Collision Integral for a Plasma in a Strong Magnetic Field*

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Starting from the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy of statistical mechanics, we derive the magnetic-field dependence of the collision integral to second order in field strength. The field is sufficiently strong to make the Larmor frequency of the same order as the plasma frequency. The calculation is carried out explicitly for a Landau plasma (small-momentum-transfer regime). The simplicity of the result makes it particularly suitable for calculating transport coefficients for a plasma.

THE charged particles of a plasma interact through the Coulomb force law. Because of the long range of this interaction the duration of a collision, i.e., the time of interaction, would be infinite. However, the Debye shielding reduces the effective range of the interaction. Thus, the duration of a collision in a plasma is determined by the time required for a particle to traverse a Debye sphere. Since the Debye radius is much larger than molecular dimensions, the duration of a collision in a plasma is much longer than that in a neutral gas. An external magnetic field, of magnitude often encountered in the laboratory, will affect the plasma in two ways. One is the effect on the trajectories of the particles between collisions; the other is the effect on the collision process itself. The first has been treated extensively[1] and appears as a force term in the kinetic equation. The second can be significant for laboratory magnetic fields because of the relatively long duration of a collision.

We are concerned here with obtaining a description of the time evolution of the one-particle distribution function for a plasma in a magnetic field in which the duration of a collision is comparable to the Larmor period. This condition is satisfied for magnetic fields such that

$$B(\text{gauss}) \sim 3\times10^{-3}[n_e(\text{cm}^{-3})]^{1/2}. \qquad (1)$$

These plasmas are of the type encountered in experimental situations.[2–4] We obtain below, from the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy, a pair of equations which determine the time evolution of the single-particle distribution function to second order in the magnetic field.

The BBGKY equations may be put in dimensionless form by use of the following dimensionless variables:

$$t^*=\omega_p t;\quad \mathbf{x}^*=\mathbf{x}/\lambda_D;\quad \mathbf{v}^*=\mathbf{v}/v_{\text{th}},$$
$$\Phi^*=\Phi(\mathbf{x}^*)/\Phi_0;\quad \mathbf{B}^*=(\omega_c/\omega_p)(\mathbf{B}/|\mathbf{B}|), \qquad (2)$$

where $v_{\text{th}}\equiv(kT/m)^{1/2}$ is approximately the thermal speed, $\lambda_D\equiv(kT/4\pi n_e e^2)^{1/2}$ is the Debye length, $\omega_p\equiv v_{\text{th}}/\lambda_D$ is the plasma frequency, $\omega_c=eB/m$ is the cyclotron frequency, and Φ_0 is a mean value of the effective interaction potential for a collision. Thus, Φ_0 is a measure of the strength of a typical two-body interaction. If B^* is of order 1, then the $\mathbf{B}$ field is of such strength that an electron under the action of the magnetic field alone would make one cyclotron revolution in a time of the order of the mean duration of a collision $(\lambda_D/v_{\text{th}})$. Thus, a $|\mathbf{B}^*|$ field of order 1 is a very strong field and may be expected to significantly affect the collision integral. We shall show that this expectation is borne out by the mathematics.

The nondimensional BBGKY equations for an electron gas in a uniform, neutralizing positive background are[5]

$$\frac{\partial F^s}{\partial t}+(K^s-I_E{}^s)F^s=\frac{\Phi_0}{kT}I^sF^s+(n\lambda_D{}^3)\frac{\Phi_0}{kT}L^sF^{s+1}, \qquad (3)$$

where we have dropped all *'s to simplify notation, and where the operators K^s, I^s, and L^s are given by

$$K^s=\sum_{i=1}^{s}\mathbf{v}_i\cdot\nabla_i,\quad I_E{}^s=\sum_{i=1}^{s}(\mathbf{v}_i\times\mathbf{B})\cdot\nabla_{v_i},$$
$$I^s=\sum_{1\le i<j\le s}I_{ij},\quad I_{ij}=\nabla_i\Phi_{ij}\cdot\nabla_{v_i}+\nabla_j\Phi_{ij}\cdot\nabla_{v_j},$$
$$L^s=\sum_{i=1}^{s}\int d\mathbf{x}_{s+1}d\mathbf{v}_{s+1}I_{i,s+1}. \qquad (4)$$

