Methods for Constructing Invariant Amplitudes Free from Kinematic Singularities and Zeros*

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The work of Hara and Wang on the kinematic singularities of helicity amplitudes for particles with spin is extended to rederive the invariant amplitudes given by Hepp and Williams. Parity-conserving invariant amplitudes are given for reactions in which at most two particles have spin.

1. INTRODUCTION

The removal of kinematic singularities is an important preliminary to most calculations that utilize the analytic properties of scattering amplitudes. The complexity of this problem depends on the number of particles with spin and the magnitude of their spin. It has been studied for two-body scattering amplitudes by two different approaches. The first approach was used by Hepp[1] and by Williams,[2] who were concerned primarily with establishing the existence of amplitudes free from kinematic singularities rather than their relation to helicity amplitudes which are useful in practice. The second approach was used by Hara[3] and by Wang[4] and is concerned with establishing singularity-free combinations of helicity amplitudes.

The kinematic singularities of a physical amplitude arise from its relation to M functions,[5,6] which are analytic functions of momenta and have only dynamical singularities. An amplitude that is free from kinematic singularities is designed to express this analyticity in terms of the scalar variables s and t. However, physically useful functions of s and t such as helicity amplitudes contain kinematic singularities. These "choice-dependent" singularities occur, for example, at $s=(m_1\pm m_2)^2$. They can be exhibited explicitly by using a Lorentz transformation to a frame where the vectors do not have this singularity. Thereby, one can express the same information at $s=(m_1\pm m_2)^2$ as contained in invariant amplitudes—but only separately for each kinematic singularity. This is sufficient for some purposes, but invariant amplitudes solve the more general problem of simultaneously removing all singularities.

Hara and Wang make use of the Trueman-Wick crossing relations[7] (hereafter denoted by TW crossing). These relate the helicity amplitudes H_s when s is the energy, to the helicity amplitude H_t when t is the energy. They expose, for example, the singularities of H_s at the thresholds $s=(m_1\pm m_2)^2$ in the rotation matrices D, since the amplitude H_t has no such singularity. By this means Hara and Wang obtain nonsingular linear combinations of H_s. However, these results do contain kinematic zeros (e.g., in πN scattering) since one is unable to prove from the analyticity of these combinations the input information that H_t was nonsingular at s thresholds.

In Sec. 3 the method of Hara and Wang is extended to remove these zeros. The invariant amplitudes obtained there are related to a particular choice of those given by Hepp and Williams. This choice is examined in Sec. 2, where direct and inverse formulas are presented. These are probably the best choice until one considers the effect of parity conservation when more than one particle has spin. Parity is considered in Sec. 4; a solution is obtained when only two particles have spin, and guidelines are given for the general case. In Sec. 5 three typical examples of parity-conserving amplitudes are given. These are $j_1=2$, $j_2=j_3=j_4=0$; any j_1, $j_2=\frac{1}{2}$, $j_3=j_4=0$; and $j_1=j_2=1$, $j_3=j_4=0$. Finally, in Sec. 6 we discuss these results.

2. SOME GENERAL FORMULAS FOR INVARIANTS WITH NO EXPLICIT PARITY CONSERVATION

A. Notation

The spinor formalism has been described elsewhere[5,6,8] and consequently only the notation to be used is stated.

In terms of a T matrix corresponding to invariantly normed states, define an M function with lower spinor indices by

$$T_{fi}=\otimes_{\text{outgoing}}D(b^{-1})\otimes_{\text{incoming}}D(Cb^{-1})M(p)\,, \qquad (2.1)$$

where (a) $\otimes$ denotes one D^{j_i} matrix, of the lower spinor representation of $SL(2,C)$, for each particle; (b) $C=-i\sigma_2$ is lowering matrix; (c) the boosts b are in the helicity convention:

$$b(\mathbf{p}_i)=r(\varphi,\theta,-\varphi)Z(\sigma_i)\,,$$

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where $(\varphi,\theta,-\varphi)$ is direction of **p** and $Z(\sigma_i)$ is a boost in z direction.

So for the s-channel reaction, for instance,

$$H_s^{\lambda_3\lambda_4:\lambda_2\lambda_1}=e^{-\lambda_3\sigma_3}e^{-\lambda_4\sigma_4}e^{\lambda_2\sigma_2}e^{\lambda_1\sigma_1}(-1)^{j_1+j_2+\lambda_1-\lambda_2} \times D_{\lambda_3\mu_3}(r(0,-\theta_s,0))D_{-\lambda_4\mu_4}(r(0,-\theta_s,0)) \times M_{\mu_3\mu_4:\lambda_2-\lambda_1}(s\text{-channel c.m. system vectors}) \quad (2.2)$$

inserting the conventional $(-1)^{j_2+j_4-\lambda_2-\lambda_4}$ factor.

M has Lorentz transformation law

$$M(p)=\otimes D(g)M(\Lambda_g^{-1}p), \quad (2.3)$$

and irreducible amplitudes are formed by operating with the Clebsch-Gordan (C.G.) series in the usual way:

$$M^{(j,m)}(p)=\sum_{\mu_1+\mu_2=m} C(j_1j_2j;\mu_1\mu_2)M_{\mu_1\mu_2}(p). \quad (2.4)$$

The s, t, u rest systems are defined with particle 1 at rest and with particles 2, $\bar{3}$, $\bar{4}$, respectively, along the z axis. The x components are specified by relating rest systems to center-of-mass (c.m.) systems by a boost in the z direction. In the s, t, u c.m. system particles 3, $\bar{2}$, 3 are, respectively, in the directions $(\varphi,\theta,-\varphi)=(0,\theta_{s,t,u},0)$ with $0\leqslant\theta_{s,t,u}\leqslant\pi$ in their physical regions.

$$M_{\mu_3\mu_4:\mu_2\mu_1}(\text{c.m. system}) = e^{(\mu_1+\mu_2+\mu_3+\mu_4)\sigma_1}M_{\mu_3\mu_4:\mu_2\mu_1}(\text{rest}). \quad (2.5)$$

In Secs. 2 and 3 particle 1 will always be at rest.

Finally, the kinematic notation will be

$$S_{ij}^2=[s-(m_i+m_j)^2][s-(m_i-m_j)^2], \quad (ij)=(12) \text{ or } (34), \quad \cosh\sigma_i^s=E_i/m_i, \quad i=1,2,3,4 \quad (2.6)$$

and

$$\sinh\sigma^s=S_{12}/2m_1m_2,$$

so that $\sigma^s=\sigma_1^s+\sigma_2^s$.

Similarly, T_{ij} and σ^t are defined by $s\leftrightarrow t$ and $2\leftrightarrow 3$ in the above.

