Ambiguities in the Solutions of Partial-Wave Dispersion Relations*

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Partial-wave dispersion relations for s-wave amplitudes are studied in the approximation where the left-hand singularities are replaced by a finite sequence of poles. Conditions are obtained under which solutions are possible and it is found that when these are satisfied several classes of solutions are obtained, with different behavior in the high-energy physical region. The case of p-wave amplitudes is also discussed.

I. DISCUSSION OF THE PROBLEM

LET $f(s)$ be a partial-wave amplitude for an elastic scattering process, s being the square of the total energy of the two particles in the center-of-mass system. From the unitarity of the S matrix we know that $f(s)$ may be represented in the form[1]

$$f(s)=\frac{\eta(s)e^{2i\delta(s)}-1}{2iq(s)},$$

where $0\leqslant\eta(s)\leqslant 1$, $\delta(s)$ is real and is defined $\bmod(n\pi)$ with n an integer, and $q(s)$ is the center-of-mass momentum of either particle. The phase shift $\delta(s)$ may be fixed by taking $\delta(s_0)=0$ and requiring $\delta(s)$ to be continuous, with $s_0=(m_1+m_2)^2$, m_1 and m_2 being the masses of the two particles. The inelasticity parameter $\eta(s)=1$ when there are no competing inelastic channels open. We may define the real-valued function $F(s)$ by

$$F(s)=\mathrm{Re}f(s)-\frac{1}{\pi}P\int_{s_0}^{\infty}\frac{\mathrm{Im}f(s')ds'}{s'-s},\quad s>s_0.$$

The integral on the right side converges since $0\leqslant\mathrm{Im}f(s)\leqslant 1/q(s)$. To ensure the existence of the principal-value integral it is sufficient to assume that $\mathrm{Im}f(s)$ satisfies a Lipschitz condition for each $s>s_0$.

By making analyticity assumptions of the form proposed by Mandelstam[2] it is possible to represent $F(s)$ as the sum of a polynomial in s, the contributions of a finite set of poles which correspond to stable particles with the same quantum numbers as the partial wave in question, integrals over certain bounded cuts in the complex s plane and the integral over a cut extending to $-\infty$ along the negative real axis.[3] To simplify the discussion that follows, we assume that no subtractions are necessary (no polynomial is then required[3]) and take only the final term in the sum, so that

$$F(s)=\frac{1}{\pi}\int_{-\infty}^{s_1}\frac{g(s')ds'}{s'-s},$$

where $g(s)$ is a continuous real-valued function of s for $s\leqslant s_1<s_0$. This gives the integral equation

$$\mathrm{Re}f(s)-\frac{1}{\pi}P\int_{s_0}^{\infty}\frac{\mathrm{Im}f(s')ds'}{s'-s}=\frac{1}{\pi}\int_{-\infty}^{s_1}\frac{g(s')ds'}{s'-s},\quad s>s_0.\quad(1)$$

Assuming the validity of the integral equation (1), how may we use it? For processes like pion-nucleon scattering, where partial-wave amplitudes are known over a wide range of energies, it is possible to calculate the left side of (1) with reasonable accuracy for a range of values of s above threshold (apart from a very slowly varying contribution from the high-energy part of the integral). The attempt is then made to use this knowledge of $F(s)$ to obtain information about the discontinuity across the unphysical cuts. This method of analysis has been extensively used by Hamilton and his collaborators[4] to study pion-nucleon scattering and has also been applied to the $Y_1^*(1385)$ resonance by Martin.[5]

But it is also possible to try to proceed in the opposite direction. That is to say, some information (in perhaps a very crude form) may be thought to be known about $F(s)$ from calculations concerning the left-hand singularities and the problem is to obtain information about $f(s)$ in the physical region. Of course, in order to define a definite problem, it is necessary to prescribe $\eta(s)$ or $\delta(s)$ or some quantity depending on both of

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[1] We take this form for a partial-wave amplitude, rather than the one often used, which includes an extra factor $s^{1/2}$, because we want to be sure that our amplitude satisfies an unsubtracted dispersion relation.

[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[3] For a detailed discussion of the form of $F(s)$, see W. S. Woolcock, Phys. Rev. **153**, 1449 (1967).

[4] See the papers by J. Hamilton and T. D. Spearman, Ann. Phys. (N. Y.) **12**, 172 (1961); J. Hamilton, P. Menotti, T. D. Spearman and W. S. Woolcock, Nuovo Cimento **20**, 519 (1961); J. Hamilton, T. D. Spearman and W. S. Woolcock, Ann. Phys. (N. Y.) **17**, 1 (1962); J. Hamilton, P. Menotti, G. C. Oades and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962); D. Atkinson, *ibid.* **128**, 1908 (1962); T. D. Spearman, *ibid.* **129**, 1847 (1963); G. C. Oades, *ibid.* **132**, 1277 (1963).

[5] B. R. Martin, Phys. Rev. **138**, B1136 (1965).

them for all $s \geqslant s_0$. In practice it is most convenient to prescribe the quantity $R(s)$, given by

$$R(s)=\frac{2[1-\eta(s)\cos 2\delta(s)]}{1-2\eta(s)\cos 2\delta(s)+[\eta(s)]^2}.$$

$R(s)=(\sigma_{\text{total}}(s)/\sigma_{\text{elastic}}(s))$ for the partial wave in question and therefore $R(s)\geqslant 1$; $\eta(s)=1$ (no inelastic channels open) if and only if $R(s)=1$.

Since all that will be known about $R(s)$ is that it equals 1 when no inelastic channels are open, the problem of obtaining any information about $f(s)$ in the physical region will in general be impossible. Certain special cases are favorable. If, for example, $F(s)$ is small for a large range of values of s above threshold, the most probable inference is that both $\mathrm{Re}f(s)$ and the principal-value integral are small in this range. This means that $\mathrm{Im}f(s)$, and therefore the principal-value integral, will be negligible, at least over a somewhat smaller range, especially for higher partial waves with the amplitude written with an extra factor q^{2l} in the denominator. Then $F(s)$ will provide a good first approximation to $\mathrm{Re}f(s)$.[6] On the other hand, if it is known that a resonance occurs in the partial wave being considered, a variational type of solution may be used.[7]

Apart from these special cases, the so-called N/D method[8] is often used, usually with $R(s)$ identically equal to 1. This method may be written as follows. Consider the inhomogeneous linear integral equation of the second kind

$$h(s)=1-\frac{(s-s_0)}{\pi}\int_{-\infty}^{s_1} ds' H(s,s')g(s')h(s'), \quad s\leqslant s_1,$$

where

$$H(s,s')=-\frac{1}{\pi}\int_{s_0}^{\infty} ds'' \frac{q(s'')R(s'')}{(s''-s_0)(s''-s)(s''-s')}.$$

The kernel of this integral equation is not square integrable, but if suitable conditions are imposed on $R(s)$ and $g(s)$[9] it can be transformed into one with a square-integrable kernel. A unique solution $h(s)$ will then in general exist.

[6] See the discussion in A. Donnachie, J. Hamilton, and A. T. Lea, Phys. Rev. **135**, B515 (1964).

[7] A. Donnachie and J. Hamilton, Phys. Rev. **133**, B1053 (1964); G. C. Oades, in *Proceedings of the Siena International Conference on Elementary Particles and High-Energy Physics*, edited by G. Bernardini and G. P. Puppi (Societa Italiana di Fisica, Bologna, 1963), Vol. I, p. 388.

[8] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[9] For example, it is sufficient to assume that the limits $\int^{\to\infty} [ds R(s)/s^{3/2}]$ and $\int_{\to-\infty} [|g(s)|ds/s]$ exist. (It is useful often to use an arrow to emphasize that the limit of an integral over a finite range is being taken.)

Having determined $h(s)$ we then define, for $s>s_0$,

$$N(s)=\frac{1}{\pi}\int_{-\infty}^{s_1} ds' \frac{g(s')h(s')}{s'-s},$$

$$\mathrm{Re}D(s)=1-\frac{(s-s_0)}{\pi}P\int_{s_0}^{\infty} ds' \frac{q(s')R(s')N(s')}{(s'-s_0)(s'-s)},$$

$$\mathrm{Im}D(s)=-q(s)R(s)N(s),$$

$$f(s)=(N(s)/D(s)).$$

It is clear from the integral equation for $h(s)$ that the integral defining $N(s)$ exists if the limit $\int^{\to\infty} ds\, R(s)/s^{3/2}$ exists. Under this condition the integral defining $\mathrm{Re}D(s)$ will also converge, so that $\mathrm{Re}D(s)$ will be defined for $s>s_0$ if $R(s)$ satisfies a Lipschitz condition for each $s>s_0$. The problem is now to prove that the function $f(s)$ defined above is a solution of the integral equation (1) which satisfies the prescribed unitarity condition $\mathrm{Im}f(s)=q(s)R(s)|f(s)|^2$ for $s>s_0$.

The attempt at proof must proceed as follows. Assume first that for *each* $s<s_1$ there is a closed interval $I=[s-\delta, s+\delta]$ $(\delta>0)$ such that, for any two points s', s'' belonging to I,

$$|g(s')-g(s'')|\leqslant K|s'-s''|^{\mu} \quad (0<\mu\leqslant 1).$$

Assume, too, that for *each* $s>s_0$ a similar condition can be written for $R(s)$. Then

$$N(s)=\frac{1}{\pi}\int_{-\infty}^{s_1} ds' \frac{g(s')h(s')}{s'-s}$$

is an analytic function, regular in the complex s plane cut along the real axis from s_1 to $-\infty$ and continuous onto the cut from above and from below,[10] the limit functions being

$$\frac{P}{\pi}\int_{-\infty}^{s_1} ds' \frac{g(s')h(s')}{s'-s} \pm ig(s)h(s).$$

Similarly,

$$D(s)=1-\frac{(s-s_0)}{\pi}\int_{s_0}^{\infty} ds' \frac{q(s')R(s')N(s')}{(s'-s_0)(s'-s)}$$

is an analytic function, regular in the complex s plane cut along the real axis from s_0 to ∞ and continuous onto the cut from above and from below, the limit functions being $\mathrm{Re}D(s)\pm i\,\mathrm{Im}D(s)$ as defined earlier. Cauchy's theorem may therefore be applied in the usual way to the function $f(s)=N(s)/D(s)$ and the desired result proved, *if* the following four results can be demonstrated:

(i) $D(s)$ has no zeros in the complex s plane away from the cut $[s_0,\infty)$,

(ii) $|f(s)|\to 0$ as $|s|\to\infty$, uniformly with respect to $\arg s$ in $0\leqslant \arg s\leqslant 2\pi$,

[10] This has a precise meaning. See Theorem 2 of Appendix II.