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We consider a Landau plasma ($n\lambda_D{}^3\approx 1$, $\Phi_0/kT \equiv \epsilon \ll 1$) in which the magnetic field is time-independent and uniform over all space; the gas is homogeneous in space. The Landau plasma regime corresponds to a dense, weakly coupled gas, and describes a situation in which small-momentum transfer collisions are dominant. For the field-free case this approximation yields the well-known Landau collision integral which is of the Fokker-Planck form. Since collective effects are not explicitly taken into account, a cutoff at the Debye distance is required. We have considered for simplicity only the electron-electron collisions. The ions must be present, however, to preserve over-all charge neutrality. Therefore our system consists of an electron gas in a neutralizing background of positive charges.

The BBGKY equations are solved here for the two-body correlation function, and the kinetic equation is obtained by means of the method of extension.[5] The s-particle distribution function F^s is expanded in the small parameter ϵ with the following special notation for $s=1$,

$$F^1=f^0+\epsilon f^1+\epsilon^2 f^2.$$

The time is expressed in terms of the independent time scales τ_n, where $\tau_n=\epsilon^n t$. Thus, the time derivative is expanded as follows:

$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial\tau_0}+\epsilon\frac{\partial}{\partial\tau_1}+\epsilon^2\frac{\partial}{\partial\tau_2}+\cdots. \quad (5)$$

We assume for simplicity that at $t=0$, F^s satisfies the molecular chaos condition. We use a cylindrical coordinate system in velocity space. The components of $\mathbf{v}$ are ($v_\perp$, θ, $v_\parallel$), where $v_\perp$ is perpendicular to $\mathbf{B}$, θ is the angle between $v_\perp$ and some fixed axis in the plane perpendicular to $\mathbf{B}$, and $v_\parallel$ is along $\mathbf{B}$. The single-particle distribution is called gyrotropic if it does not depend on θ. The gyrotropy assumption need not be made at this point, but is required to obtain our final result.

In first order (ϵ^1) the equation for the single-particle distribution function yields

$$\partial f^0/\partial\tau_1=0 \quad\text{and}\quad f^1(\tau_0)=0. \quad (6)$$

The two-body correlation function, defined by $F_{12}{}^2 \equiv f_1f_2+g_{12}$ vanishes in lowest order, i.e., $g_{12}{}^0=0$. In first order $g_{12}{}^1$ satisfies the equation

$$\partial g_{12}{}^1/\partial\tau_0+(K^2-I_E{}^2)g_{12}{}^1=I_{12}{}^2f_1{}^0f_2{}^0. \quad (7)$$

Solving Eq. (7) using the initial condition $g_{12}{}^1(0)=0$, we obtain

$$g_{12}{}^1(x_{12},y_{12},z_{12},\theta_1,\theta_2,\tau_0)=\int_0^{\tau_0}d\lambda\frac{\partial\Phi}{\partial x_{12j}}(\tilde{x}_{12},\tilde{y}_{12},\tilde{z}_{12})[D_jf_1{}^0f_2{}^0](\tilde{\theta}_1,\tilde{\theta}_2,\tilde{\tau}_0), \quad (8)$$

where

$$\tilde{x}_{12}=x_{12}+\frac{1}{B}\sum_{i=1}^{2}(-1)^i v_{i\perp}\{\sin\theta_i-\sin\tilde{\theta}_i\},$$

$$\tilde{y}_{12}=y_{12}-\frac{1}{B}\sum_{i=1}^{2}(-1)^i v_{i\perp}\{\cos\theta_i-\cos\tilde{\theta}_i\},$$

$$\tilde{z}_{12}=z_{12}-(v_{1z}-v_{2z})\lambda,\quad \tilde{\theta}_i=\theta_i-B\lambda,$$

$$\tilde{\tau}_0=\tau_0-\lambda,\quad D_j=(\partial/\partial v_{1j}-\partial/\partial v_{2j}).$$

The coordinate system has been chosen so that the z axis is along $\mathbf{B}$ and $\mathbf{x}_{12}=\mathbf{x}_1-\mathbf{x}_2$. The result in Eq. (8) does *not* depend on the assumption of gyrotropy.