The Trueman-Wick[7] crossing angle for particle 1 for $s\rightarrow t$ channel crossing is defined by

$$\cos X_1=\frac{(S+m_1^2-m_2^2)(t+m_1^2-m_3^2)+2m_1^2(m_3^2-m_1^2+m_2^2-m_4^2)}{S_{12}T_{13}}, \quad \sin X_1=\frac{2m_1[\phi]^{1/2}}{T_{13}S_{12}}, \quad (2.6')$$

with

$$\phi=stu-s(m_1^2m_2^2+m_3^2m_4^2)-t(m_1^2m_3^2+m_2^2m_4^2) -u(m_1^2m_4^2+m_2^2m_3^2)+2m_1^2m_2^2m_3^2m_4^2(\textstyle\sum 1/m_i^2).$$

B. Construction of Covariant Polynomials

Both Hepp[1] and Williams[2] construct their covariant polynomials out of the elementary covariants:

$$M_{\alpha_1\alpha_2}[Z_a,Z_b]=\underset{\alpha}{\mathbf{S}}(Z_a)_{\alpha_1\dot\beta_1}C^{\dot\beta_1\dot\beta_2}(Z_b)_{\alpha_2\dot\beta_2}, \quad \alpha=\pm\tfrac{1}{2}.$$

Here

$$Z_a^{(s,t,u)}=k_a^{(s,t,u)}\cdot\sigma_{\alpha\dot\beta}/m_a, \quad \sigma_\mu=(\boldsymbol{\sigma},1), \quad (2.7)$$

and

$$k_a^s=p_a^s, \quad k_a^t=[(-1) \text{ if } a=2 \text{ or } 3]p_a^t, \quad k_a^u=[(-1) \text{ if } a=2 \text{ or } 4]p_a^u, \quad (2.7')$$

while

$$\underset{\alpha_1\cdots\alpha_n}{\mathbf{S}}=\frac{1}{n!}\sum_{\text{all permutations of } \alpha_1\cdots\alpha_n}.$$

There are 3 independent $M[Z_i,Z_j]$ i, $j=1\cdots4$, one of which is to be eliminated whenever it occurs more than once in a covariant polynomial by the constraint given by Eq. (2.19) of Ref. 1. The analogous equation in the lower-dotted formalism is written out in (2.18). The M's are to be chosen so as to simplify the requirements of crossing and parity, but we will not consider the latter until Sec. 4.

With particle 1 at rest we have, for any vector q with $q_y=0$,

$$M[Z_1,q]=\begin{bmatrix} q_x & -q_z \\ -q_z & -q_x \end{bmatrix}, \quad (2.8)$$

not distinguishing between q and $\sigma\cdot q$.

So in the s channel $M[Z_1,Z_2]$ is antidiagonal, which simplifies calculations. $M[Z_1,Z_3]$ has this property in the t channel, and so it is convenient to form covariant polynomials from $M[Z_1,Z_2]$ and $M[Z_1,Z_3]$, which also gives a definite behavior on $2\leftrightarrow 3$ crossing. Alternatively, $M[Z_1,Z_2]$ and $M[Z_1,Z_3-Z_4]$ are useful for $3\leftrightarrow 4$ crossing.

For the covariant that is allowed to appear only once, it is simplest—until one considers parity for more than one spinning particle—to take, for some vector w,

$$W(w)=M[w,Z_2]+(Z_1\cdot Z_2)M[Z_1,w]-(w\cdot Z_1)M[Z_1,Z_2] =\sinh\sigma^s w_x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ in the } s \text{ channel}. \quad (2.9)$$

W is proportional to $M[Z_1,\sigma_\mu\epsilon^{\mu\nu\lambda\rho}p_{1\nu}p_{2\lambda}w_\rho]$ used in Ref. 2.

C. Evaluation of Covariant Polynomials

In order to be able to discuss the previously chosen special covariants in all 3 channels, we take the model M's:

$$A=\alpha\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

$$B=\beta\begin{bmatrix}\sin\theta_b & \cos\theta_b\\ \cos\theta_b & -\sin\theta_b\end{bmatrix}, \qquad (2.10)$$

$B'=B$ with a prime added,

$$C=\gamma\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

and correspondingly take as our $2j+1$ covariants:

$$Q_{(b'b)}{}^{(j,r)}=\underset{\alpha}{\mathbf{S}}B_{\alpha_1\alpha_2}'\cdots B_{\alpha_{2r-1}\alpha_{2r}}'B_{\alpha_{2r+1}\alpha_{2r+2}}\cdots B_{\alpha_{2j-1}\alpha_{2j}},$$

$$Q_{(b'bc)}{}^{(j,r)}=\underset{\alpha}{\mathbf{S}}\,B_{\alpha_1\alpha_2}'\cdots B_{\alpha_{2r-1}\alpha_{2r}}'B_{\alpha_{2r+1}\alpha_{2r+2}}\cdots B_{\alpha_{2j-3}\alpha_{2j-2}} \times C_{\alpha_{2j-1}\alpha_{2j}}. \qquad (2.11)$$

$Q_{(ab)}{}^{(j,r)}$, $Q_{(abc)}{}^{(j,r)}$ are as above, with $B'\to A$, and results will be given for this simple case only. The more general covariants with b' replacing a may be obtained from

$$Q_{(b'b)\text{ or }(b'bc)}{}^{(j,r)}(m)=\sum_{\lambda}Q_{(ab'')\text{ or }(ab''c)}{}^{(j,r)}(\lambda)\times D_{\lambda m}{}^{j}(\theta_{b'}), \qquad (2.12)$$

where $\theta_{b'}+\theta_{b''}=\theta_b$ and $Q\ldots^{\cdot\cdot}(m)$ is matrix element of $Q\ldots^{\cdot\cdot}{}_{\alpha_1\cdots\alpha_{2j}}$ for a state $|j,m\rangle$.

If we choose $M[Z_1,Z_2]$, $M[Z_1,Z_3]$, and W as B', B, and C, respectively, the covariants (2.11) are of type $Q_{(ab)}$, $Q_{(abc)}$ in the s and t channels, but of type $Q_{(b'b)}$, $Q_{(b'bc)}$ in the u channel.

On evaluating (2.11) with $l+l'=j$ we find

$$Q_{(ab)}{}^{(j,l')}(m)=\frac{\beta^l\alpha^{l'}2^j l!}{[{}_{2j}C_{j+m}(l+m)!(l-m)!]^{1/2}}D_{0m}{}^{l}(\theta_b), \qquad (2.13)$$

$$Q_{(abc)}{}^{(j,l')}(m)=\frac{\gamma}{\sin\theta_b}\frac{\alpha^{l'}\beta^{l-1}2^j(l-1)!}{[{}_{2j}C_{j+m}(l+m)!(l-m)!]^{1/2}}mD_{0m}{}^{l}(\theta_b). \qquad (2.14)$$

Note that

$$Q_{(b'b)\text{ or }(ab)}{}^{(j,l')}(m)=(-1)^mQ_{(b'b)\text{ or }(ab)}{}^{(j,l')}(-m)$$

and (2.15)

$$Q_{(b'bc)\text{ or }(abc)}{}^{(j,l')}(m)=(-1)^{m+1}Q_{(b'bc)\text{ or }(abc)}{}^{(j,l')}(-m).$$

D. Inverse Relations

Writing

$$M^{(j)}=\sum_{i=1}^{2j+1}q_iQ_i$$

for a linearly independent set Q_i, we have now obtained M as a function of the invariant amplitudes $q_i(s,t)$. Conversely the q_i may be expressed in terms of $M^{(j,m)}$ $[m=-j\cdots+j]$.

A powerful method of obtaining such inverse formulas has been given in Refs. 2 and 6, and applied by Williams[2] to the Q_i of (2.11). Although this method appears the best in the majority of physically interesting cases (i.e., with explicit parity conservation, cf. Secs. 2E and 4) it happens that a simpler explicit result may be obtained for the covariants (2.11).