(iii) $|f(s)(s-s_0)|\to 0$ as $s\to s_0$, uniformly with respect to $\arg(s-s_0)$ in $0\leqslant\arg(s-s_0)\leqslant 2\pi$, together with the same result with s_0 replaced by s_1,

(iv)

$$\int_{-\infty}^{s_1} ds' g(s')h(s')\int_{s_0}^{\infty} ds'' \frac{q(s'')R(s'')}{(s''-s_0)(s''-s)(s''-s')} = \int_{s_0}^{\infty} ds'' \frac{q(s'')R(s'')}{(s''-s_0)(s''-s)} \int_{-\infty}^{s_1} ds' \frac{g(s')h(s')}{s''-s'}.$$

The problem of establishing by the N/D method the existence of a solution of the integral equation (1) satisfying the prescribed unitarity condition is thus beset with great difficulties in the case when there is a cut extending to $-\infty$.[11]

In this paper therefore we propose to treat a simpler problem, in which the left-hand cut is replaced by a *finite number of poles*. From now on we use the variable $x=(s-s_0)$. The poles are at the points $-x_1, -x_2, \cdots, -x_n$, where $0<x_1<x_2<\cdots<x_n$, with residues $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ respectively. We therefore consider the system

$$f_1(x)-\frac{P}{\pi}\int_0^{\infty}\frac{f_2(t)dt}{t-x}=\sum_{i=1}^{n}\frac{\Gamma_i}{x+x_i}, \tag{2}$$

$$f_2(x)=q(x)R(x)[f_1^2(x)+f_2^2(x)], \tag{3}$$

and look for pairs of real-valued functions $f_1(x)$, $f_2(x)$ which satisfy these equations for all $x>0$. The function $q(x)$ is given by

$$q(x)=\tfrac{1}{2}x^{1/2}\left(\frac{x+4m_1m_2}{x+(m_1+m_2)^2}\right)^{1/2}, \quad x\geqslant 0.$$

Throughout the paper, the function $R(x)$ will be assumed to satisfy the following conditions:

(α) $R(x)\geqslant 1$,
(β) the limit $\int^{\to\infty} dxR(x)/x^{3/2}$ exists,
(γ) for each $x>0$, there is an interval $I=[x-\xi, x+\xi]$ ($\xi>0$) such that, for any two points x', x'' belonging to I, $|R(x')-R(x'')|\leqslant K|x'-x''|^{\mu}(0<\mu\leqslant 1)$. For $x=0$, there must be an interval $[0,\xi]$. The quantities ξ, μ, and K may depend on x.

There are two reasons for restricting ourselves to this simplified problem. The first is that all practical dynamical calculations which attempt the solution of dispersion relations make the approximation of replacing the left-hand singularities by a finite set of poles. The second is that it is possible to find conditions on the poles for which the existence of solutions can be demonstrated with proper mathematical rigor. Moreover, it is possible to discuss in detail the question of the *uniqueness* of the solution. It seems to us that insufficient attention has been paid to this question.[12] What we show is that, under suitable conditions on the poles, there exist several *classes* of solutions of Eqs. (2), (3). Each class consists of a single or double infinity of solutions and is characterized by a different behavior of the functions $f_1(x)$, $f_2(x)$ as $x\to\infty$.

The source of these ambiguities is well known. For example, in connection with the N/D solution for the general case, Amati and Fubini[13] remark that it is possible to add to the function $D(s)$ an arbitrary meromorphic function, provided that the resulting function does not have zeros in the complex s plane away from the cut $[s_0,\infty)$. However in this general case it is impossible to tell whether any such meromorphic functions exist, so that nonuniqueness remains a theoretical possibility. In the case of a finite number of poles, however, nonuniqueness of the solution can be demonstrated explicitly. Whenever the standard solution of the N/D type exists, it is possible, by adding to $D(z)$ a suitable pole on the negative real axis or, in some cases, a linear function of z, to construct other solutions.

Our solutions will be obtained in the usual way by constructing analytic functions $f(z)$ satisfying the following conditions:

(a) $f(z)$ is regular in the whole complex plane cut along the real axis from 0 to ∞, except for poles at $-x_1,\cdots,-x_n$ with residues $\Gamma_1,\cdots,\Gamma_n$; (b) $f(z^*)=f^*(z)$ when z is not a singular point; (c) $f(z)$ is continuous onto the cut[10] from above (and therefore from below) for each $x>0$ and the (continuous) limit function $f(x)=f_1(x)+if_2(x)$ satisfies Eq. (3) for all $x>0$; (d) $|f(z)|\to 0$ as $|z|\to\infty$, uniformly with respect to $\arg z$ for $0\leqslant\arg z\leqslant 2\pi$; (e) $|zf(z)|\to 0$ as $z\to 0$, uniformly with respect to $\arg z$ for $0\leqslant\arg z\leqslant 2\pi$; (f) the function $f_2(x)$ defined in (c) above satisfies a Lipschitz condition for each $x>0$. That is, there exist numbers $\xi>0$, $K>0$, $\mu(1\geqslant\mu>0)$ (all depending on x) such that

$$|f_2(x+h)-f_2(x)|\leqslant K|h|^{\mu}$$

for $|h|\leqslant\xi$. To prove that Eq. (2) is satisfied by the pair of real-valued functions $f_1(x)$, $f_2(x)$ it is necessary to use Cauchy's theorem of residues in the form given, for example, by Copson.[14] This states that if $F(\zeta)$ is an analytic function, continuous in the closed region bounded by a simple closed rectifiable curve C and regular in the interior of the region, except for a finite number of poles, then $\int_C F(\zeta)d\zeta=2\pi i$[sum of residues

[11] A study of this problem, with a very special assumption made about the form of the discontinuity across the left-hand cut, has been made by D. Atkinson and A. P. Contogouris, Nuovo Cimento **39**, 1082 (1965).

[12] There is, however, a brief discussion of ambiguities of the type which we consider in S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963). The existence of ambiguities in the solutions of partial-wave dispersion relations derived from a static model was first pointed out by L. Castillejo, R. H. Dalitz, and F. J. Dyson [Phys. Rev. **101**, 453 (1956)]. Their method was applied by Shirkov and his collaborators to obtain different classes of solutions of partial-wave dispersion relations for pion-pion scattering. For an excellent summary, see D. V. Shirkov, Nucl. Phys. **34**, 510 (1962).

[13] D. Amati and S. Fubini, Ann. Rev. Nucl. Sci. **12**, 359 (1962).

[14] E. T. Copson, *An Introduction to the Theory of Functions of a Complex Variable* (Oxford University Press, London, 1935), p. 117.

of $F(\zeta)$ at its poles within C]. The application is obvious on taking $F(\zeta)=f(\zeta)/\zeta-z$, where z is not a singular point of $f(\zeta)$. To finally obtain Eq. (2), fix $x(>0)$ and let $y\to 0$ from above. Using condition (f) above, we see from Theorem 1 of Appendix II that Eq. (2) is satisfied.

It is desirable to point out what we do *not* do in this paper. We make no attempt to find *all* solutions of the type given above, but only those with a particularly simple form. Further, for the case of more than one pole, we shall impose *sufficient* conditions for solutions to exist; these may clearly be too strong. This is a different approach to that of Martin[15] who, for the general case of a left-hand cut plus bound state poles, finds a *necessary* condition for solutions to exist at all. It is clear, too, that we cannot include crossing symmetry. Lovelace[16] has emphasized this defect of the N/D method. Finally, we do not consider solutions for which $f(x)$ has a zero in the physical region; $f_2(x)$ is always positive. Solutions with a zero of $f(x)$ at an energy below the inelastic threshold, through which $f_1(x)$ changes from negative to positive as x increases, can easily be constructed. But the existence of such a zero is, in principle, a matter for experimental test and we therefore simplify the discussion by not considering this possibility.

In Sec. II we consider the case of a single pole with negative residue. In the case of a single pole with positive residue, treated in Sec. III, a new type of solution is easily found. The case of a single pole contains all the features of the general case with any finite number of poles, which is discussed in Sec. IV. Finally, in Sec. V we consider a number of questions of physical interest, like the behavior of $f(x)$ for large positive x, the sign of the scattering length $f_1(0)$ and the possible existence of zeros of $f_1(x)$. We also look at the case of higher partial waves.

II. SINGLE POLE WITH NEGATIVE RESIDUE

In this case Eq. (2) becomes

$$f_1(x)-\frac{P}{\pi}\int_0^\infty \frac{f_2(t)dt}{t-x}=\frac{\Gamma_1}{x+x_1} \quad \text{for} \quad x>0,$$

$$\text{with} \quad \Gamma_1<0. \qquad (4)$$

Define

$$\lambda(z)=-\frac{1}{\pi}\int_0^\infty \frac{q(t)R(t)}{(t+x_1)(t-z)}dt; \qquad (5)$$

then $\lambda(z)$ is an analytic function, regular in the whole complex z plane cut along the real axis from 0 to ∞, and $\lambda(z^*)=\lambda^*(z)$ when z is not a singular point. By Theorem 2 of Appendix II the boundary value of $\lambda(z)$ as z approaches the cut from above is[17]

$$\lambda(x)=-\frac{P}{\pi}\int_0^\infty \frac{q(t)R(t)}{(t+x_1)(t-x)}dt-i\frac{q(x)R(x)}{x+x_1} \text{ for } x>0. \quad (6)$$

