Mass-Spin Relation in a Lagrangian Model

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It is shown that in Lagrangian models based on homogeneous spaces of the Poincaré group a mass-spin relation of the form $m^2=a+bs$ can be derived in a natural way.

MANY wave equations combining particles with different masses m and spins s in one multiplet have recently been proposed.[1] These equations, like the first one of that kind found by Majorana,[2] exhibit some rather unphysical properties; for example, when the spin increases, the mass tends to an accumulation point.

For mesons, experimental results[3] seem to suggest a formula of the type

$$m^2=a+bs\,, \tag{1}$$

with suitably chosen parameters a and b. Such a formula could also be approximately valid for baryons.

In this paper, from general considerations on the Poincaré group, we show that one can construct Lagrangian models which lead quite naturally to Eq. (1). One particular model is investigated at the one-particle level, and some remarks about other possible examples are given.

In our model, the wave functions no longer have indices describing the spin,[4] but they are functions f on the direct product $\mathcal{E}$ of the Minkowski space $\mathfrak{M}$ with a two-dimensional (complex) spinor space $\mathcal{S}$. Every point of $\mathcal{E}$ can be written in the form (x,ξ), where x belongs to $\mathfrak{M}$ and ξ to $\mathcal{S}$.

A transformation $[a,\Lambda]$ of the covering of the Poincaré group $\bar{\mathcal{P}}$ will act on $\mathcal{E}$ according to

$$(x,\xi)\overset{[a,\Lambda]}{\longrightarrow}(x',\xi')=(\Lambda x\Lambda^\dagger+a,\ \Lambda\xi)\,, \tag{2}$$

where Λ is a unimodular 2×2 matrix and x and a are Hermitian 2×2 matrices. It may easily be verified that if we discard all the points of the form $(x,0)$, the remaining space $\mathcal{E}^*$ is a homogeneous space for $\bar{\mathcal{P}}$, i.e., given two points of $\mathcal{E}^*$ there always exists a transformation $[a,\Lambda]$ mapping one point on the other.

The transformation (2) induces the following representation of $\bar{\mathcal{P}}$ on the function $f(x,\xi)$:

$$U_{[a,\Lambda]}f(x,\xi)=f([a,\Lambda]^{-1}(x,\xi))\,. \tag{3}$$

From this equation one readily gets the infinitesimal generators of $\bar{\mathcal{P}}$ as differential operators and, consequently, the differential form of the two fundamental invariants P^2 and W^2 of the group. One obtains

$$P^2=-\Box \tag{4}$$

and

$$W^2=\Box D(D+1)\,, \tag{5}$$

where

$$D=\tfrac{1}{2}\xi^\alpha(\partial/\partial\xi^\alpha)\,. \tag{6}$$

The index α takes the values 1 and 2. Equation (5) is valid only when we require the analyticity of $f(x,\xi)$ in the variables ξ.

Let $\chi_{m,s}(x,\xi)$ be an eigenfunction of

$$P^2\chi_{m,s}=m^2\chi_{m,s}\,, \tag{7}$$

$$W^2\chi_{m,s}=-m^2s(s+1)\chi_{m,s}\,. \tag{8}$$

(We suppose m^2 strictly positive.)

We define the following scalar product:

$$(\chi_{m,s},\chi_{m,s}')=\int_{p_0=+(\mathbf{p}^2+m^2)^{1/2}}\tilde{\chi}_{m,s}{}^*(p,\xi^*)\tilde{\chi}_{m,s}{}'(p,\xi)\times\delta(\xi p\xi^*-m\lambda^2)\frac{d^3p}{p_0}d^4\xi\,, \tag{9}$$

where $\tilde{\chi}_{m,s}(p,\xi)$ is the Fourier transform of $\chi_{m,s}(x,\xi)$ with respect to the variables x, and $d^4\xi$ is the invariant measure on $\mathcal{S}$:

$$d^4\xi=d\,\mathrm{Re}\xi_1\,d\,\mathrm{Im}\xi_1\,d\,\mathrm{Re}\xi_2\,d\,\mathrm{Im}\xi_2\,. \tag{10}$$

In the argument of the delta function, $\xi^\alpha p_{\alpha\dot\beta}\xi^{\dot\beta}$ ($\xi^{\dot\beta}=\xi^{*\beta}$) is invariant under the Poincaré transformations, and λ^2 is an arbitrary positive number; henceforth $\lambda^2=1$.

By introducing the δ function in (9), the integration over ξ for each value of the momentum p is restricted to a compact manifold of $\mathcal{S}$. This can easily be seen in choosing the new variables

$$\xi'=\Lambda_{0\leftarrow p}\xi\,, \tag{11}$$

where $\Lambda_{0\leftarrow p}$ is the pure Lorentz transformation which maps p into the vector $(m,0,0,0)$. In turn, ξ' can be expressed conveniently as

$$\xi'^1=\rho e^{i\psi}e^{i\varphi}\sin\tfrac{1}{2}\theta\,,\qquad \xi'^2=\rho e^{i\psi}e^{-i\varphi}\cos\tfrac{1}{2}\theta\,, \tag{12}$$

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[1] Harish-Chandra, Proc. Roy. Soc. (London) **A192**, 195 (1948); I. M. Gelfand and A. M. Yaglom, Zh. Eksperim. i Teor. Fiz. **18**, 70 (1948); J. M. Lévy-Leblond, Nuovo Cimento **45**, 772 (1966); G. Feldman and P. T. Matthews, Phys. Rev. **154**, 5 (1967); **154**, 1241 (1967); Y. Nambu (to be published); C. Fronsdal, Phys. Rev. (to be published).