Taking the linearly independent set

$$Q_{(ab)}{}^{(j,r)},\quad r=0\cdots j;\qquad Q_{(abc)}{}^{(j,r)},\quad r=0\cdots j-1$$

the results (2.15) imply that if we form

$$M^{(j,m)}\pm(-1)^mM^{(j,-m)},$$

the problem at once splits into two.

Also, since

$$Q_{(ab)\text{ or }(abc)}{}^{(j,r)}(m)=0\quad\text{for}\quad j-r<|m|,$$

the relation between M and q is triangular and so the inversion is rather easy to do by hand. Since this will be exploited in Sec. 3 just the answer is presented here.

Simplifying the result of Sec. 3 obtained in terms of $D_{jm}{}^{j}(\theta_b)$, for $m\geqslant 0$ let

$$X_m{}^l(\cot\theta_b)=\frac{d^l}{d^l(\cot\theta_b)}\left[\frac{(1+\cos\theta_b)^m+(\cos\theta_b-1)^m}{(\sin\theta_b)^m}\right]\times\frac{1}{2^l l!}\,(\times\tfrac{1}{2},m=0), \qquad (2.16)$$

and then, again with $l+l'=j$, we find

$$q_{(ab)}{}^{(j,l')}=\sum_{m\geqslant 0}({}_{2j}C_{j+m})^{1/2}\frac{1}{2^{l'+1}\alpha^{l'}}\frac{(-1)^{l-m}}{(\beta\sin\theta_b)^l}X_m{}^l(\cot\theta_b)\times[M^{(j,m)}+(-1)^mM^{(j,-m)}], \qquad (2.17)$$

$$q_{(abc)}{}^{(j,l')}=\sum_{m>0}({}_{2j}C_{j+m})^{1/2}\frac{(-1)^{l-m}}{2^{l'+1}\gamma\,\alpha^{l'}(\beta\sin\theta_b)^{l-1}}\frac{l}{m}\times X_m{}^l(\cot\theta_b)[M^{(j,m)}-(-1)^mM^{(j,-m)}]. \qquad (2.17')$$

This concludes the treatment of the covariants (2.11). The above will also form the basis of the discussion in Sec. 4 of parity-conserving amplitudes. However, the simplest description of some practical problems lies in the formalism of Barut, Muzinich, and Williams,[6] which does not use the C.G. series. Deviating from the main line of the argument, in the next section are given the details in an especially simple case.

E. The Method of Barut, Muzinich, and Williams

In order to consider the practical problem of finding parity-conserving amplitudes in the $j_3=j_4=0$, $j_1=j_2$ case, it is most convenient to take the linear combinations of Q's suggested by a lower-lower dotted formalism (Refs. 5 and 6). For particle 1, define lower-dotted states by

$$|p_1\dot{\lambda}_1\rangle=|p_1\lambda_1\rangle D^{\lambda_1}{}_{\dot{\lambda}_1}(C^{-1}Z_1).$$

Covariant polynomials are formed[6,9] from elementary spinors with one α_2 and one $\dot{\alpha}_1$ index. We are allowed any number of Z_1, Z_2, and q, and one of a covariant, which we take as the parity conjugate of q

$$\pi q=Z_2C^{-1}q^TCZ_1.$$

Two πq's are eliminated whenever they appear in a covariant by the constraint

$$\underset{\alpha\dot{\alpha}}{\mathsf{S}}\{\pi q\pi q+qq+2(Z_1\cdot Z_2)\pi qq+q^2[Z_1Z_1+Z_2Z_2]+2[2(Z_1\cdot q)(Z_2\cdot q)-(Z_1\cdot Z_2)q^2]Z_1Z_2-2(Z_1\cdot q)\times[qZ_1+\pi qZ_2]-2(Z_2\cdot q)[qZ_2+\pi qZ_1]\},\quad(2.18)$$

where Z_1, Z_2, πq, and q each have one lower and one lower-dotted index wherever they appear otherwise than in a scalar product [which is denoted, for instance, by $(Z_1\cdot Z_2)$].

In fact, to get explicit parity conservation one would judiciously choose more than one πq in a singularity-free manner as dictated by (2.18).

Denote the matrix elements of the covariant containing l_1 Z_1's, l_2 Z_2's, l_3 q's, and l_4 πq's by $F_{\lambda_2\dot{\lambda}_1}(l_1l_2l_3l_4)$. Then we may expand

$$M_{\lambda_2\dot{\lambda}_1}=\sum_{\text{all }l_i}\beta_{l_1l_2l_3l_4}F_{\lambda_2\dot{\lambda}_1}(l_1l_2l_3l_4)\,,\quad(2.19)$$

and as long as the sum runs over all the l_i we may invert to find[2,6] (in the rest frame of particle 1)

$$\beta_{l_1l_2l_3l_4}=\binom{n}{l_1l_2l_3l_4}\frac{1}{(2q_x\sinh\sigma^s)^n}(-1)^{l_3+l_4}\sum_{\mu_2\dot{\mu}_1}(-1)^{s_2+\mu_2}F_{-\mu_2\dot{\mu}_1}[l_4l_3l_2l_1]M_{\mu_2\dot{\mu}_1}\,,\quad(2.20)$$

where $n=l_1+l_2+l_3+l_4$ and

$$\binom{n}{l_1l_2l_3l_4}$$

is a multinomial coefficient.

[9] K. Hepp, Helv. Phys. Acta **36**, 355 (1963).

As the inverse coefficients are essentially the same as the direct ones, not much extra work is involved in getting the inverse relations except that all the F's have to be calculated and not just the $(2j_1+1)^2$ independent ones.

Finally, we reduce the sum over all l_i to a sum over the chosen linearly independent set by means of (2.18). Then is even easier (in the s channel) than it appears, since the β's to be eliminated have large l_3l_4 and small l_1l_2: Their corresponding inverse F therefore has, from (2.20), large l_1l_2 and hence many zero elements.

As an example for $j_1=j_2=\frac{3}{2}$ there are, for each parity, 8 linearly independent F's and ten F's in (2.19), while in the inverse relation, the coefficients of three M's are unaffected by the constraint (2.18), while three have one extra term and two have two extra terms.

Finally, no useful explicit formula could be found for the F's like (2.13), but a little of the tiresome insertion of elementary 2-spinors is avoided by noting that

$$F_{\lambda_2\dot{\lambda}_1}(l_1l_2l_3l_4)=\binom{n}{l_1l_2l_3l_4}^{-1}\times\{\text{coefficient of }x_1^{l_1}x_2^{l_2}x_3^{l_3}x_4^{l_4}\text{ in }D_{\lambda_2\dot{\lambda}_1}{}^{j_1=j_2}(x_1Z_1+x_2Z_2+x_3q+x_4\pi q)\}.$$

In Sec. 5, $j_1=j_2=1$ is given as an example of the above.