The following simple results will be used later:

$$\operatorname{Im}\lambda(z)=-\operatorname{Im}z\frac{1}{\pi}\int_0^\infty \frac{q(t)R(t)}{(t+x_1)|t-z|^2}dt \quad \text{for } \operatorname{Im}z\neq 0, \quad (7)$$

$$(d\lambda(x)/dx)<0 \quad \text{for} \quad x<0. \qquad (8)$$

Next we define

$$D^{(-)}(z)=A+\gamma/(z+x_0)+\lambda(z). \qquad (9)$$

$D^{(-)}(z)$ is an analytic function, regular in the whole z plane cut along the real axis from 0 to ∞, except for a pole at $z=-x_0$ with residue γ. In order that $D^{(-)}(z^*)=D^{(-)*}(z)$ when z is not a singular point the constants A, γ, and x_0 are chosen to be real. We require further that

$$D^{(-)}(-x_1)=A+\gamma/(x_0-x_1)+\lambda(-x_1)=\Gamma_1^{-1}. \quad (10)$$

In order to prove that $D^{(-)}(z)$ does not have zeros for $\operatorname{Im}z\neq 0$ it is sufficient to restrict γ so that

$$\gamma\geqslant 0. \qquad (11)$$

It is clear from Eqs. (9), (7), and (11) that $\operatorname{Im}D^{(-)}(z)\neq 0$ for $\operatorname{Im}z\neq 0$. Further, Eqs. (9), (8), and (11) imply that

$$dD^{(-)}(x)/dx<0 \quad \text{for} \quad x<0. \qquad (12)$$

It now follows from Eqs. (12) and (10), together with $\Gamma_1<0$, that if $x_0>x_1$ then $D^{(-)}(x)$ will have a zero between $-x_0$ and $-x_1$. We therefore require that

$$x_0<x_1. \qquad (13)$$

Next, since

$$\lim_{x\to-\infty} D^{(-)}(x)=A,$$

it again follows from Eqs. (12) and (10) that, for $D^{(-)}(x)$ *not* to have a zero for $x<-x_1$, it is necessary (and sufficient) that

$$A\leqslant 0. \qquad (14)$$

Since the first two terms of the sum in Eq. (10) are nonpositive by Eqs. (11), (13), and (14), it is necessary that

$$\lambda(-x_1)>\Gamma_1^{-1} \qquad (15)$$

for solutions of the type we are trying to construct to exist at all.[18] Equation (15) is a condition on the input parameters x_1, Γ_1 of Eq. (4) and the prescribed function $R(x)$. The conditions laid down already are sufficient to ensure that $D^{(-)}(z)$ can have zeros only for $y=0$, $x>-x_0$. Since the boundary value of $D^{(-)}(z)$ as z ap-

[15] A. Martin, Nuovo Cimento **38**, 1326 (1965).

[16] C. Lovelace, CERN 66/1041/66-TH.689, 1966 (unpublished).

[17] The function $\lambda(x)$ is denoted in Appendix I by $\lambda(x+i0)$, since it is necessary there to distinguish the boundary values of $\lambda(z)$ as z approaches the cut from above and from below.

[18] It is possible for equality to hold in Eq. (15), and then there is just one solution with $A=\gamma=0$.

proaches the cut from above is

$$D^{(-)}(x)=A+\gamma/(x+x_0)+\mathrm{Re}\lambda(x)+i\,\mathrm{Im}\lambda(x),$$

and since $\mathrm{Im}\lambda(x)\neq 0$ for $x>0$ by Eq. (6), $D^{(-)}(z)$ can have zeros only for $y=0$, $-x_0<x\leqslant 0$. We shall shortly state the conditions on A, γ and x_0 for which $D^{(-)}(z)$ has no zeros in this range. When these conditions are satisfied we define

$$N(z)=1/(z+x_1),$$
$$f^{(-)}(z)=N(z)/D^{(-)}(z).$$

Then $f^{(-)}(z)$ satisfies conditions (a)–(f) of Sec. I and, in particular, its boundary value $f_1^{(-)}(x)+if_2^{(-)}(x)$ as z approaches the cut from above satisfies Eqs. (3) and (4). Conditions (d)–(f) are proved in Appendix I. Condition (a) follows from Eq. (10) and the fact that $D^{(-)}(z)$ has no zeros by construction. Condition (b) is obvious from the corresponding properties of $N(z)$ and $D^{(-)}(z)$ and the first part of condition (c) follows directly from Theorem 2 of Appendix II. The second part of (c) holds since for $x>0$

$$\mathrm{Im}f^{(-)}(x)=-\frac{\mathrm{Im}D^{(-)}(x)}{|D^{(-)}(x)|^2}N(x)=\frac{q(x)R(x)}{|D^{(-)}(x)|^2}[N(x)]^2$$
$$=q(x)R(x)|f^{(-)}(x)|^2.$$

We now discuss the conditions for which $D^{(-)}(z)$ has no zeros for $y=0$, $-x_0<x\leqslant 0$. There are three cases to consider.

(1) $\gamma=0$. Equation (10) then gives

$$A=\Gamma_1^{-1}+\frac{1}{\pi}\int_0^\infty\frac{q(t)R(t)}{(t+x_1)^2}dt. \tag{16}$$

Equation (15) shows that $A<0$. Hence

$$D^{(-)}(0)=A-\frac{1}{\pi}\int_0^\infty\frac{q(t)R(t)}{t(t+x_1)}dt$$

is also negative; indeed from Eq. (12) $D^{(-)}(x)$ decreases monotonically as x increases from $-\infty$ to 0 and $D^{(-)}(z)$ therefore has no zeros for $y=0$, $x\leqslant 0$. Inserting Eq. (16) into Eq. (9) we get

$$D_1^{(-)}(z)=\Gamma_1^{-1}-\frac{(z+x_1)}{\pi}\int_0^\infty\frac{q(t)R(t)}{(t+x_1)^2(t-z)}dt, \tag{17}$$

and the resulting function $f_1^{(-)}(z)=N(z)/D_1^{(-)}(z)$ gives the standard N/D solution.

(2) $A=0$. From Eq. (10) we have

$$\gamma/(x_1-x_0)=-\Gamma_1^{-1}+\lambda(-x_1). \tag{18}$$

Now as $x\to x_0$ from the right $D^{(-)}(x)\to\infty$ by Eq. (11) and therefore $D^{(-)}(x)$ does not have a zero for $-x_0<x\leqslant 0$ if and only if

$$D^{(-)}(0)=\gamma/x_0-\frac{1}{\pi}\int_0^\infty\frac{q(t)R(t)}{t(t+x_1)}dt>0. \tag{19}$$

Eliminating γ from Eqs. (18) and (19) gives

$$x_0<\frac{-\Gamma_1^{-1}+\lambda(-x_1)}{-\Gamma_1^{-1}+\chi}x_1, \tag{20}$$

where

$$\chi=\frac{x_1}{\pi}\int_0^\infty\frac{q(t)R(t)}{t(t+x_1)^2}dt. \tag{21}$$

Hence it is possible, for each pair of pole parameters x_1, Γ_1 allowed by Eq. (15), to find a range of values of x_0 for which Eq. (19) holds. For *each* x_0 in this range [given by Eq. (20)], Eq. (18) gives the corresponding value of γ and we therefore have a single infinity of functions $f^{(-)}(z)$ satisfying the conditions of Sec. I, characterized by the continuous parameter x_0. We shall denote a function of this class by

$$f_2^{(-)}(z)=N(z)/D_2^{(-)}(z).$$

(3) $A\neq 0$, $\gamma\neq 0$. Again $D^{(-)}(x)$ does not have a zero for $-x_0<x\leqslant 0$ if and only if

$$D^{(-)}(0)=A+\gamma/x_0-\frac{1}{\pi}\int_0^\infty\frac{q(t)R(t)}{t(t+x_1)}dt>0. \tag{22}$$

Equation (10) continues to hold and eliminating A from Eqs. (10) and (22) gives

$$\gamma>\frac{x_0}{x_1}(x_1-x_0)(-\Gamma_1^{-1}+\chi). \tag{23}$$

But since $A<0$, Eq. (10) implies that

$$\gamma<(x_1-x_0)(-\Gamma_1^{-1}+\lambda(-x_1)). \tag{24}$$

For Eqs. (23) and (24) to hold simultaneously, it is clear that x_0 has once again to satisfy the inequality of Eq. (20). For each x_0 in this range, there is a range of values of γ given by Eqs. (23) and (24) and for each such γ the value of A is given by Eq. (10). We therefore have a double infinity of functions satisfying the conditions of Sec. I, characterized by the continuous parameters x_0 and γ. We shall denote a function of this class by $f_3^{(-)}(z)=N(z)/D_3^{(-)}(z)$.

III. SINGLE POLE WITH POSITIVE RESIDUE

We now look for pairs of functions $f_1(x)$, $f_2(x)$ satisfying Eqs. (3) and (4) with $\Gamma_1>0$. Define

$$D^{(+)}(z)=A+\gamma/(z+x_0)+Bz+\lambda(z), \tag{25}$$

where the constants A, B, γ, and x_0 are real and $\lambda(z)$ is defined in Eq. (5).[19] The function $D^{(+)}(z)$ has the same analytic properties as $D^{(-)}(z)$. In this case the equation corresponding to (10) is

$$D^{(+)}(-x_1)=A+\gamma/(x_0-x_1)-Bx_1+\lambda(-x_1)=\Gamma_1^{-1}. \tag{26}$$

[19] Note that, for the case $\Gamma_1<0$, we see no simple way of showing that $D^{(-)}(z)$ has no zeros for $\mathrm{Im}z\neq 0$ without assuming that $B\leqslant 0$, while the condition $B\geqslant 0$ is necessary to avoid a zero in $D^{(-)}(x)$ for $x<-x_1$. We therefore took $B=0$ in Sec. II.

To avoid zeros of $D^{(+)}(z)$ for $\mathrm{Im}z\neq 0$ it is sufficient to impose the conditions

$$\gamma\geqslant 0,\quad B\leqslant 0. \tag{27}$$

Again it is easy to see that

$$dD^{(+)}(x)/dx<0 \quad \text{for} \quad x<0. \tag{28}$$

Any solution $f^{(+)}(z)$ of the type we are going to construct will also satisfy Eq. (4) for $x<0$. It follows that, for $\Gamma_1>0$, $f_1(x)>0$ for $-x_1<x<0$. Since $f_1(x)$ will have a zero at $x=x_0$ when $\gamma>0$, we must choose

$$x_0>x_1. \tag{29}$$

For $\gamma>0$ Eqs. (26), (28), and (29) imply that $D^{(+)}(x)$ does not have a zero for $x_0<x\leqslant 0$ if and only if

$$D^{(+)}(0)=A+\gamma/x_0-\frac{1}{\pi}\int_0^\infty\frac{q(t)R(t)}{t(t+x_1)}dt>0. \tag{30}$$

Moreover, if $\gamma>0$ and $B<0$, then $D^{(+)}(x)\to-\infty$ as $x\to -x_0$ from the left and $D^{(+)}(x)\to\infty$ as $x\to-\infty$. Therefore $D^{(+)}(x)$ has a zero for $x<-x_0$. This case must be excluded and so $\gamma>0$ implies $B=0$. Further, with $\gamma>0$, $B=0$, $D^{(+)}(x)$ will not have a zero for $x<-x_0$ if and only if $A\leqslant 0$. $D^{(+)}(z)$ therefore has no zeros in the whole z plane under the conditions

$$\gamma>0,\quad B=0,\quad A\leqslant 0, \tag{31}$$

together with (29) and (30).

