New Correlation Length in Bose-Gas Models of Liquid Helium

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A series of calculations is reported on the thermodynamics of both ideal and hard-core Bose gases in thin-film geometry. These calculations indicate that a Bose gas at liquid-helium densities possesses a statistical correlation length of about 70 Å, representing the thickness beyond which the thermodynamic behavior of the film closely approximates that of the bulk system. Some indication of this behavior may be implicit in recent experimental results both on superfluid flow in thin helium films and on the properties of liquid helium confined to small pores.

IN following up Ziman's suggestion[1] that liquid helium may, in some approximate sense, be thought of as consisting of a large number of "maximal assemblies," we have investigated extensively[2] the properties of an ideal Bose gas confined within a container of size $L\times L\times D$, (with $D\leq L$) in an attempt to understand the nature of the superfluid transition in thin films.

We would like to report here on a series of calculations dealing with both ideal and hard-core Bose gases confined to "films" of thickness D but infinite lateral extent. These calculations indicate that the Bose gas at approximate liquid-helium densities possesses a statistical correlation length of about 70 Å. That this correlation length is of statistical rather than of dynamical origin is evidenced by the fact that it occurs in both the ideal gas and the strongly interacting hard-sphere gas. Roughly speaking, it is the length beyond which the finite-size system closely approximates the bulk system.

Let us first consider an ideal Bose gas confined to a "film" of thickness D but infinite lateral extent. The specific boundary conditions obtaining at the faces of the film are of only minor consequence to our present considerations and we adopt vanishing wave functions for convenience. The energy eigenstates are characterized by the eigenvalues

$$E_n(p_x,p_y)=\frac{1}{2m}\left(p_x^2+p_y^2+\frac{n^2h^2}{4D}\right),$$

and the total number of particles per unit volume is given by

$$\frac{N}{V}=\frac{1}{Dh^2}\int_{-\infty}^{\infty}dp_x\int_{-\infty}^{\infty}dp_y\sum_{n=1}^{\infty}N_n(p_x,p_y)$$
$$=\left(\frac{2\pi mkT}{h^2D}\right)\sum_{j=1}^{\infty}\frac{e^{j\mu/kT}}{j}\sum_{n=1}^{\infty}e^{-j\alpha n^2},$$

where $\alpha=h^2/8mkTD^2$. In this expression the level density,

$$N_n=\left\{\exp\left(\frac{E_n-\mu}{kT}\right)-1\right\}^{-1},$$

has been treated as the sum of a geometric series.[1,2] Using the Jacobi transformation[3] as described previously[2] together with the relation

$$\sum_{m=1}^{\infty}\frac{x^m}{m}=-\ln(1-x),\quad \text{valid for } -1<x<1,$$

this expression can be rewritten in a convenient form for numerical computations:

$$\frac{N}{V}=\rho=\left(\frac{2\pi mkT}{Dh^2}\right)\sum_{j=1}^{j_1}\frac{e^{j\mu/kT}}{j}\left[\frac{1}{2}\left(\frac{\pi}{j\alpha}\right)^{1/2}(1+2e^{-\pi^2/j\alpha})-\frac{1}{2}-e^{-j\alpha}-e^{-4j\alpha}\right]-\ln[1-e^{(\mu/kT-\alpha)}]-\ln[1-e^{(\mu/kT-4\alpha)}],$$

where $\mu(T)$ is the chemical potential and j_1 is the integer lying between $(1.68/\alpha)$ and $(1.68/\alpha)-1$.

We have solved this equation for $\mu(T)$ numerically, and also a similar equation for the internal energy $U(T)$ from which the specific heat follows by differentiation. The results are shown in Figs. 1 and 2. In Fig. 1, C_v is plotted against T for a variety of values of the

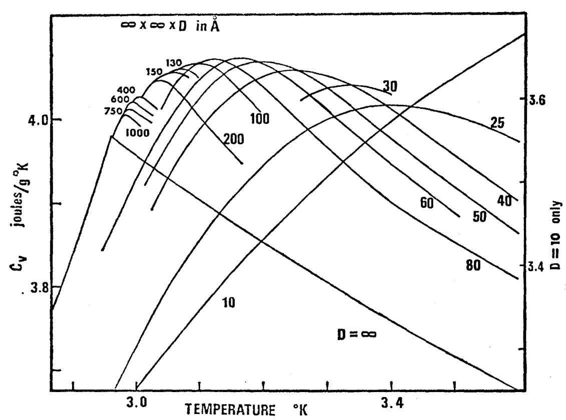

FIG. 1. Specific heat C_v plotted against temperature T for ideal Bose-gas films of thickness D and infinite lateral extent. The gas density is 2.00×10^{22} particles/cm².