3. DIRECT DERIVATION OF INVERSE RELATIONS FROM THE TRUEMAN-WICK CROSSING RELATIONS

A. Statement of Problem

We take the TW crossing relation[7] when there is only one particle (No. 1) with spin. This is

$$M^{(j,m)}(s\text{ rest})=(-1)^mM^{(j,\mu)}(t\text{ rest})\quad D_{\mu m}{}^j(X_1)\,,\quad(3.1)$$

where:

(i) As in Sec. 2, $M(s$ rest) denotes the M function with particle 1 at rest and particle 2 along the z axis and $M(t$ rest) has particle 3 along the z axis;

(ii) We have used the C.G. series to treat any number of spinning particles, as the use of $M(s,t$ rest) ensures that all particles transform with the same rotation as in the one-spinning-particle case.

(iii) The TW crossing relation relates the analytic continuation of $M(s$ rest) to $M(t$ rest) in the t physical region. The phase in (3.1) is stated for the customary route of continuation for the s rest vectors so that $S_{12}\to+|S_{12}|$ and $[\phi]^{1/2}\to+[\phi]^{1/2}$ in the t physical region.

As described in the Introduction, the M functions are analytic functions of the momenta and kinematic singularities occur through making the choice $p_i=p_i(s,t)$ to get a function of s and t.

There is the unavoidable singularity at the physical-region boundary and $M^{(j,m)}$ is proportional to $\{[\phi]^{1/2}\}^{|m|}$ times a function regular at the physical-region boundary.

The other singularities are choice-dependent, and in our case,

$$M(s\text{ rest})\ \text{is singular at}\ S_{12}=0,$$
$$M(t\text{ rest})\ \text{is singular at}\ T_{13}=0.$$

Following Hara[3] and Wang,[4] the nature of this singularity is given by the TW crossing relation (3.1) which, as written, exhibits the S_{12} singularity of $M(s\text{ rest})$ in the D matrix, since $M(t\text{ rest})$ has no singularity there.

We will try to find some nonsingular linear combinations of $M(s\text{ rest})$ which are not only analytic at S_{12} but whose analyticity at S_{12} enables one to deduce that $M(t\text{ rest})$ is analytic there. These linear combinations should also enable one to prove the physical-region boundary behavior of M and that $M(s\text{ rest})$ is analytic at T_{13}.

B. Solution

This problem is soluble if one notes (using the results of Sec. 2) the triangular relation, in both the s and t channels, between M and a suitable set of invariant amplitudes.

Consider the equation—for any θ and $\eta=\pm1$

$$D_{jm}{}^j(\theta)+\eta D_{-jm}{}^j(\theta)=[D_{j\mu}{}^j(\theta-X_1)+\eta D_{-j\mu}{}^j(\theta-X_1)]\times D_{\mu m}{}^j(X_1). \quad (3.2)$$

Then consider the following two cases:

1. $\eta=(-1)^j$

Operate on both sides of (3.2) with

$$\frac{(\sin^{j-l}X_1)}{l!}\frac{d^l}{d^l(\cot\theta)}\frac{1}{\sin^j\theta}\cdots\Bigg|_{\theta=X_1},\quad l=0\cdots j$$

and write the resulting equation:

$$A_l{}^{jm}=B_l{}^{j\mu}D_{\mu m}{}^j(X_1). \quad (3.3)$$

We have

$$A_l{}^{jm}=(-1)^{j-m}B_{j-l}{}^{jm}$$

and (3.4)

$$A_l{}^{jm}=0\quad |m|<l.$$

Then, forming

$$\sum_m A_l{}^{jm}(-1)^m M^{(j,m)}(s\text{ rest})=\sum_\mu B_l{}^{j\mu}M^{(j,\mu)}(t\text{ rest}), \quad (3.5)$$

and letting l increase through integral values from 0 to j, we get linear combinations of $M(s\text{ rest})$ (of one less in number each time) equal to linear combinations of $M(t\text{ rest})$ (of one more in number each time).

We must now multiply by factors to remove the singularities and so we form the invariant amplitudes

$$I_l{}^j=\sum_{m\geqslant0}\frac{A_l{}^{jm}(-1)^m[M^{(j,m)}(s\text{ rest})+(-1)^mM^{(j,-m)}(s\text{ rest})](\times\tfrac12\ \text{if}\ m=0)}{(\sin^jX_1)S_{12}{}^{j-l}T_{13}{}^l},\quad l=0\cdots j, \quad (3.6)$$

which are proved analytic at T_{13} from (3.6), and at S_{12} from the expression in terms of $M(t\text{ rest})$ (3.5), and at $\phi=0$ from either (3.5) or (3.6). But, as they are triangular, these relations may be inverted. Taking, for example, the T_{13} singularity, we start at $m=l=j$ and work downwards in m, inverting to find $M^{(j,m)}(s\text{ rest})$ in terms of $I_l{}^j$ ($l\geqslant m$). At each stage the new $M^{(j,m)}(s\text{ rest})$ has a simple coefficient in $I_{l=m}{}^j$ proportional to $(\sin X_1)^{j-m}$, the T_{13} singularity of which is canceled by the division factors in (3.6). Thus we may prove $M(s\text{ rest})$ analytic at $T_{13}=0$. S_{12} and $\phi=0$ are dealt with similarly, and so $I_l{}^j$ are the desired singularity-free amplitudes.

2. $\eta=-(-1)^j$

The remaining j invariant amplitudes are given by $\eta=-(-1)^j$. This time operate on both sides of (3.2) with

$$\frac{\sin^{(j-l)}X_1}{(l-1)!}\frac{d^{l-1}}{d^{l-1}(\cot\theta)}\frac{1}{\sin^{j-1}\theta}\cdots\Bigg|_{\theta=X_1},\quad l=1\cdots j$$

and write the resulting equation:

$$\bar{A}_l{}^{jm}=\bar{B}_l{}^{j\mu}D_{\mu m}{}^j(X_1), \quad (3.7)$$

where

$$\bar{A}_l{}^{jm}=(-1)^{j-m}\bar{B}_{j-l+1}{}^{jm}. \quad (3.8)$$

Then similarly we get the invariant amplitudes

$$\bar{I}_l{}^j=\sum_{m>0}\frac{\bar{A}_l{}^{jm}(-1)^m[M^{(j,m)}(s\text{ rest})-(-1)^mM^{(j,-m)}(s\text{ rest})]}{(\sin^jX_1)(S_{12})^{j-l+1}T_{13}{}^l}. \quad (3.9)$$

C. Relation to Sec. 2

In Sec. 2C take

$$A=M[Z_1,Z_2],$$
$$B=M[Z_1,Z_3],$$
$$C=W(Z_3).$$

Then

$$I_l^j \text{ and } q_{(ab)}{}^{(j,l')},\ \bar{I}_l^j \text{ and } q_{(abc)}{}^{(j,l')} \text{ for } l+l'=j,$$

have exactly the same structure in the s and t channels. They are thus identical up to a proportionality constant, and one may verify that

$$q_{(ab)}{}^{(j,l')}=(2m_1m_2)^{l'}(2m_1m_3)^{l}\tfrac{1}{2}(-1)^jI_l^j,$$
$$q_{(abc)}{}^{(j,l')}=(2m_1m_2)^{l'+1}(2m_1m_3)^{l}\tfrac{1}{2}(-1)^j\bar{I}_l^j. \qquad (3.10)$$

4. PARITY-CONSERVING INVARIANT AMPLITUDES

A. Introduction

The general results we have obtained explicitly conserve parity only when just one particle has spin. If more than one particle has spin, the nonphysical nature of the C.G. series is exposed since it is not preserved by the parity transformation.