For $\gamma=0$, Eqs. (26) and (28) show immediately that $D^{(+)}(x)$ has no zero for $x\leqslant 0$ if and only if Eq. (30) holds. Clearly $A=0$ is impossible in this case for then Eq. (30) would not hold. It follows from Eqs. (26), (27), and (30) that

$$\Gamma_1^{-1}-\chi-\frac{\gamma x_1}{x_0-x_1}>0, \tag{32}$$

where χ is defined in Eq. (21). Since the third term on the left side of (32) is nonpositive by Eqs. (27) and (29), it is necessary that

$$\Gamma_1^{-1}>\chi \tag{33}$$

for solutions of the type we are trying to construct to exist at all. Again Eq. (33) is a condition on the input parameters x_1, Γ_1 and the prescribed function $R(x)$.

We have now written down the conditions under which $D^{(+)}(z)$ has no zeros in the whole z plane. When these conditions are satisfied we define

$$N(z)=1/(z+x_1),$$
$$f^{(+)}(z)=N(z)/D^{(+)}(z),$$

and it may be shown exactly as in Sec. II that $f^{(+)}(z)$ satisfies conditions (a)–(f) of Sec. I, so that its boundary value $f_1^{(+)}(x)+if_2^{(+)}(x)$ as z approaches the cut from above satisfies Eqs. (3) and (4). For convenience we distinguish four separate cases.

(1) $B=0$, $\gamma=0$. In this case Eq. (26) becomes $A=\Gamma_1^{-1}-\lambda(-x_1)$ and therefore, from Eq. (25),

$$D_1^{(+)}(z)=\Gamma_1^{-1}-\frac{1}{\pi}(z+x_1)\int_0^\infty\frac{q(t)R(t)}{(t+x_1)^2(t-z)}dt.$$

The resulting function $f_1^{(+)}(z)=N(z)/D_1^{(+)}(z)$ gives the standard N/D solution.

(2) $B=0$, $A=0$. Now Eq. (26) becomes

$$\gamma/(x_0-x_1)=\Gamma_1^{-1}-\lambda(-x_1), \tag{34}$$

while from Eq. (30) we have

$$\gamma/x_0>\frac{1}{\pi}\int_0^\infty\frac{q(t)R(t)}{t(t+x_1)}dt. \tag{35}$$

Eliminating γ between Eqs. (34) and (35) gives

$$x_0>\frac{\Gamma_1^{-1}-\lambda(-x_1)}{\Gamma_1^{-1}-\chi}x_1. \tag{36}$$

Hence it is possible, for each pair of pole parameters x_1, Γ_1 allowed by Eq. (33), to find a range of values of x_0 [namely, that given in Eq. (36)] for which Eqs. (26), (29), (30), and (31) hold. For *each* x_0 in this range Eq. (34) gives the corresponding value of γ. We therefore have a single infinity of functions $f^{(+)}(z)$ satisfying the conditions of Sec. I, characterized by the continuous parameter x_0. A function of this class will be denoted by $f_2^{(+)}(z)=N(z)/D_2^{(+)}(z)$.

(3) $B=0$, $A\neq 0$, $\gamma\neq 0$. Equation (26) now reads

$$A+\gamma/(x_0-x_1)+\lambda(-x_1)=\Gamma_1^{-1}. \tag{37}$$

But $A<0$ by Eq. (31) and so Eq. (37) implies that

$$\gamma>(x_0-x_1)[\Gamma_1^{-1}-\lambda(-x_1)]. \tag{38}$$

Eliminating A from Eqs. (30) and (37) gives

$$\gamma<(x_0/x_1)(x_0-x_1)(\Gamma_1^{-1}-\chi). \tag{39}$$

Equations (38) and (39) can hold simultaneously if and only if Eq. (36) again holds. For each x_0 in this range, there is a range of values of γ permitted by Eqs. (38) and (39) and for each such γ the value of A is given by Eq. (37). Hence we have a double infinity of functions satisfying the conditions of Sec. I, characterized by the continuous parameters x_0 and γ, for which we use the notation $f_3^{(+)}(z)=N(z)/D_3^{(+)}(z)$.

(4) $\gamma=0$, $B\neq 0$. In this case Eqs. (26) and (30) become

$$A-Bx_1+\lambda(-x_1)=\Gamma_1^{-1}, \tag{40}$$

$$A-\frac{1}{\pi}\int_0^\infty\frac{q(t)R(t)}{t(t+x_1)}dt>0. \tag{41}$$

Since $B<0$ by Eq. (27), Eq. (40) implies that

$$A<\Gamma_1^{-1}-\lambda(-x_1). \tag{42}$$

Thus for each pair of pole parameters x_1, Γ_1 allowed by Eq. (33), there is a range of values of A permitted by

Eqs. (41) and (42) and for each A in this range the value of B is given by Eq. (40). Hence we have a single infinity of functions satisfying the conditions of Sec. I, characterized by the continuous parameter A and denoted by $f_4^{(+)}(z)=N(z)/D_4^{(+)}(z)$.

IV. FINITE NUMBER OF POLES

We consider now the general problem with n poles which is formulated in Eqs. (2) and (3). The construction of functions $f(z)$ from which solutions are obtained proceeds as follows. Define

$$N(z)=\sum_{i=1}^{n} \nu_i/(z+x_i), \tag{43}$$

where

$$\sum_{i=1}^{n} \nu_i=1, \tag{44}$$

$$D(z)=A+\gamma/(z+x_0)+Bz+\lambda(z), \tag{45}$$

with

$$\lambda(z)=-\frac{1}{\pi}\int_0^\infty \frac{q(t)R(t)N(t)}{t-z}dt. \tag{46}$$

In order that $D(z^*)=D^*(z)$ when z is not a singular point of $D(z)$ the constants A, B, γ, and x_0 are chosen to be real. If $D(x)$ is the boundary value of $D(z)$ as z approaches the cut $[0,\infty)$ from above, then

$$\operatorname{Im}D(x)=-q(x)R(x)N(x),$$

and $\operatorname{Im}D(x)\neq 0$ for $x>0$ provided $N(x)\neq 0$ for $x>0$. Define further $f(z)=N(z)/D(z)$. Since $f(z)$ must have residues Γ_i at the poles $-x_i$ we require that

$$D(-x_i)=\nu_i\Gamma_i^{-1}, \quad i=1, 2, \cdots, n. \tag{47}$$

Using Eqs. (45), (46), and (43), Eq. (47) becomes

$$\sum_{j=1}^{n} \alpha_{ij}\nu_j=A+\gamma/(x_0-x_i)-Bx_i, \tag{48}$$

where

$$\alpha_{ij}=\delta_{ij}\Gamma_j^{-1}+\frac{1}{\pi}\int_0^\infty \frac{q(t)R(t)}{(t+x_i)(t+x_j)}dt$$

and δ_{ij} is the Kronecker delta. Let (β_{ij}) be the matrix inverse to (α_{ij}); we must of course assume that the matrix (α_{ij}) is nonsingular. Then the linear equations (48) give

$$\nu_i=\sum_{j=1}^{n} \beta_{ij}(A+\gamma/(x_0-x_j)-Bx_j), \quad i=1, 2, \cdots, n, \tag{49}$$

and therefore from Eq. (43),

$$N(z)=A\sum_{i=1}^{n} \frac{a_i}{(z+x_i)}+B\sum_{i=1}^{n} \frac{b_i}{(z+x_i)}+\gamma\sum_{i=1}^{n} \frac{c_i}{(z+x_i)}, \tag{50}$$

where

$$a_i=\sum_{j=1}^{n} \beta_{ij}, \quad b_i=-\sum_{j=1}^{n} \beta_{ij}x_j, \quad c_i=\sum_{j=1}^{n} \frac{\beta_{ij}}{x_0-x_j}. \tag{51}$$

Similarly, inserting Eq. (49) into Eq. (44), we have

$$A\sum_{i=1}^{n} a_i+B\sum_{i=1}^{n} b_i+\gamma\sum_{i=1}^{n} c_i=1. \tag{52}$$

From Eqs. (45) and (46) it follows that, for $\operatorname{Im}z\neq 0$,

$$\operatorname{Im}D(z)=\operatorname{Im}z\left(-\frac{\gamma}{|z+x_0|^2}+B-\frac{1}{\pi}\int_0^\infty \frac{q(t)R(t)N(t)}{|t-z|^2}dt\right). \tag{53}$$

Since, for real x sufficiently large, $N(x)>0$ it is clear from Eq. (53) that sufficient conditions for $D(z)$ to have no zeros for $\operatorname{Im}z\neq 0$ are[20]

$$N(x)=A\sum_{i=1}^{n} \frac{a_i}{(x+x_i)}+B\sum_{i=1}^{n} \frac{b_i}{(x+x_i)}+\gamma\sum_{i=1}^{n} \frac{c_i}{(x+x_i)}>0 \quad \text{for all } x\geqslant 0, \tag{54}$$

$$\gamma\geqslant 0, \quad B\leqslant 0. \tag{55}$$

Under these conditions we see that

$$dD(x)/dx<0 \quad \text{for} \quad x<0. \tag{56}$$

As in Sec. III, if $\gamma>0$ then we must have $B=0$ to avoid a zero in $D(x)$ for $x<-x_0$. We therefore distinguish four cases in the same way as in Sec. III and use the subscripts 1 to 4 and the superscripts $(\pm)$ to denote solutions which correspond to those of Secs. II and III for the case of a single pole. We consider the first and fourth cases to begin with.

(1) $\gamma=0$, $B=0$. From Eqs. (45), (46), and (56) it is clear that there are two different conditions under which $D(x)$ has no zero for $x\leqslant 0$, namely,

$$A>-\lambda(0) \quad \text{or} \quad A<0.$$

But Eqs. (52) and (54) in this case become

$$A=(\sum_{i=1}^{n} a_i)^{-1}, \tag{57}$$

$$A\sum_{i=1}^{n} \frac{a_i}{(x+x_i)}>0 \quad \text{for all } x\geqslant 0. \tag{58}$$

Now if

$$\sum_{i=1}^{n} \frac{a_i}{(x+x_i)}<0 \quad \text{for all } x\geqslant 0 \text{ and } \sum_{i=1}^{n} a_i\neq 0, \tag{59}$$

then $\sum^{n}_{i=1} a_i<0$, the constant A determined by Eq. (57) is negative and Eq. (58) is satisfied. We thus obtain a function $f_1^{(-)}(z)$ which gives the standard N/D

[20] Actually $N(x)$ could have isolated zeros of even order for $x>0$; we do not consider this exceptional possibility.

solution. The condition (59) reduces to (15) for the case of a single pole with negative residue.