The kinetic equation is obtained in second order using the assumptions of a homogeneous gas and a gyrotropic single-particle distribution function. Equating coefficients of ϵ^2 we obtain

$$\partial f^2/\partial\tau_0+\partial f^0/\partial\tau_2=n\lambda_D{}^3L^1g_{12}{}^1. \quad (9)$$

We write $g_{12}{}^{1*}\equiv g_{12}{}^1(\tau_0=\infty)$, and write Eq. (8) as an integral from 0 to ∞ (i.e., $g_{12}{}^{1*}$) minus an integral from τ_0 to ∞ [which we call $h(\tau_0)$]. The correlation function $g_{12}{}^{1*}$ can be shown to be independent of τ_0 so that when Eq. (9) is integrated over τ_0 we obtain

$$f^2(\tau_0)=\tau_0\left(\frac{\partial f^0}{\partial\tau_2}+n\lambda_D{}^3L^1g_{12}{}^{1*}\right)-\int_0^{\tau_0}h(\lambda)d\lambda. \quad (10)$$

The freedom introduced by the multiple time-scale expansion is now exploited to require that ϵ^2f^2/f^0 be small for all τ_0. The last term on the right-hand side of Eq. (10) does not increase indefinitely as τ_0 increases. Therefore, in order for f^2 not to grow with τ_0, we must have

$$\partial f^0/\partial t=(n\lambda_D{}^3)(\Phi_0/kT)^2L^1g_{12}{}^{1*}, \quad (11)$$

which is the kinetic equation in a general form. Related results have been reported previously.[6]

We shall now put Eq. (11) into an explicit form by doing a *second* perturbation expansion in which we treat the dimensionless B as a small parameter. The expansions for $g_{12}{}^1$ and f^0 are

$$g_{12}{}^1=g_{12}{}^{(0)}+Bg_{12}{}^{(1)}+B^2g_{12}{}^{(2)}+\cdots,$$
$$f^0=f^{00}+Bf^{01}+B^2f^{02}+\cdots. \quad (12)$$

The initial conditions applied are $g_{12}{}^1(\tau_0=0)=0$ and $f^{01}(\tau_0=0)=0$.

We obtain the following results for the single-particle distribution function. The f^{01} term is zero and f^{00} and

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f^{02} are determined by the following equations:

$$\frac{\partial f_1^{00}}{\partial t}=A^{(3)}\frac{\partial}{\partial v_{1i}}\int d\mathbf{v}_2\frac{T_{ij}}{|\mathbf{v}_{12}|}D_j f_1^{00} f_2^{00}, \tag{13}$$

$$\frac{\partial f_1^{02}}{\partial t}=A^{(3)}\frac{\partial}{\partial v_{1i}}\int d\mathbf{v}_2\frac{T_{ij}}{|\mathbf{v}_{12}|}D_j(f_1^{02}f_2^{00}+f_1^{00}f_2^{02}) -A^{(1)}\frac{\partial}{\partial v_{1i}}\int d\mathbf{v}_2\frac{N_{ij}}{|\mathbf{v}_{12}|^3}D_j f_1^{00}f_2^{00}, \tag{14}$$

where

$$D_j=(\partial/\partial v_{1j}-\partial/\partial v_{2j}),$$

$$A^{(\alpha)}=(n\lambda_D^3)\left(\frac{\Phi_0}{kT}\right)^2 8\pi^5\int_0^\infty \tilde{\Phi}^2(k)k^\alpha dk,$$