It has only been possible to obtain linear combinations of the invariants of Sec. 2 for which the constraint of parity conservation takes a simple form in the two cases: Any j_1j_2 and $j_3=j_4=0$, or $j_3=\frac{1}{2}$, $j_4=0$. Only the result for the first case is presented below since it has features that are probably common with the general case. The proof involves the study of the matrices

$$T_{jj'}{}^m(\sigma)=\sum_{\mu_1+\mu_2=m} C(j_1j_2j:\mu_1\mu_2)e^{-2\mu_2\sigma}C(j_1j_2j':\mu_1\mu_2)$$

which are proportional to the representative of the boost e^σ in the (j_1,j_2) representation of the homogeneous Lorentz group. In Sec. 3, as the energies were nonsingular, we only needed to consider the rotation group in order to find invariant amplitudes. It appears that parity causes difficulties because it involves the study of the full Lorentz group.

Williams and Guertin have also considered the same problem.

B. Parity-Conserving Amplitudes when Two Particles Have Spin

After doing a parity transformation and a rotation through π about the y axis to bring the vectors back to their original values (since they have no y components), define the parity-conjugate M function by

$$P_{\lambda_1\lambda_2}(p)=(-1)^{j_1+j_2-m}M_{-\lambda_1-\lambda_2}(p)e^{-2\lambda_2\sigma}, \qquad (4.1)$$

where $m=\lambda_1+\lambda_2$. Then, if parity is conserved,

$$P_{\lambda_1\lambda_2}(p)=\eta_pM_{\lambda_1\lambda_2}(p).$$

By the same argument as in Sec. 3, we may derive a set of parity-conjugate invariant amplitudes—called $\overset{(-)}{P}{}_l^j$—which are the same linear combinations of P as $\overset{(-)}{I}{}_l^j$ were of M. Then a parity-definite, linearly independent, singularity-free subset of $\overset{(-)}{P}$ and $\overset{(-)}{I}$ may be found as follows.

1. General Case

For fixed l let j_{mid} be the midpoint of the allowed values of j, i.e.,

$$j_{\text{mid}}=\tfrac{1}{2}\{j_1+j_2+\max[l,|j_1-j_2|]\}.$$

Then keep $\overset{(-)}{I}{}_l^j$ for $j>j_{\text{mid}}$ and replace $\overset{(-)}{I}{}_l^j$ for $j<j_{\text{mid}}$ by $\overset{(-)}{P}{}^j$ for $j>j_{\text{mid}}$.

2. Anomalous Cases

This is a complete prescription except when j_{mid} is an integer. In this case one replaces $I_l^{j_{\text{mid}}}$ and $\overset{(-)}{I}{}_l^{j_{\text{mid}}}$ by (taking $j_1\geqslant j_2$):

(i) $l=0$,

$$I_0^{j_{\text{mid}}}+(-1)^{j_1+j_2}P_0^{j_{\text{mid}}};$$

(ii) $0<l\leqslant|j_1-j_2|$,

$$I_l^{j_{\text{mid}}}+(-1)^{j_1+j_2}P_l^{j_{\text{mid}}}$$
$$\bar{I}_l^{j_{\text{mid}}}-(-1)^{j_1+j_2}\bar{P}_l^{j_{\text{mid}}}$$

(In the above $j_{\text{mid}}=j_1$ and this case only occurs if j_i is integral);

(iii) $l>|j_1-j_2|$, j_i integral,

$$\cosh k\sigma I_l^{j_{\text{mid}}}+S_{12}\sinh k\sigma\bar{I}_l^{j_{\text{mid}}}+(-1)^{j_1+j_2}\times[\cosh k\sigma P_l^{j_{\text{mid}}}+S_{12}\sinh k\sigma\bar{P}_l^{j_{\text{mid}}}]$$

$$\cosh k\sigma\bar{I}_l^{j_{\text{mid}}}+\frac{\sinh k\sigma}{S_{12}}I_l^{j_{\text{mid}}}-(-1)^{j_1+j_2}\times\left[\cosh k\sigma\bar{P}_l^{j_{\text{mid}}}+\frac{\sinh k\sigma}{S_{12}}P_l^{j_{\text{mid}}}\right]$$

with $k=\frac{1}{2}(l-|j_1-j_2|)$, $j_{\text{mid}}=k+j_1$;

(iv) $l>|j_1-j_2|$, j_i half-odd integral,

$$\left(\cosh k\sigma\bar{I}_l^{j_{\text{mid}}}+\frac{\sinh k\sigma}{S_{12}}I_l^{j_{\text{mid}}}\right)\pm\left(\cosh k\sigma\bar{P}_l^{j_{\text{mid}}}+\frac{\sinh k\sigma}{S_{12}}P_l^{j_{\text{mid}}}\right)$$

with $k=\frac{1}{2}(l-|j_1-j_2|-1)$, $j_{\text{mid}}=j_1+k+\frac{1}{2}$. (4.2)

C. Analogous Results for Covariant Polynomials

The above results are stated in terms of invariant amplitudes. Similarly one could give the answer in terms of the covariant polynomials Q defined in Sec. 2. The prescription for these is to replace Q for $j>j_{\text{mid}}$ by the parity conjugates of those for $j<j_{\mathrm{mid}}$. The midpoint anomalies (4.2) are unaltered save for some factors given by (3.10) and $\sigma \rightarrow -\sigma$ in (iii) and (iv).

D. General Method for More than Two Spinning Particles

The situation in (C) is indicative of the general method of finding parity-conserving covariant polynomials. Namely, one starts at the lowest j and works upwards in j, replacing the Q of higher j by parity conjugates of those of lower j. The C.G. series greatly complicates the parity transformation and it is helpful to unsymmetrize the Q's (which is possible by working up from the lowest j). This should be done so as to get covariants made up from the elementary covariants of the type $M_{\alpha_1\alpha_2}[Z_1,q]$ with one index belonging to the same particle (here particle 1) as one of its momentum arguments. This has parity conjugate

$$[(Z_1\cdot Z_2)(Z_1\cdot q)-(Z_2\cdot q)]C_{\alpha_1\alpha_2}+M_{\alpha_1\alpha_2}[q,Z_2] -(Z_1\cdot q)M_{\alpha_1\alpha_2}[Z_1,Z_2], \quad (4.3)$$

which may nonsingularly replace $M_{\alpha_1\alpha_2}[q,Z_2]$.

An example of this procedure is Hepp's treatment[1] of the $NN\rightarrow NN$ case.

E. Derivation of Both Direct and Inverse Formulas

In a practical case, the writing down of a set of covariant polynomials which have definite behavior under parity and any desired crossing, is the easiest part of the work. One may do it either as in Sec. 4B so that the relation of the new q_i to H_s is almost immediate or as in Sec. 4C where the relation of H_s to the new q_i is easy. If one chooses unsymmetrized covariants this relation will be similar to Sec. 2E.

However, having chosen a particular way the inverse relation will be more tedious. The results of Sec. 4B are quite easy to invert in two steps: First invert for $M^{(j,m)}$, $j>j_{\mathrm{mid}}$, and their parity conjugates and then find the physical amplitudes from these. The case of $j_3=j_4=0$, $j_2=\frac{1}{2}$, any j_1 is given in Sec. 5 as an example of this. The difficulty in the any j_1j_2 case is proportional to the lower spin.