If, on the other hand,

$$\sum_{i=1}^{n} \frac{a_i}{(x+x_i)} > 0 \quad \text{for all } x \geqslant 0 \text{ and } \sum_{i=1}^{n} a_i \neq 0, \qquad (60)$$

then $\sum^n{}_{i=1} a_i > 0$, the constant A determined by Eq. (57) is positive and Eq. (58) is again satisfied. However, we require the further restriction $A > -\lambda(0)$, which, on using Eqs. (46) and (50), becomes

$$\sum_{i=1}^{n} a_i \lambda_i < 1, \qquad (61)$$

where

$$\lambda_i = -\frac{1}{\pi} \int_0^\infty \frac{q(t)R(t)}{t(t+x_i)} dt, \quad i=1,2,\cdots,n. \qquad (62)$$

We therefore obtain a function $f_1^{(+)}(z)$ which gives the standard N/D solution if the two conditions (60) and (61) hold. For a single pole with positive residue, Eq. (60) is no restriction at all while Eq. (61) reduces to Eq. (33).

(4) $\gamma = 0$, $B \neq 0$. Since $B<0$, $D(x) \rightarrow \infty$ as $x \rightarrow -\infty$. Equation (56) then shows that $D(x)$ has no zeros for $x \leqslant 0$ if and only if

$$D(0) = A + \lambda(0) > 0. \qquad (63)$$

Equations (52) and (54) become

$$A \sum_{i=1}^{n} a_i + B \sum_{i=1}^{n} b_i = 1, \qquad (64)$$

$$A \sum_{i=1}^{n} \frac{a_i}{(x+x_i)} + B \sum_{i=1}^{n} \frac{b_i}{(x+x_i)} > 0 \quad \text{for all } x \geqslant 0. \qquad (65)$$

We restrict ourselves to functions corresponding to small negative values of B. If then Eq. (59) holds it follows from either Eq. (64) or Eq. (65) that we must take $A<0$. But then Eq. (63) cannot hold, since $\lambda(0)<0$. Therefore we can have only functions $f_4^{(+)}(z)$, for the case when Eq. (60) holds. When Eq. (60) is satisfied it is clear that there is a positive constant k_1 such that

$$\left| \sum_{i=1}^{n} \frac{b_i}{(x+x_i)} \right| < k_1 \sum_{i=1}^{n} \frac{a_i}{(x+x_i)} \quad \text{for all } x \geqslant 0;$$

then Eq. (65) holds for

$$A>0, \quad |B| < k_1^{-1} A.$$

Using Eqs. (46) and (50), Eq. (63) becomes

$$A\left(1 - \sum_{i=1}^{n} \lambda_i a_i\right) - B \sum_{i=1}^{n} \lambda_i b_i > 0, \qquad (66)$$

where λ_i is given by Eq. (62). Hence, provided condition (61) holds once again, Eq. (66) is satisfied for $A>0$, $|B| < k_2^{-1} A$, where

$$k_2 = \left(\left| \sum_{i=1}^{n} \lambda_i b_i \right| \Big/ \left(1 - \sum_{i=1}^{n} \lambda_i a_i\right) \right).$$

Since $B<0$ it follows that we have a single infinity of functions $f_4^{(+)}(z)$ for which

$$A>0, \quad 0 < -B < A \min\{k_1^{-1}, k_2^{-1}\},$$

$$A \sum_{i=1}^{n} a_i + B \sum_{i=1}^{n} b_i = 1.$$

The conditions to be satisfied by the input parameters and the function $R(x)$ are just (60) and (61) as in case (1).

We turn now to the other two cases.

(2) $B=0$, $A=0$. Equations (52) and (54) become

$$\gamma \sum_{i=1}^{n} c_i = 1, \qquad (67)$$

$$\sum_{i=1} \frac{c_i}{(x+x_i)} > 0 \quad \text{for all } x \geqslant 0, \qquad (68)$$

since $\gamma > 0$ by Eq. (55). If $\sum^n{}_{i=1} c_i \neq 0$, Eq. (68) implies that $\sum^n{}_{i=1} c_i > 0$ and so Eq. (67) will correctly give a value of γ greater than 0. From Eq. (51) it is seen that the constants c_i depend on x_0. There are two cases where we can be certain that values of x_0 can be found for which Eq. (68) is satisfied, namely,

$$\sum_{i,j=1}^{n} \frac{\beta_{ij}}{x_j(x+x_i)} < 0 \quad \text{for all } x \geqslant 0 \text{ and } \sum_{i,j=1}^{n} \frac{\beta_{ij}}{x_j} \neq 0, \qquad (69)$$

and

$$\sum_{i=1}^{n} \frac{a_i}{x+x_i} > 0 \quad \text{for all } x \geqslant 0 \text{ and } \sum_{i=1}^{n} a_i \neq 0. \qquad (60)$$

In either case there will be a further condition which ensures that $D(x)$ has no zeros for $x \leqslant 0$, namely, $D(0) > 0$, which reduces to

$$1/x_0 - \sum_{i=1}^{n} \lambda_i c_i > 0 \qquad (70)$$

on using Eqs. (45), (46), and (50).

When condition (69) holds, Eqs. (68) and (70) are satisfied for all sufficiently small x_0. For we can find a *positive* constant k_3 such that

$$\sum_{i,j=1}^{n} \frac{|\beta_{ij}|}{x_j(x_j - \frac{1}{2}x_1)(x+x_i)} < -k_3 \sum_{i,j=1}^{n} \frac{\beta_{ij}}{x_j(x+x_i)} \text{ for all } x \geqslant 0.$$

Then, if

$$k_4 = \sum_{i,j=1}^{n} \frac{|\lambda_i|\,|\beta_{ij}|}{x_j - \frac{1}{2}x_1},$$

it follows from the definition of the constants c_i in Eq. (51) that Eqs. (68) and (70) are satisfied and that $\sum^n{}_{i=1} c_i>0$ for $x_0<\min\{k_3^{-1},k_4^{-1},\frac{1}{2}x_1\}$. There is therefore a range of values of x_0 for which there exist functions $f_2^{(-)}(z)$ satisfying the conditions of Sec. I.

Similarly, if Eqs. (60) and (61) hold, then Eqs. (68) and (70) are satisfied and $\sum^n{}_{i=1} c_i>0$ for

$$x>x_n+\max\{k_5,k_6\}\,,$$

where

$$\sum_{i,j=1}^{n}\frac{x_j|\beta_{ij}|}{x+x_i}<k_5\sum_{i=1}^{n}\frac{a_i}{x+x_i}\quad\text{for all } x\geqslant 0$$

and

$$k_6=\sum_{i,j=1}^{n}|\beta_{ij}|\,|\lambda_i|x_j/(1-\sum_{i=1}^{n}\lambda_i a_i).$$

Again there is a range of values of x_0 for which there are functions $f_2^{(+)}(z)$ satisfying the conditions of Sec. I.

(3) $B=0$, $\gamma\neq 0$, $A\neq 0$. We write Eqs. (52) and (54) once more for this case:

$$A\sum_{i=1}^{n}a_i+\gamma\sum_{i=1}^{n}c_i=1\,,\tag{71}$$

$$A\sum_{i=1}^{n}\frac{a_i}{(x+x_i)}+\gamma\sum_{i=1}^{n}\frac{c_i}{(x+x_i)}>0\quad\text{for all } x\geqslant 0.\tag{72}$$

Since $\gamma>0$ it is clear that to avoid a zero of $D(x)$ for $x<x_0$ we must have

$$A<0.\tag{73}$$

Finally, in view of Eq. (56), the function $D(x)$ has no zeros for $x\leqslant 0$ if and only if

$$D(0)=\gamma/x_0+A+\lambda(0)>0.\tag{74}$$

Now there are three cases under which it is possible to obtain functions $f_3(z)$ in a straightforward way. The simplest case to consider is that for which Eqs. (60) and (61) hold. A little work then shows that Eqs. (72) and (74) are satisfied if

$$(1+Ax_0/\gamma)(x_0-x_n)>\max\{k_5,k_6\}=X\,,\quad\text{say}.$$

Hence for each $x_0>x_n+X$, there exist functions of the type we have called $f_3^{(+)}(z)$. For fixed x_0 in this range, there is a single infinity of functions for which A, γ are restricted by Eq. (71) and

$$-\gamma/A>\frac{x_0(x_0-x_n)}{x_0-x_n-X}.$$

Next, if the condition (69) holds we can find functions which join on to the functions $f_2^{(-)}(z)$. If

$$\left|\sum_{i=1}^{n}\frac{a_i}{x+x_i}\right|<-k_7\sum_{i,j=1}^{n}\frac{\beta_{ij}}{x_j(x+x_i)}\quad\text{for all } x\geqslant 0$$

and

$$k_8=|1-\sum_{i=1}^{n}\lambda_i a_i|\,,$$

then it is found after some work that Eqs. (72) and (74) are satisfied for

$$0<-A/\gamma<\min\{k_7^{-1}(1-k_3x_0),\,k_8^{-1}(x_0^{-1}-k_4)\}\,,\tag{75}$$

the constants k_3 and k_4 having been defined earlier. Thus for each $x_0<\min\{k_3^{-1},k_4^{-1},\frac{1}{2}x_1\}$ there exist functions $f_3^{(-)}(z)$ satisfying the conditions of Sec. I. For each x_0 in this range there is a single infinity of functions for which A, γ are restricted by Eqs. (71) and (75).

The interesting point is that there is another condition under which we can obtain functions $f_3^{(-)}(z)$, namely Eq. (59). This was the condition under which the function $f_1^{(-)}(z)$ could be found. As above, Eq. (74) is satisfied if

$$-\gamma/A>k_8/(x_0^{-1}-k_4).\tag{76}$$

Moreover, we can find a positive constant k_9 such that

$$\sum_{i,j=1}^{n}\frac{|\beta_{ij}|}{(x_j-\frac{1}{2}x_1)(x+x_i)}<-k_9\sum_{i=1}^{n}\frac{a_i}{x+x_i}\quad\text{for all } x\geqslant 0,$$

and then Eq. (72) holds if

$$x_0<\tfrac{1}{2}x_1,\quad -\gamma/A<k_9^{-1}.\tag{77}$$

Thus, for $x_0<\min\{\frac{1}{2}x_1,\,(k_4+k_8k_9)^{-1}\}$, Eqs. (76) and (77) permit a range of values of γ/A for which Eqs. (72) and (74) hold and functions $f_3^{(-)}(z)$ exist.

