thus

$$d\langle M[b, D]\rangle/dt\propto\langle\text{operators}\times[b, D]\rangle=0.$$

Hence, if $[b, D]=0$ at time t, then $\langle M[b, D]\rangle=0$ at $t+\Delta t$ for arbitrary M, which implies that $[b, D]=0$ for all time. Similar remarks apply to $[b, b^\dagger]=1$.