In most practical cases, however, since one unsymmetrizes to get a simple parity transformation, it is more convenient not to use inversion methods based on the C.G. series but to invert by the methods of Refs. 6 and 2.

F. Approach via Wigner Amplitudes

Finally we note that parity-conserving amplitudes may not be trivially attained by replacing the M functions used in Sec. 3, by Wigner amplitudes with particle 1 at rest. These retain the same transformation law under rotations which was all that was needed there, but the argument that the t amplitudes were free of s singularities fails at $s=(m_1-m_2)^2$ because of the $E=-m$ singularity in the particle-2 boost.

This is perhaps most clearly seen by comparing the results of Refs. 9 and 10 for the $\pi B'B''$ vertex. Vertex factors are discussed in the same way as scattering amplitudes except we are only allowed to use $C_{\alpha_1\alpha_2}$ and $M[Z_1,Z_2]$ to form covariants. Both references take C.G. series; that in Ref. 10 is in terms of helicity amplitudes, thereby being physically meaningful, but only attaining the canonical p^J behavior at $s=(m_1+m_2)^2$ or $s=(m_1-m_2)^2$ (depending on channel considered); that in Ref. 9 is in terms of spinor amplitudes and behaves like p^J at both $s=(m_1\pm m_2)^2$.

5. EXAMPLES

A. Introduction

In this section three examples will be given. In Sec. 5B $j_1=2$, $j_2=j_3=j_4=0$, which is an example of the results in Secs. 2 and 3. In Sec. 5C any j_1, $j_2=\frac{1}{2}$, $j_3=j_4=0$, which is an example of the method given in Sec. 4 for finding parity-conserving amplitudes when only two particles have spin. In Sec. 5D $j_1=j_2=1$, $j_3=j_4=0$, which is an example of the method of Sec. 2E (this is better than Sec. 4 for $j_1=j_2$).

Direct and inverse relations for the s-channel c.m. system helicity amplitudes are given, where

$$H_s^{\lambda_2\lambda_1}=(-1)^{j_1+j_2+\lambda_1-\lambda_2}e^{\lambda_2\sigma^s}M_{\lambda_2-\lambda_1}(\mathrm{rest}) =(-1)^{j_2-\lambda_2}e^{\lambda_2\sigma^s}M_{\lambda_2\dot{\lambda}_1}(\mathrm{rest})\,.$$

Put

$$H_s^{\lambda_2\lambda_1}(\eta)=H_s^{\lambda_2\lambda_1}+\eta H_s^{-\lambda_2,-\lambda_1}$$

and

$$M_s=\textstyle\sum_i e_iE_i+\sum_j n_jN_j\,,$$

a sum over e_i and n_j—invariant amplitudes of even and odd relative parity, respectively. (E_i and N_j are corresponding covariant polynomials.)

An arbitrary vector q linearly independent of p_1/m_1 and p_2/m_2 is left in the results. It may for example be chosen as p_3/m_3 or p_3-p_4, depending on the crossing desired. q's components are those in particle 1's rest system and in Sec. 5D the definitions $A=q_0+q_z$, $B=q_0-q_z$ are used.

The results are given only in the s channel. The t- and u-channel formulas are similar but have fewer zero elements.

π denotes parity conjugation so that

$$\pi H_s^{\lambda_2\lambda_1}=(-1)^{j_1+j_2+\lambda_1-\lambda_2}H_s^{-\lambda_2,-\lambda_1}.$$

B. $j_1=2$, $j_2=j_3=j_4=0$

Take covariant polynomials

$$E_1=M[Z_1,Z_2]M[Z_1,Z_2]\,,\quad N_1=M[Z_1,Z_2]W[Z_3]\,,$$
$$E_2=M[Z_1,Z_2]M[Z_1,Z_3]\,,\quad N_2=M[Z_1,Z_3]W[Z_3]\,,$$
$$E_3=M[Z_1,Z_3]M[Z_1,Z_3]\,.$$

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[10] L. Durand, P. C. DeCelles, and R. B. Marr, Phys. Rev. **126**, 1882 (1962).

TABLE I. Case (B): H_s in terms of invariant amplitudes.

	e_1	e_2	e_3
H_s^0	$(4/\sqrt{6})\sinh^2\sigma^s$	$-(4/\sqrt{6})\sinh\sigma^s \sinh\sigma^t \cos X_1$	$(2/\sqrt{6})\sinh^2\sigma^t(3\cos^2 X_1-1)$
$H_s^1(-)$	0	$2\sinh\sigma^s \sinh\sigma^t \sin X_1$	$-4\sinh^2\sigma^t \cos X_1 \sin X_1$
$H_s^2(+)$	0	0	$2(\sin X_1 \sinh\sigma^t)^2$
	n_1	n_2	
$H_s^1(+)$	$-\phi^{1/2}\sinh\sigma^s/m_1m_2m_3$	$(\phi^{1/2}\sinh\sigma^t/m_1m_2m_3)\cos X_1$	
$H_s^2(-)$	0	$-(\phi^{1/2}\sinh\sigma^t/m_1m_2m_3)\sin X_1$	

TABLE II. Case (B): The invariant amplitudes in terms of H_s.

	H_s^0	$H_s^1(-)$	$H_s^2(+)$
e_1	$\sqrt{6}/4\sinh^2\sigma^s$	$\cos X_1/2\sinh^2\sigma^s \sin X_1$	$(1+\cos^2 X_1)/4\sinh^2\sigma^s \sin^2 X_1$
e_2	0	$1/2\sinh\sigma^s \sinh\sigma^t \sin X_1$	$\cos X_1/\sinh\sigma^s \sinh\sigma^t \sin^2 X_1$
e_3	0	0	$1/2(\sin X_1 \sinh\sigma^t)^2$
	$H_s^1(+)$	$H_s^2(-)$	
n_1	$-m_1m_2m_3/\phi^{1/2}\sinh\sigma^s$	$-m_1m_2m_3\cos X_1/\phi^{1/2}\sinh\sigma^s \sin X_1$	
n_2	0	$-m_1m_2m_3/\phi^{1/2}\sinh\sigma^t \sin X_1$	

TABLE III. Case (C): H_s in terms of the invariant amplitudes.