This completes the three cases. Note that for a single pole with negative residue the conditions (59) and (69) are equivalent. In that case the functions $f_3^{(-)}(z)$ for $-\gamma/A$ large and those for $-\gamma/A$ small continue into each other; there are functions for all

$$-\gamma/A>x_0\left[1+\frac{\lambda(0)x_1}{(x_1-x_0)(-\Gamma_1^{-1}+\chi)}\right]^{-1}$$

for each x_0 given by Eq. (20). However, for more than one pole the conditions (59) and (69) are no longer equivalent. A simple numerical example quickly shows this. Take $R(x)=1$ for all $x\geqslant 0$, $m_1=m_2=1$ and two poles with $x_1=16$, $x_2=25$. Then for $\Gamma_1=144/23$, $\Gamma_2=-10$, say, it is easily verified that Eq. (59) holds but Eq. (69) does not. On the other hand, if say $\Gamma_1=-15$, $\Gamma_2=20$, Eq. (69) holds but Eq. (59) does not. Thus *all* the functions $f^{(+)}(z)$ exist under the same conditions (60) and (61), while the functions $f^{(-)}(z)$ divide into two classes which exist under the two *inequivalent* conditions (59) and (69), respectively.

V. DISCUSSION OF THE RESULTS

In order to apply the mathematical results of Secs. II–IV to partial-wave amplitudes for elastic scattering it is necessary to assume that the lowest inelastic threshold is above the elastic threshold. If an inelastic channel is open at the elastic threshold, $R(x)\rightarrow\infty$ (like $x^{-1/2}$) as $x\rightarrow 0$ and $R(x)$ cannot satisfy a uniform Lipschitz condition in any finite interval $[0,\xi]$. Further, the results as they stand apply only to the case of an s wave and for the present we consider this case only.

For an s-wave amplitude below the inelastic threshold $f(x)=\sin\delta(x)e^{i\delta(x)}/q(x)$ and the scattering length a is given by

$$a=\lim_{x\rightarrow 0}\frac{\sin\delta(x)}{q(x)}=f(0).$$

We therefore look at the sign of $f(0)=N(0)/D(0)$ for each of the functions $f(z)$ constructed in Secs. II–IV. In every case, $D(0)$ has a definite sign and $N(0)>0$. It is readily seen that $f_1^{(-)}(0)<0$ but that $f(0)>0$ in all the other cases. Thus, when the left-hand cut is approximated by a finite number of poles, a solution which gives a negative scattering length is obtained only when the pole parameters satisfy Eq. (59); then only the usual N/D solution has a negative scattering length. On the other hand there are clearly various conditions and several types of solution which give a positive scattering length.

We next consider the asymptotic behavior of $f(x)$ for large x. To make a definite statement about the asymptotic behavior of $D(x)$ we need to postulate some definite behavior of $R(x)$ for large x. We want to apply Theorem 3 of Appendix 2 and therefore assume that

$$x^{-\mu}R(x)=r_0+r(x),$$

where $r_0>0$, $0\leqslant\mu<\frac{1}{2}$ and the function $r(x)$ satisfies the conditions laid down in the theorem for $g_1(x)$. The function $q(x)R(x)N(x)$ then satisfies the conditions on $g(x)$, with $A=r_0$ and $\alpha=\frac{1}{2}-\mu$. Applying the theorem, then, we see that, for large x,

$$\operatorname{Im}\lambda(x)\sim -r_0x^{\mu-\frac{1}{2}}, \tag{78}$$

$$\operatorname{Re}\lambda(x)\sim -r_0\cot\pi(\tfrac{1}{2}-\mu)x^{\mu-\frac{1}{2}}. \tag{79}$$

From Eq. (78) it follows that

$$\operatorname{Im}D(x)\sim -r_0x^{\mu-\frac{1}{2}} \tag{80}$$

in all cases. Also, from Eqs. (43) and (44) we have $N(x)\sim 1/x$ for large x. We can now distinguish four separate cases, with a different asymptotic behavior of $\operatorname{Re}f(x)$ and $\operatorname{Im}f(x)$ for each.

(1) $A=B=0$, $\mu=0$ [$f_2^{(\pm)}(x)$ with $\mu=0$]. From Eqs. (45) and (79),

$$\lim_{x\rightarrow\infty}x^{1/2}\operatorname{Re}D(x)=0. \tag{81}$$

It follows from Eqs. (80) and (81) that

$$\lim_{x\rightarrow\infty}x^{1/2}\operatorname{Re}f(x)=0,$$
$$\operatorname{Im}f(x)\sim r_0^{-1}x^{-1/2}.$$

It is not possible in this case to say anything about the sign of $\operatorname{Re}f(x)$ for large x.

(2) $A=B=0$, $0<\mu<\frac{1}{2}$ [$f_2^{(\pm)}(x)$ with $\mu>0$]. Now Eqs. (45) and (79) give $\operatorname{Re}D(x)\sim -r_0\cot\pi(\frac{1}{2}-\mu)x^{\mu-\frac{1}{2}}$, so that, using Eq. (80),

$$\operatorname{Re}f(x)\sim -\frac{\cot\pi(\frac{1}{2}-\mu)x^{-\mu-\frac{1}{2}}}{r_0\csc^2\pi(\frac{1}{2}-\mu)},$$

$$\operatorname{Im}f(x)\sim\frac{x^{-\mu-\frac{1}{2}}}{r_0\csc^2\pi(\frac{1}{2}-\mu)}.$$

(3) $B=0$, $A\neq0$, $0\leqslant\mu<\frac{1}{2}$ [$f_1^{(\pm)}(x)$, $f_3^{(\pm)}(x)$]. It follows from Eqs. (45) and (79) that in this case $\lim_{x\rightarrow\infty}\operatorname{Re}D(x)=A$ and so, from Eq. (80),

$$\operatorname{Re}f(x)\sim A^{-1}x^{-1},$$
$$\operatorname{Im}f(x)\sim r_0A^{-2}x^{\mu-\frac{3}{2}}.$$

(4) $B\neq0, A\neq0, 0\leqslant\mu<\frac{1}{2}$ [$f_4^{(+)}(x)$]. Using Eqs. (45), (79), and (80) once more, we have, for large x,

$$\operatorname{Re}D(x)\sim Bx,$$
$$\operatorname{Re}f(x)\sim B^{-1}x^{-2},$$
$$\operatorname{Im}f(x)\sim r_0B^{-2}x^{\mu-\frac{7}{2}}.$$

The different classes of solutions we have constructed thus exhibit quite different behavior of $\operatorname{Re}f(x)$ and $\operatorname{Im}f(x)$ for large x.

It will be seen that, in cases (2)–(4) above, $\operatorname{Re}f(x)$ has a definite sign for all sufficiently large x. On taking account of the sign of the scattering length $f(0)$ it is clear that $\operatorname{Re}f_2^{(\pm)}(x)$ (for $0<\mu<\frac{1}{2}$) and $\operatorname{Re}f_3^{(\pm)}(x)$ and $\operatorname{Re}f_4^{(+)}(x)$ (for $0\leqslant\mu<\frac{1}{2}$) must change sign at least once on the positive real axis. However, no such statement can be made about $\operatorname{Re}f_1^{(\pm)}(x)$ or about $\operatorname{Re}f_2^{(\pm)}(x)$ for $\mu=0$.

Finally we consider the case of higher partial waves. Here one expects, from the theory of potential scattering and from empirical experience, that

$$\lim_{x\rightarrow 0}\frac{x^l}{f_l(x)}\neq0, \tag{82}$$

where l is the orbital angular momentum of the amplitude $f_l(x)$. Our solution in Secs. II–IV were constructed so that $f(0)\neq0$ and so they do not satisfy Eq. (82) if $l>0$. To try to obtain solutions which satisfy Eq. (82) for the case of an amplitude with $l>0$, it would be necessary to approximate the left-hand cut by more than one pole (the number of poles will increase with increasing l) and to impose restrictions on the pole parameters. This difficulty is a very serious one in practice.

For the case of a p wave, however, a modification of the previous method can be used. If $f(x)=f_1(x)+if_2(x)$ is a p-wave amplitude we define $F_1(x)=f_1(x)/x$,

$F_2(x)=f_2(x)/x$ and consider the system of equations

$$F_1(x)-\frac{P}{\pi}\int_0^\infty \frac{F_2(t)dt}{t-x}=\sum_{i=1}^n \frac{\Gamma_i'}{x+x_i}, \tag{83}$$

$$F_2(x)=xq(x)R(x)[F_1^2(x)+F_2^2(x)]. \tag{84}$$

One then looks for an analytic function $F(z)$ satisfying conditions (a)–(f) of Sec. I, except that, in condition (c), Eq. (3) is replaced by Eq. (84). By using the methods of Sec. IV it is not difficult to show that to avoid zeros of $D(z)$ the constant γ must be zero and that possible functions $F(z)$ will be of the form $F(z)=N(z)/N(z)$, with $N(z)$ as in Eqs. (43) and (44) and

$$D(z)=A+Bz+z\lambda(z), \tag{85}$$

$$A>0, \quad B\leqslant 0. \tag{86}$$

The essential point to notice is that $x\lambda(x)\to\infty$ as $x\to-\infty$. Following through a calculation exactly like that in Sec. IV, we find that functions $F(z)$ exist, for a range of small non-negative values of $-B/A$, which satisfy the correct conditions if

$$\sum_{i,j=1}^n \frac{\beta_{ij}'}{x+x_i}>0 \quad \text{for all } x\geqslant 0, \tag{87}$$

where (β_{ij}') is the matrix inverse to (α_{ij}') and

$$\alpha_{ij}'=\delta_{ij}\Gamma_j'^{-1}-\frac{x_i}{\pi}\int_0^\infty \frac{q(t)R(t)}{(t+x_i)(t+x_j)}dt.$$

For a single pole, Eq. (87) becomes simply

$$\Gamma_1'^{-1}>\frac{x_1}{\pi}\int_0^\infty \frac{q(t)R(t)}{(t+x_1)^2}dt.$$

Since a range of values of $-B/A$ is allowed, the solution to the problem is not unique for this p-wave case also.