[2] E. Majorana, Nuovo Cimento **9**, 335 (1932).

[3] M. N. Focacci *et al.*, Phys. Rev. Letters **17**, 890 (1966).

[4] F. Lurçat has suggested a model based on the Poincaré group manifold itself as a homogeneous space. F. Lurçat, Physics **1**, 95 (1964); F. Lurçat and Nghiem Xuan Hai (unpublished).

where ρ is positive and real, and the angles ψ, φ, and θ vary, respectively, between the limits $[0,2\pi]$, $[0,2\pi]$, and $[0,\pi]$.[5] With these changes of variables, the δ function becomes $\delta(\rho^2-1)$ and the scalar product becomes

$$\int \tilde{\chi}_{m,s}{}^*(p,\xi^*[p,\theta,\varphi,\psi])\tilde{\chi}_{m,s}{}'(p,\xi[p,\theta,\varphi,\psi]) \times\frac{d^3p}{p_0}\sin\theta d\theta d\varphi d\psi. \quad (13)$$

The eigenfunctions $\tilde{\chi}_{m,s}$ are conveniently written as homogeneous polynomials in ξ'^1 and ξ'^2 of degree s:

$$\tilde{\chi}_{m,s,q}(p,\xi)=a_q(p)(\xi'^1)^{s+q}(\xi'^2)^{s-q}. \quad (14)$$

The number q is the eigenvalue of the operator $-\frac{1}{2}i\partial_\varphi$, which is the third component of the spin operator in the rest frame. The number s is the eigenvalue[6] of the operator D of Eq. (6). The condition of unitarity of the representation with respect to the scalar product (13) implies some support properties of $a_q(p)$ on the mass hyperboloid (existence of a wave packet) and restricts the values of s and q to be integral or half-integral with the usual spectrum $q=(-s, -s+1, \cdots, s)$.

Our Hilbert space for free one-particle states is spanned by the functions $\chi_{m,s}(x,\xi)$. Let $f(x,\xi)$ be one element of this space and $f^*(x,\xi^*)$ the conjugate function; and associate with them a Hermitian Lagrangian density of the form

$$\mathfrak{L}=-af^*f+f_\mu{}^*f^\mu+b(Df)^*(Df), \quad (15)$$

which is the most general Lagrangian quadratic in f and in its first derivatives and whose Lagrange equations are compatible with the analyticity requirement.[7]

Under variation of f^*, one obtains the Lagrange equation[8]

$$(-\Box-a-bD)f(x,\xi)=0, \quad (16)$$

which implies the desired relation (1) between mass and spin.

Analogous conclusions can be reached for many analogous models with a homogeneous space of the form (x,η), where the Poincaré group acts as usual on x, while only the homogeneous Lorentz group acts on the variables η. As in this model, in order to be able to write a Lagrangian similar to (15) leading to an equation similar to (1), it is necessary that in the expression

$$W^2=-\tfrac{1}{2}P^2M_{\mu\nu}M^{\mu\nu}-M_{\nu\rho}P^\nu P_\gamma M^{\rho\gamma} \quad (17)$$

the second term vanishes identically, so that a factor of P^2 can be factorized explicitly. This is achieved in the model discussed above by the analyticity condition.[9]

It is interesting to mention the possibility of generalizing the Lagrangian formalism in order to include internal degrees of freedom by the addition of new variables corresponding to a homogeneous space of the internal-symmetry group. As an example we can use an isotopic spinor ζ and deduce a mass formula

$$m^2=a+bs+cT(T+1). \quad (18)$$

Note added in proof. After the present paper was submitted for publication we learned about an article by D. Finkelstein in which a similar approach is suggested leading, however, to a different mass formula. We thank Professor Finkelstein for calling his paper to our attention. [D. Finkelstein, Phys. Rev. **100**, 924 (1955).]

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[5] Note that the two variables ρ and ψ are invariant under the Poincaré transformations.

[6] The number s is also the eigenvalue of the operator $-\frac{1}{2}i\partial_\psi$, which could be interpreted as the generator of a "spin gauge."

[7] The natural definition of the parity operator P is

$$Pf(x_0,\mathbf{x},\xi)=\epsilon f(x_0,-\mathbf{x},\xi^*),$$

where ϵ corresponds to the intrinsic parity.

[8] Note in the derivation of the Lagrange equations that $D^*=\xi^{\alpha*}\partial/\partial\xi^{\alpha*}$.

[9] Examples of other interesting models:

(a) Replace η by a vector ω and choose the class of functions by

$$\chi_m(x,\omega)=\int_{p0=(p^2+m^2)^{1/2}}e^{ip\cdot x}\tilde{\chi}(p,\omega)\delta(\omega^2+\lambda)\delta(\omega\cdot p)d^3p/p_0.$$

(b) Replace η by an antisymmetric tensor W and choose the class of functions by

$$\chi_m(x,W)=\int_{p0=(p^2+m^2)^{1/2}}e^{ip\cdot x}\tilde{\chi}(p,W)\delta(W^{\mu\nu}P_\nu)\delta(W^{\mu\nu}W_{\mu\nu}-\lambda)d^3p/p_0.$$