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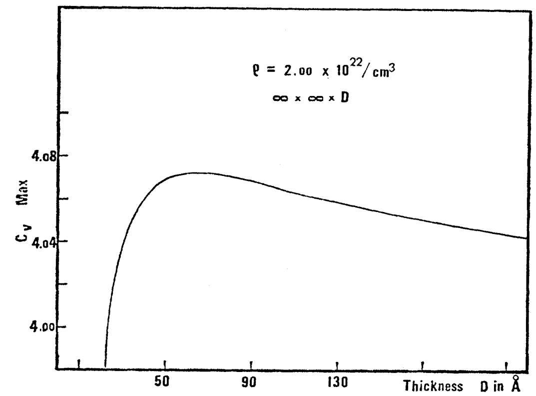

FIG. 2. Height of specific-heat maximum versus film thickness for ideal Bose-gas films of infinite lateral extent and thickness *D*.

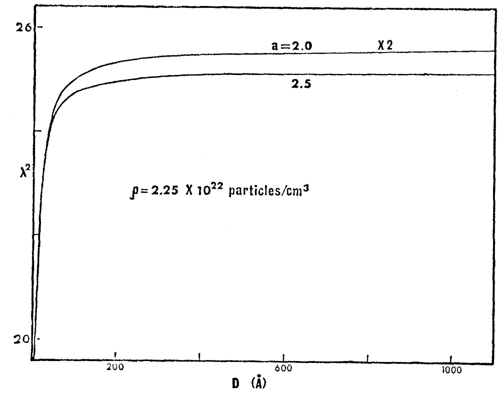

FIG. 3. Effective interaction strength λ^2 versus film thickness *D* for a hard-sphere Bose gas. The upper curve, which has been multiplied by a factor of 2, corresponds to a hard-core radius of 2.0 Å; the lower curve, to a hard-core radius of 2.5 Å.

film thickness *D* and for a particle density of 2.00×10^{22} particles/cm^3. (A similar set of curves shifted to higher temperatures by 0.20°K is obtained for a density of 2.25×10^{22} particles/cm^3.) It is interesting to observe that C_v has an absolute maximum which lies above the bulk condensation temperature and which occurs for a film thickness of about 70 Å. It is clear from Fig. 1, and from plots of other thermodynamic functions which we have obtained, that 70 Å represents a transition from thin-film to bulk-gas behavior. This conclusion is highlighted in Fig. 2 where the height of the specific-heat maximum on an expanded scale has been plotted against film thickness *D*.

It should be emphasized that all of these systems, as pointed out by Ziman,[1] behave two dimensionally in the sense that no accumulation occurs into the ground state at finite temperatures. We shall discuss in more detail elsewhere the connection between the shift of the specific-heat maxima to higher temperatures with decreasing thickness of film and the "effective level density" in these systems.

We have also carried out some numerical calculations for a finite system of hard-sphere bosons by adapting the method of Brueckner and Sawada[4] as modified by Parry and ter Haar[5] in order to take into account the depletion of the zero-momentum state. The results are shown in Fig. 3 where we have plotted λ^2, the effective interaction strength, as a function of film thickness for assumed hard-core radii of 2.0 and 2.5 Å and a total particle density of 2.25×10^{22} particles/cm^3. (An alternative way to represent these results is to plot *D* verus ρ_0, where ρ_0 is the particle density in the zero-momentum state.) In either case, it can be seen that the film changes from thin-film to bulk-gas behavior at a thickness of about 70 Å.

The correspondence between these results and those obtained for the ideal gas, suggest that the purely statistical correlation length occurring in the latter is carried over to the former with little modification due to the dynamics.

A few remarks in conclusion: (1) While the Brueckner-Sawada approach suffers from several questionable approximations, similar results which we have obtained using the pseudopotential method[6] suggest that our conclusions about the persistence of a statistical correlation length will remain valid nonetheless. (2) In actual helium films, wall effects may play an important role in modifying the behavior of the first few layers; such effects are additional to the effects described here. In practice, bulk behavior may be achieved for *D* values a few tens of angstroms greater than the 70 Å correlation length suggested in this letter. (3) Preliminary evidence of such effects as we have discussed may be implicit in the experimental work on superfluid flow by the Leiden group, as reported by Matheson,[7] and may also provide the explanation for the results of Brewer *et al.*[8] dealing with the properties of liquid helium confined to small pores (of diameter about 65 Å). Further experimental work in this direction appears to us desirable.

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