$$T_{ij}=\delta_{ij}-v_{12i}v_{12j}/|\mathbf{v}_{12}|^2,$$

$$N_{ij}=-\frac{B_iB_j}{B^2}+(B_jv_{12i}+B_iv_{12j})\frac{B_kv_{12k}}{B^2|\mathbf{v}_{12}|^2}+\tfrac{3}{4}T_{ij} +\frac{1}{4}\left(\delta_{ij}-5\frac{v_{12i}v_{12j}}{|\mathbf{v}_{12}|^2}\right)\frac{(B_kv_{12k})^2}{B^2|\mathbf{v}_{12}|^2}-\tfrac{3}{2}\epsilon_{ipm}\epsilon_{jnl}\frac{B_pB_n}{B^2}\frac{v_{12m}v_{12l}}{|\mathbf{v}_{12}|^2}.$$

Here $\tilde{\Phi}(k)$ is the Fourier transform of $\Phi(|\mathbf{x}_{12}|)$ and ϵ_{ijk} is the completely skew-symmetric Levi-Civita density[7] which equals unity for $i=1$, $j=2$, $k=3$.

Equation (13) is the usual Landau equation.[8] It is known to have an H theorem and to have the Maxwellian as the unique equilibrium distribution. The tensor T_{ij} has a principal axis in the direction of the relative velocity $\mathbf{v}_{12}$ and is axisymmetric about $\mathbf{v}_{12}$. The field-dependent correction, Eq. (14), is determined by the tensor N_{ij} whose principal axes are in the directions of the relative velocity $\mathbf{v}_{12}$, the acceleration due to the magnetic field $\mathbf{v}_{12}\times\mathbf{B}$, and the orthogonal direction $\mathbf{v}_{12}\times(\mathbf{v}_{12}\times\mathbf{B})$. We further note that the collision current densities $\mathbf{J}^{(m)}$, defined through the equations $\partial f^{0m}(\mathbf{v}_1)/\partial t =(\partial/\partial v_{1i})\int d\mathbf{v}_2 J_i^{(m)}$, are orthogonal to the relative velocity $\mathbf{v}_{12}$ because the $\mathbf{J}^{(m)}$ are calculated to lowest order in the momentum transfer. This property is used in the demonstration that f^0 becomes Maxwellian for sufficiently large times.

By virtue of Eq. (13) f^{00} will become Maxwellian, M, in due time. Consider now Eq. (14) at large times, that is when f^{00} is Maxwellian. The term containing N_{ij} vanishes since $N_{ij}D_jM_1M_2=0$. The T_{ij} term in Eq. (14) drives f^{02} to zero. This is seen by noting that the T_{ij} term corresponds to the linearized version of the Landau equation for $f^{00}+B^2f^{02}$. Since f^{00} is Maxwellian, then f^{02} must eventually vanish. We have thus shown that $f^0=f^{00}+B^2f^{02}$ will become the Maxwellian distribution for sufficiently large times. This is so in spite of the fact that $H\equiv\int f^0\ln f^0 d\mathbf{v}$ does not have a negative definite time rate of change for all f^0 (except when $B\rightarrow 0$).[9]

Equations (13) and (14) provide a pair of equations for determining the single-particle distribution function in the presence of a strong magnetic field. We note that the magnetic-field corrections to order B^2 yield a remarkably simple collision integral, and that this collision integral, furthermore, has the same form as the standard Landau collision integral. The Landau collision integral has been extensively used for the study of transport properties.[8] The techniques developed there should, therefore, be applicable to calculate the effects of the strong field as well.

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[9] The two collision integrals of Eqs. (13) and (14) can yield competing contributions to $\partial H/\partial t$ inside the lobes of the region defined by

$$|v_{12}|^2<(A^{(1)}/A^{(3)})(B^2/4)(1-5\cos^2\Theta),$$

where Θ is the angle between the direction of the magnetic field and the relative velocity $\mathbf{v}_{12}$.