$$H_s^{\frac{1}{2},m_1}[(-1)^{j_1+m_1}]=\frac{1}{2\sinh\sigma[(j_1-m_1+1)/(2j_1+1)]^{1/2}}\Big[\sum_{r=0}^{n-1}(x^r+\pi x^r)(-1)^n[e^{\sigma/2}-(-1)^n e^{-\sigma/2}]Q_{(ab)}^{(n,r+1)}(m_1-\tfrac{1}{2})$$

$$+\sum_{s=0}^{n-2}(y^s+\pi y^s)(-1)^{n+1}[e^{\sigma/2}+(-1)^n e^{-\sigma/2}]Q_{(abc)}^{(n,s+1)}(m_1-\tfrac{1}{2})+(z+\pi z)\sinh\sigma(-1)^n[e^{\sigma/2}+(-1)^n e^{-\sigma/2}]\left(1-\frac{(m_1-\frac{1}{2})}{n}\right)Q_{(ab)}^{(n,0)}(m_1-\tfrac{1}{2}),$$

$$H_s^{\frac{1}{2},m_1}(-(-1)^{j_1+m_1})=\frac{1}{2\sinh\sigma[(j_1-m_1+1)/(2j_1+1)]^{1/2}}\Big[\sum_{r=0}^{n-1}(x^r-\pi x^r)(-1)^n[e^{\sigma/2}+(-1)^n e^{-\sigma/2}]Q_{(ab)}^{(n,r+1)}(m_1-\tfrac{1}{2})+\sum_{s=0}^{n-2}(y^s-\pi y^s)$$

$$\times(-1)^{n+1}[e^{\sigma/2}-(-1)^n e^{-\sigma/2}]Q_{(abc)}^{(n,s+1)}(m_1-\tfrac{1}{2})+(z-\pi z)(-1)^n\sinh\sigma[e^{\sigma/2}-(-1)^n e^{-\sigma/2}]\left(1-\frac{(m_1-\frac{1}{2})}{n}\right)Q_{(ab)}^{(n,0)}(m_1-\tfrac{1}{2})$$

TABLE IV. Case (C): The invariant amplitudes in terms of H_s. For $z\pm\pi z$ use y rules for $s=0$.

With $l+r=n$:

$$\underset{1\leqslant r\leqslant n}{x^{r-1}\pm\pi x^{r-1}}=(-1)^n\sum_{m\geqslant 0}\frac{({}_{2n}C_{n+m})^{1/2}(-1)^{l-m}X_m{}^l(-q_z/q_x)}{2^{r+1}(\sinh\sigma)^r q_x{}^l}$$

$$\times[e^{\frac{1}{2}\sigma}\pm(-1)^n e^{-\frac{1}{2}\sigma}]\left[\left(\frac{j_1-m+\frac{1}{2}}{2j_1+1}\right)^{1/2}H_s^{\frac{1}{2},m+\frac{1}{2}}[\pm(-1)^{n+m}]+(-1)^m\left(\frac{j_1+m+\frac{1}{2}}{2j_1+1}\right)^{1/2}H_s^{\frac{1}{2},-m+\frac{1}{2}}[\pm(-1)^{n+m}]\right].$$

With $l+s=n$:

$$\underset{1\leqslant s\leqslant n-1}{y^{s-1}\pm\pi y^{s-1}}=(-1)^{n+1}\sum_{m>0}\frac{({}_{2n}C_{n+m})^{1/2}(-1)^{l-m}}{2^{s+1}(\sinh\sigma)^{s+1}q_x{}^l}\frac{l}{m}X_m{}^l(-q_z/q_x)$$

$$\times[e^{\frac{1}{2}\sigma}\pm(-1)^{n+1}e^{-\frac{1}{2}\sigma}]\left[\left(\frac{j_1-m+\frac{1}{2}}{2j_1+1}\right)^{1/2}H_s^{\frac{1}{2},m+\frac{1}{2}}[\pm(-1)^{n+m}]+(-1)^{m+1}\left(\frac{j_1+m+\frac{1}{2}}{2j_1+1}\right)^{1/2}H_s^{\frac{1}{2},-m+\frac{1}{2}}[\pm(-1)^{n+m}]\right]$$

TABLE V. Case (D): H_s in terms of invariant amplitudes.

	e_1	e_2	e_3	e_4	e_6
$H_s^{0,0}$	-1	$-\cosh\sigma$	$-\frac{1}{2}(A+B)$	$-\frac{1}{2}(Ae^{\sigma}+Be^{-\sigma})$	$-\frac{1}{2}[A^2e^{\sigma}+B^2e^{-\sigma}-2\cosh\sigma q_x^2]$
$H_s^{1,1}(+)$	$2\cosh\sigma$	2	$Ae^{\sigma}+Be^{-\sigma}$	$A+B$	$2AB$
$H_s^{1,0}(-)$	0	0	$\sqrt{2}q_x\sinh\sigma$	0	$\sqrt{2}q_x(B-A)$
$H_s^{0,1}(-)$	0	0	0	$\sqrt{2}q_x\sinh\sigma$	$\sqrt{2}q_x(Ae^{\sigma}-Be^{-\sigma})$
$H_s^{1,-1}(+)$	0	0	0	0	$-2q_x^2$

	n_1	n_3	n_4	n_5
$H_s^{1,1}(-)$	$2\sinh\sigma$	$Ae^{\sigma}-Be^{-\sigma}$	$B-A$	$A^2e^{\sigma}-B^2e^{-\sigma}$
$H_s^{1,0}(+)$	0	$\sqrt{2}q_x\cosh\sigma$	$-\sqrt{2}q_x$	$\sqrt{2}q_x(Ae^{\sigma}+Be^{-\sigma})$
$H_s^{0,1}(+)$	0	$-\sqrt{2}q_x$	$\sqrt{2}q_x\cosh\sigma$	$-\sqrt{2}q_x(A+B)$
$H_s^{1,-1}(-)$	0	0	0	$2q_x^2\sinh\sigma$

TABLE VI. Case (D): The invariant amplitudes in terms of H_s.

	$H_s^{0,0}$	$H_s^{1,1}(+)$	$H_s^{1,0}(-)$	$H_s^{0,1}(-)$	$H_s^{1,-1}(+)$
e_1	$\frac{1}{\sinh^2\sigma}$	$\frac{\cosh\sigma}{2\sinh^2\sigma}$	$-\frac{(Ae^{\sigma}-Be^{-\sigma})}{2\sqrt{2}q_x\sinh^2\sigma}$	$\frac{(A-B)}{2\sqrt{2}q_x\sinh^2\sigma}$	$\frac{[A^2e^{\sigma}+B^2e^{-\sigma}-2\cosh\sigma q^2]}{4q_x^2\sinh^2\sigma}$
e_2	$-\frac{\cosh\sigma}{\sinh^2\sigma}$	$-\frac{1}{2\sinh^2\sigma}$	$\frac{(A-B)}{2\sqrt{2}q_x\sinh^2\sigma}$	$-\frac{(Ae^{\sigma}-Be^{-\sigma})}{2\sqrt{2}q_x\sinh^2\sigma}$	$\frac{[-(A+B)(Ae^{\sigma}+Be^{-\sigma})\cosh\sigma+2\cosh^2\sigma q^2+2AB]}{+4q_x^2\sinh^2\sigma}$
e_3	0	0	$\frac{1}{\sqrt{2}q_x\sinh\sigma}$	0	$-\frac{(A-B)}{2q_x^2\sinh\sigma}$
e_4	0	0	0	$\frac{1}{\sqrt{2}q_x\sinh\sigma}$	$\frac{(Ae^{\sigma}-Be^{-\sigma})}{2q_x^2\sinh\sigma}$
e_6	0	0	0	0	$\frac{-1}{2q_x^2}$

	$H_s^{1,1}(-)$	$H_s^{1,0}(+)$	$H_s^{0,1}(+)$	$H_s^{1,-1}(-)$
n_1	$\frac{1}{2\sinh\sigma}$	$-\frac{(Ae^{\sigma}+Be^{-\sigma})}{2\sqrt{2}q_x\sinh^2\sigma}$	$-\frac{(A+B)}{2\sqrt{2}q_x\sinh^2\sigma}$	$\frac{(A^2e^{\sigma}-B^2e^{-\sigma})}{4q_x^2\sinh^2\sigma}$
n_3	0	$\frac{\cosh\sigma}{\sqrt{2}q_x\sinh^2\sigma}$	$(\sqrt{2}q_x\sinh^2\sigma)^{-1}$	$-\frac{(Ae^{\sigma}-Be^{-\sigma})}{2q_x^2\sinh^2\sigma}$
n_4	0	$(\sqrt{2}q_x\sinh^2\sigma)^{-1}$	$\frac{\cosh\sigma}{\sqrt{2}q_x\sinh^2\sigma}$	$\frac{(B-A)}{2q_x^2\sinh^2\sigma}$
n_5	0	0	0	$\frac{1}{2q_x^2\sinh\sigma}$

Then Tables I and II give the resulting relation between H_s and the invariant amplitudes.