There are two further points to notice in connection with the p-wave case. First, the scattering length is always positive; it is impossible to obtain a p-wave amplitude with a negative scattering length by solving Eqs. (83) and (84) for $F_1(x)$ and $F_2(x)$. Second, even if it is believed that the N/D solution is the "correct" physical solution there is a practical difficulty in the p-wave case, namely that the convergence of the integral defining $\lambda(z)$ is very slow (even if, say, $R(x)\to$ constant as $x\to\infty$) and therefore the solution $F(x)$ depends very strongly on the values of $R(x)$ for large x. For example, from actual numerical calculations on the pion-nucleon scattering amplitude with $l=1$, $J=\frac{3}{2}$, $I=\frac{3}{2}$, using a realistic three-pole approximation to the left-hand singularities, it is known[21] that small changes in $R(x)$ in the GeV region can move the position of the N_{33}^* resonance given by the N/D solution through a range of several hundred MeV. This difficulty of the N/D solution for the p-wave case is not shared by the N/D solution $f_1^{(\pm)}$ for an s wave; this can be seen by writing $D(z)$ in the form

$$D(z)=A'-\frac{z}{\pi}\int_0^\infty \frac{q(t)R(t)N(t)}{t(t-z)}dt.$$

But, in view of the variety of solutions which have been constructed in this paper, it is clear that additional criteria are required to decide which solution, if any, is an approximation to the physical s-wave amplitude under consideration.

[21] G. C. Oades (private communication).

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APPENDIX I

We prove here that the function $f(z)$ constructed in Secs. II–IV satisfies the conditions (d), (e), and (f) of Sec. I. Now $f(z)=N(z)/D(z)$; $N(z)$ is an analytic function, regular in the whole z plane except for poles at $-x_1, \cdots, -x_n$, and $N(z)\sim 1/z$ as $|z|\to\infty$, uniformly with respect to θ.[22] $D(z)$ is defined as follows:

$$D(z)=A+Bz+\frac{\gamma}{z+x_0}+\lambda(z),$$

where

$$\lambda(z)=-\frac{1}{\pi}\int_0^\infty \frac{dtq(t)R(t)N(t)}{t-z} \quad \text{for} \quad 0<\theta<2\pi,$$

$$\lambda(x\pm i0)=-\frac{P}{\pi}\int_0^\infty \frac{dtq(t)R(t)N(t)}{t-x}\mp iq(x)R(x)N(x) \quad \text{for} \quad \theta=\begin{pmatrix}0\\2\pi\end{pmatrix}.$$

The constants B, γ satisfy $B\leqslant 0$, $\gamma\geqslant 0$. The angle θ is defined by $z=re^{i\theta}$, $0\leqslant\theta\leqslant 2\pi$; it is necessary to distinguish the upper and lower sides of the cut by the values 0, 2π of θ.

To prove condition (e), we first show that

$$\lambda(z)\to-\frac{1}{\pi}\int_0^\infty \frac{dtq(t)R(t)N(t)}{t}$$

[22] Henceforth, "uniformly with respect to θ" will mean that the range $0\leqslant\theta\leqslant 2\pi$ is intended, unless the contrary is specifically stated.

as $z\to 0$, uniformly with respect to θ. The integral on the right clearly converges at both limits. In fact, we need consider only the range of integration $[0,\xi]$ and $|z|<\xi$. Now

$$\int_0^\xi dt\, q(t)R(t)N(t)\left(\frac{1}{t-z}-\frac{1}{t}\right)=z\int_0^\xi (dt/t^{1/2})(g(t)/t-z),$$

where $g(t)=(q(t)R(t)N(t)/t^{1/2})$. Since $R(t)$ satisfies a Lipschitz condition of order μ, uniformly in the interval $[0,\xi]$, the same is true for $g(t)$. Let

$$M=\max\{g(t):t\in[0,\xi]\}.$$

Define $\phi(z)$ by

$$\phi(z)=\int_0^\xi \frac{dt}{t^{1/2}}\frac{g(t)}{t-z} \qquad \text{for} \quad 0<\theta<2\pi,$$

$$\phi(x\pm i0)=P\int_0^\xi \frac{dt}{t^{1/2}}\frac{g(t)}{t-x}\pm i\pi\frac{g(x)}{x^{1/2}} \quad \text{for} \quad \theta=\begin{pmatrix}0\\2\pi\end{pmatrix}.$$

First restrict z to the sector $\beta\leqslant\theta\leqslant 2\pi-\beta$, where $0<\beta<\pi/2$. Then

$$\begin{aligned}|z\phi(z)|&\leqslant rM\int_0^\xi \frac{dt}{t^{1/2}(t^2+r^2-2tr\cos\beta)^{1/2}}\\ &=r^{1/2}M\int_0^{\xi/r}\frac{du}{u^{1/2}(u^2+1-2u\cos\beta)^{1/2}}\\ &<r^{1/2}M\int_0^\infty \frac{du}{u^{1/2}(u^2+1-2u\cos\beta)^{1/2}}.\end{aligned}$$

Now consider the sectors $0\leqslant\theta\leqslant\beta$ and $2\pi-\beta\leqslant\theta\leqslant 2\pi$. Here

$$\begin{aligned}|z\,\mathrm{Im}\phi(z)|&\leqslant x\sec\theta\, M|y|\int_0^\xi \frac{dt}{t^{1/2}[(t-x)^2+y^2]} \qquad (y\neq 0)\\ &=x^{1/2}\sec\theta\, M|\tan\theta|\\ &\quad\times\int_0^{\xi/x}\frac{du}{u^{1/2}[(u-1)^2+\tan^2\theta]} \qquad (\tan\theta\neq 0)\\ &<x^{1/2}\sec\theta\, M|\tan\theta|\int_0^\infty \frac{du}{u^{1/2}[(u-1)^2+\tan^2\theta]}\\ &<\pi Mx^{1/2}\sec\beta,\end{aligned}$$

while, for $y=0$, $|x\,\mathrm{Im}\phi(x\pm i0)|=\pi g(x)x^{1/2}\leqslant\pi Mx^{1/2}$. Finally,

$$\begin{aligned}\mathrm{Re}\phi(z)&=\int_0^\xi \frac{dt(t-x)(g(t)-g(x))}{t^{1/2}[(t-x)^2+y^2]}\\ &\qquad+g(x)\int_0^\xi \frac{dt(t-x)}{t^{1/2}[(t-x)^2+y^2]}\\ &=\phi_1(z)+\phi_2(z), \quad \text{say.}\end{aligned}$$

Then

$$\begin{aligned}|z\phi_1(z)|&\leqslant x\sec\beta\, K\int_0^\xi \frac{dt|t-x|^{1+\mu}}{t^{1/2}(t-x)^2}\\ &=x^{\frac12+\mu}\sec\beta\, K\int_0^{\xi/x}\frac{du}{u^{1/2}|u-1|^{1-\mu}}\\ &<x^{\frac12+\mu}\sec\beta\, K\int_0^\infty \frac{du}{u^{1/2}|u-1|^{1-\mu}}.\end{aligned}$$

For the last integral to exist we must take $\mu<\frac12$, as we certainly can. Also

$$\begin{aligned}|z\phi_2(z)|&\leqslant x\sec\theta\, M\left|\int_0^\xi \frac{dt(t-x)}{t^{1/2}[(t-x)^2+y^2]}\right|\\ &=x^{1/2}\sec\theta\, M\left|\int_0^{\xi/x}\frac{du(u-1)}{u^{1/2}[(u-1)^2+\tan^2\theta]}\right|\\ &<2\ln(\sqrt{2}+1)Mx^{1/2}\sec\beta \quad \text{for} \quad x<\xi/2.\end{aligned}$$

Hence

$$D(z)\to A+\frac{\gamma}{x_0}-\frac{1}{\pi}\int_0^\infty \frac{dt\, q(t)R(t)N(t)}{t} \quad \text{as} \quad z\to 0,$$

uniformly with respect to θ. The limit on the right side we have denoted by $D(0)$. In all our solutions, $D(0)\neq 0$ and (e) follows.

To prove (d), consider first the right half plane $x\geqslant 0$. Here, for $y\neq 0$,

$$\begin{aligned}|D(z)|&\geqslant|\mathrm{Im}D(z)|\\ &=\left|\frac{-\gamma y}{(x+x_0)^2+y^2}-\frac{y}{\pi}\int_0^\infty \frac{dt\, q(t)R(t)N(t)}{(t-x)^2+y^2}+By\right|\\ &\geqslant\frac{|y|}{\pi}\int_0^\infty \frac{dt\, q(t)R(t)N(t)}{(t-x)^2+y^2}\geqslant\frac{|y|}{\pi}\int_0^\infty \frac{dt\, q(t)N(t)}{(t-x)^2+y^2}\\ &=|\mathrm{Im}\psi(z)|,\end{aligned}$$

where

$$\psi(z)=\frac{1}{\pi}\int_0^\infty \frac{dt\, q(t)N(t)}{t-z} \quad \text{for} \quad 0<\theta<2\pi.$$

If we define

$$\psi(x\pm i0)=\frac{P}{\pi}\int_0^\infty \frac{dt\, q(t)N(t)}{t-x}\pm iq(x)N(x) \qquad \text{for} \quad \theta=\begin{pmatrix}0\\2\pi\end{pmatrix},$$

then we have $|D(z)|\geqslant|\mathrm{Im}\psi(z)|$ for all z such that $x\geqslant 0$, $z\neq 0$.

Now

$$q(t)N(t)=\frac{1}{2t^{1/2}}+h(t),$$

where $h(t)$ satisfies the conditions of Theorem 3 of Appendix II. Therefore

$$\psi(z)=\tfrac{1}{2}iz^{-1/2}+\frac{\chi(z)}{z},$$

where

$$\chi(z)\to-\int_0^\infty h(t)dt \quad\text{as}\quad |z|\to\infty,$$

uniformly with respect to θ. Hence

$$\begin{aligned}|z^{1/2}D(z)|&\geqslant|\tfrac{1}{2}\cos\tfrac{1}{2}\theta+r^{1/2}\operatorname{Im}[\chi(z)/z]|\\&\geqslant|\tfrac{1}{2}\cos\tfrac{1}{2}\theta|-r^{-1/2}|\chi(z)|,\end{aligned}$$

provided this last expression is non-negative. Hence we can choose R such that $|z^{1/2}D(z)|\geqslant\frac{1}{4}$, say, for all z such that $r\geqslant R$, $x\geqslant 0$.