In the above $j_{\rm mid}=j_i$ and this case only occurs if j_i is integral.

C. Any j_1, $j_2=\frac{1}{2}$, $j_3=j_4=0$

Putting $j_1=\frac{1}{2}(2n-1)$, take as covariant polynomials

$$0\leqslant r\leqslant n-1:\ X^r=C_{\alpha_1\alpha_2}\underbrace{M[Z_1,Z_2]\cdots}_{r\text{ times}}\ \underbrace{M[Z_1,q]\cdots}_{n-r-1\text{ times}},$$

$$0\leqslant s\leqslant n-2:\ Y^s=C_{\alpha_1\alpha_2}\underbrace{M[Z_1,Z_2]\cdots}_{s\text{ times}}\ \underbrace{M[Z_1,q]\cdots}_{n-s-2\text{ times}}\ W(q)$$

$$Z=\underbrace{M_{\alpha_1\alpha_2}[Z_1,q]M[Z_1,q]\cdots M[Z_1,q]}_{n\text{ times}},$$

where indices not shown explicitly belong to particle 1.

A complete set of covariants is formed by X^r, Y^s, Z, and their parity conjugates. Let the corresponding invariant amplitudes be x^r, y^s, z, πx^r, πy^s, and πz.

Then by taking matrix elements between states of $j=n$ and their parity conjugates (as mentioned in Sec. 4, this is easier than taking $j=n$ and $j=n-1$), one finds the results given in Tables III and IV. Here the answer is left in terms of the Q introduced in Sec. 2.

D. $j_1=j_2=1$; $j_3=j_4=0$

In the notation of Sec. 2E let

$$Q_1=Z_1|_{\alpha_2\dot{\alpha}_1}Z_1|_{\beta_2\dot{\beta}_1},\quad Q_2=Z_1Z_2,\quad Q_3=Z_1q,\quad Q_4=Z_1\pi q,$$
$$Q_5=qq,\quad Q_6=q\pi q,$$

and put

$$2E_i = Q_i + \pi Q_i, \quad i=1,2,3,4,6$$
$$2N_i = Q_i - \pi Q_i. \quad i=1,3,4,5$$

Then the results are presented in Tables V and VI.

6. DISCUSSION

Invariant amplitudes free from kinematic singularities provide a convenient way of stating the analyticity of $2 \rightarrow 2$ scattering amplitudes. In the preceding sections we have given the relation of these to physically useful amplitudes in some special cases but no completely general formula has been obtained. However, in many practical cases one may use the methods of Hara and Wang to derive results whose incompleteness is either irrelevant or commensurate with the dynamical approximations to be employed. I give below four examples of this:

(i) In the form-factor example at the end of Sec. 4, if one is interested in $s \lesssim 0$ one may consider it unnecessary to get the threshold behavior correct at $s=(m_1+m_2)^2$. Then the simple analysis of Ref. 10 will be sufficient.

(ii) When one writes dispersion relations with one of the invariants (say t) fixed, t helicity amplitudes, when their physical-region boundary behavior has been divided out, express the same analyticity in s as invariant amplitudes. This simple (to state) prescription is applicable in surprisingly many cases.[11]

(iii) Hara and Wang by considering the "parity-conserving" helicity amplitudes find functions whose singularities may be removed by a simple multiplicative factor. However, these nonsingular combinations have kinematic zeros at $s=0$ and $s=(m_i \pm m_j)^2$. It is sometimes important to remove some of these zeros in a practical calculation. At $s=0$ (in the general mass case) one can use the fact that the original helicity amplitudes were nonsingular, while at $s=(m_i \pm m_j)^2$ the perpendicular amplitudes introduced by Kotanski[12] diagonalize the behavior. These perpendicular amplitudes seem useful only at these special points because they are singular on the physical-region boundary.

(iv) It should be noted that the kinematic singularities of partial-wave amplitudes are more easily discussed by the method of Hara and Wang,[13] than from particular invariant amplitudes.

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[11] T. L. Trueman, Phys. Rev. Letters **17**, 1198 (1966); A. Bialas and B. E. Y. Svensson, Nuovo Cimento **42A**, 672 (1966).

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Errata

Mandelstam Iteration in a Realistic Bootstrap Model of the Strong-Interaction *S* Matrix, NAREN F. BALI, GEOFFREY F. CHEW, AND SHU-YUAN CHU [Phys. Rev. **150**, 1352 (1966)]. Lines 5 through 17 of the first paragraph of Sec. I should read: "emphasis of these regions is experimental. It has been observed that four-line connected parts are large within three narrow strips, as shown in Fig. 1. The strip labeled M^s manifests itself in two ways: (a) In the s physical region there may be strong peaks in low-energy cross sections; these peaks are associated with s poles of definite J_s whose residues have a corresponding polynomial dependence on $z_s = \cos\theta_s$ and thus on t (or u). The inevitable dying out of such peaks above about 2 GeV in center-of-mass energy indicates that even if resonances continue at high s, the partial widths for individual two-particle channels are small. (b) When there exist low-s poles on or near the physical sheet,"

Pion Production in π^--p Interactions at Energies 790, 830, and 870 MeV, N. M. CASON, I. DERADO, J. W. LAMSA, V. P. KENNEY, J. A. POIRIER, W. D. SHEPHARD, C. N. VITTITOE, AND J. L. STAUTBERG [Phys. Rev. **150**, 1134 (1966)]. Reference 3 should read: R. A. Burnstein, G. R. Charlton, T. B. Day, G. Quareni, A. Quareni-Vignudelli, G. B. Yodh, and I. Nadelhaft, Phys. Rev. **137**, B1044 (1965).

Conformal Group in Space Time, H. A. KASTRUP [Phys. Rev. **142**, 1060 (1966)]: Formula (20) contains misprints. It should read

$$\int_0^\infty da \sinh a P_{iq-1/2}{}^{-l-1/2}(\cosh a) P_{iq'-1/2}{}^{-l-1/2}(\cosh a) = |\Gamma(iq)|^2 |\Gamma(iq+l+1)|^{-2} \delta(q-q').$$

General Theory of Dispersion Sum Rules with Special Emphasis on the High-Energy Contribution, I. J. MUZINICH [Phys. Rev. **151**, 1206 (1966)]. Equation (3.7) should read:

$$P^2(A_2-A_1) - m\nu(A_5-A_4) = P_t T_1 \qquad (3.7)$$