We consider finally the left half plane $x\leqslant 0$, and take the various cases in turn. If $B\neq 0$, then $(D(z)/z)\to B$, $f(z)=N(z)/D(z)\sim B^{-1}z^{-2}$ as $|z|\to\infty$, uniformly with respect to θ for $\pi/2\leqslant\theta\leqslant 3\pi/2$. Next, if $B=0$, $A\neq 0$, then $D(z)\to A$, $f(z)\sim A^{-1}z^{-1}$ as $|z|\to\infty$, uniformly as before. Lastly, if $B=0$, $A=0$, we have

$$-\operatorname{Re}\lambda(z)\geqslant\operatorname{Re}\psi(z),\quad|\operatorname{Im}\lambda(z)|\geqslant|\operatorname{Im}\psi(z)|,$$

so that

$$|\lambda(z)|\geqslant|\psi(z)|,$$

$$|z^{1/2}D(z)|\geqslant\tfrac{1}{2}-\frac{|\chi(z)|}{r^{1/2}}-\frac{\gamma r^{1/2}}{(r-x_0)},$$

provided this last expression is non-negative. Hence we can choose R' such that $|z^{1/2}D(z)|\geqslant\frac{1}{4}$, say, for all z such that $r\geqslant R'$, $x\leqslant 0$.

Since $N(z)\sim 1/z$ as $|z|\to\infty$, uniformly with respect to θ, it is clear that (d) holds for $f(z)=N(z)/D(z)$.

Finally, note that,

$$\text{for } x>0,\quad f_2(x)=-\frac{N(x)\operatorname{Im}D(x+i0)}{|D(x+i0)|^2},$$

and that $|D(x+i0)|\neq 0$ for all $x>0$. Condition (f) on the function $f_2(x)$ then follows from condition (γ) on $R(x)$ and Theorem 2(a) of Appendix II. [In fact a stronger condition than (f) clearly holds.]

APPENDIX II

We collect here three theorems which are used elsewhere in this work.[23] In each theorem, $g(x)$ is a real-valued function defined for $x>0$ and satisfying the following conditions:

(i) $g(x)\in L([a,b])$ for any choice of a, b with $0<a<b$;
(ii) the limits $\int_{\to 0} g(x)dx$ and $\int^{\to\infty} g(x)dx/x$ exist.

[23] Theorems 1 and 2 are proved by straightforward analysis; the techniques are given, for example, by N. Muskheliskvili, in *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953). The results of Theorem 3 are due to W. S. Woolcock (to be published).

Theorem 1. Suppose that $g(x)$ satisfies a Lipschitz condition at $x_0(>0)$; that is, there exist numbers $K>0$, $\mu(1\geqslant\mu>0)$, $\xi(x_0>\xi>0)$ such that

$$|g(x_0+h)-g(x_0)|\leqslant K|h|^\mu$$

for all $|h|\leqslant\xi$.

Then

(a) $$P\int_{\to 0}^{\to\infty}\frac{g(t)dt}{t-x_0}\text{ exists;}$$

(b) $$\int_{\to 0}^{\to\infty}\frac{g(t)dt}{(t-x_0)-iy}\to P\int_{\to 0}^{\to\infty}\frac{g(t)dt}{t-x_0}+i\pi g(x_0)\quad\text{as}\quad y\downarrow 0.$$

Theorem 2. Suppose that there exists a closed interval $I=[x_0-\xi, x_0+\xi]$ $(0<\xi<x_0)$ such that, for any two points x_1, $x_2\in I$,

$$|g(x_1)-g(x_2)|\leqslant K|x_1-x_2|^\mu,\quad(0<\mu\leqslant 1).$$

Then

(a) $$f_1(x)=P\int_{\to 0}^{\to\infty}\frac{g(t)dt}{t-x}$$

(which exists for $x_0-\xi<x<x_0+\xi$) satisfies the uniform Lipschitz condition

$$|f_1(x_1)-f_1(x_2)|\leqslant K'|x_1-x_2|^{\mu'},\quad(0<\mu'<\mu)$$

for any two points x_1, $x_2\in I'=[x_0-\xi', x_0+\xi']$, where $0<\xi'<\xi$. (The constant K' will depend on the value of ξ' chosen.)

(b) $$f(x,y)=\int_{\to 0}^{\to\infty}\frac{g(t)dt}{t-x-iy}\to f_1(x_0)+i\pi g(x_0)\quad\text{as}\quad x\to x_0,\quad y\downarrow 0.$$

[In full, given $\epsilon>0$, there exists $\delta>0$ such that

$$|f(x,y)-f_1(x_0)-i\pi g(x_0)|<\epsilon$$

for all x, y such that $0<|x-x_0|<\delta$, $0<y<\delta$.]

Theorem 3. Suppose that $x^\alpha g(x)=A+g_1(x)$, where A, α are constants $(0<\alpha<1)$ and $g_1(x)$ is a finite sum of terms, each of which satisfies *one* of the three following conditions:

(i) (a) $g_1(x)$ satisfies a Lipschitz condition, uniformly for all sufficiently large x. More precisely, there exist constants $K>0$, $\xi>0$, μ $(0<\mu\leqslant 1)$, $a>\xi$ such that $|g_1(x+h)-g_1(x)|\leqslant K|h|^\mu$ for all $x\geqslant a$ and all $|h|\leqslant\xi$; (b) $g_1(x)\ln x\to 0$ as $x\to\infty$.

(ii) Given $\epsilon>0$, there exists X (depending on ϵ) such that

$$\frac{|g_1(x_2)-g_1(x_1)|}{x_2-x_1}<\frac{\epsilon}{x_1}$$

for all x_1, x_2 for which $x_2>x_1\geqslant X$.

(iii) $g_1(x)=1/p(x)$, where $p(x)$ satisfies the following conditions:

(a) $p(x)>0$ for all $x\geqslant a(>0)$,
(b) $p(x)$ is *concave* in $[a,\infty)$,
(c) $p(x)\to\infty$ as $x\to\infty$.

Then

$$z^\alpha f(z)\to A\pi(\cot\pi\alpha+i)$$

as $|z|\to\infty$ in any direction, the convergence being uniform with respect to θ for $0\leqslant\theta\leqslant 2\pi$. Here

$$z=re^{i\theta},\quad (r>0,\ 0\leqslant\theta\leqslant 2\pi)$$

$$z^\alpha=r^\alpha e^{i\alpha\theta},$$

$$f(z)=\int_{\to 0}^{\to\infty}\frac{g(t)dt}{t-z}\quad\text{for}\quad 0<\theta<2\pi,$$

$$f(x\pm i0)=P\int_{\to 0}^{\to\infty}\frac{g(t)dt}{t-x}\pm i\pi g(x)\quad\text{for}\quad\theta=\binom{0}{2\pi}.$$

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Neutral Semileptonic Decays of K Mesons*

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The question of the existence of neutral leptonic currents coupled to the neutral strangeness-changing current is discussed in the light of recent experimental limits on $K_2^0\to\mu^+\mu^-$ and $K_2^0\to e^+e^-$ decay rates.

I. INTRODUCTION

NEW experimental results on K_2^0 decays into lepton pairs have been reported recently.[1] At present, the total branching ratio corresponding to the mode $K_2^0\to\mu^+\mu^-$ is[2]

$$\frac{\Gamma(K_2^0\to\mu^+\mu^-)}{\Gamma(K_2^0\to\text{all modes})}<2.5\times10^{-6};\qquad(1)$$

and for $K_2^0\to e^+e^-$,[3]

$$\frac{\Gamma(K_2^0\to e^+e^-)}{\Gamma(K_2^0\to\text{all modes})}<5\times10^{-5}.\qquad(2)$$

Such decay modes are expected to occur from electromagnetic induction of neutral leptonic currents. However, the question arises whether or not they could also appear as a consequence of the existence of direct weak couplings[4] between neutral leptonic currents and neutral strangeness changing current. We should like to discuss here some implications of the new experimental upper limits mentioned above, upon the possible existence of such weak couplings.

By analogy to the usual semileptonic weak Hamiltonian, one expects neutral strangeness-changing semileptonic decays to be described by an effective Hamiltonian of the type

$$H(\text{neutral})=\frac{G}{\sqrt{2}}\sum_l g_l(J_3{}^2)^\mu\bar{l}i\gamma_\mu(1+i\gamma_5)l+\text{H.c.},\qquad(3)$$

$$l=\nu_e,\ e^-,\ \nu_\mu,\ \mu^-.$$

Here, we have assumed that neutral leptonic currents have V-A structure, like the charged currents, and that $(J_3{}^2)^\mu$ is the $\Delta S=1$, $\Delta Q=0$ component of the usual octet of hadronic currents, consisting of a vector part plus an axial-vector part: $(J_3{}^2)^\mu=(V_3{}^2)^\mu+(A_3{}^2)^\mu$. The constant G is the Fermi coupling constant: $G=1.02\times10^{-5}/m_p{}^2$; and g_l are dimensionless unknown parameters (in principle different for each lepton pair) which depress the intensity of the neutral decay rates with respect to the corresponding charged modes. We assume that the parameters g_l are real or pure imaginary, and we shall discuss the physical implications accordingly. We shall also comment on some implications of the upper limits given above upon the predictions of a

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[1] See N. Cabibbo, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, California, 1967).

[2] W. Vernon *et al.* (communication at the Berkeley Conference). Other recent experiments on this branching ratio give the following results: $[\Gamma(K_2^0\to\mu^+\mu^-)/\Gamma(K_2^0\to\text{all modes})]<8\times10^{-6}$ [M. Bott-Bodenhausen, X. de Bouard, D. G. Cassel, D. Dekkers, R. Felst, R. Mermod, I. Savin, P. Scharff, M. Vivargent, T. R. Willits, and K. Winter, Phys. Letters 23, 277 (1966)]; $\Gamma(K_2^0\to\mu^+\mu^-/\Gamma(K_2^0\to\text{all modes})<5\times10^{-5}$ [A. Abashian *et al.*, University of Illinois Report (unpublished)]. All these figures are 90% confidence limits.

[3] M. Bott-Bodenhausen *et al.*, see Ref. 2. See also A. Abashian *et al.*, Ref. 2. The latter group finds $(\Gamma(K_2^0\to e^+e^-)/\Gamma(K_2^0\to\text{all modes}))<5\times10^{-5}$ (90% confidence limit).

[4] Perhaps mediated by neutral intermediate vector boson(s).