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Current Algebra and Non-Regge Behavior of Weak Amplitudes. II

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Certain weak amplitudes exhibit non-Reggeistic behavior. These amplitudes have fixed poles in the complex angular-momentum plane which have the dual property of allowing a sum rule of the Dashen–Gell-Mann–Fubini type to hold, although one might naively expect a superconvergence relation for this amplitude, *and* insuring that spin-one particle poles are reproduced correctly in the left-hand side of the sum rule. We demonstrate the existence of the fixed pole directly by comparing the sum rule with the Froissart-Gribov continuation to the complex J plane. We also study some models which exhibit this behavior.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

ON the basis of the postulated equal-time commutation relations of the isovector current densities,

$$[j_0^\alpha(x,t),j_\nu^\beta(y,t)]=i\epsilon_{\alpha\beta\gamma}j_\nu^\gamma(x,t)\delta^3(x-y)+\text{Schwinger terms},\quad(1)$$

Fubini, and Dashen and Gell-Mann[1] have suggested sum rules, the simplest one of which is

$$\frac{2}{\pi}\int_0^\infty ds'\,\mathrm{Im}A_1{}^{\alpha\beta}(s',t,k_1^2,k_2^2)=+\frac{4}{(2\pi)^3}F_\pi(t)\epsilon_{\alpha\beta\gamma}\epsilon_{fi\gamma}.\quad(2)$$

Here $F_\pi(t)$ is the electromagnetic form factor for the pion, and $A_1{}^{\alpha\beta}(s,t,k_1^2,k_2^2)$ is one of the scalar invariant amplitudes of the matrix element

$$T_{\mu\nu}{}^{\alpha\beta}\equiv(4p_0p_0')^{1/2}i\int d^4x\times e^{ik\cdot x}\langle\pi_f(p_2)|T\{j_\mu{}^\alpha(x),j_\nu{}^\beta(0)\}|\pi_i(p_1)\rangle,\quad(3)$$

where i and f are the isospin indices of the initial and final pions. Many applications of the type of sum rules which follow from local current commutation relations have appeared in the literature.[2]

The questions we wish to discuss in this paper[3] have to do with the analytic continuation of Eq. (2) from the region[4]

$$k_1^2>0,\quad k_2^2>0,\quad t=-(p_2-p_1)^2<0,$$

to the region where these invariants are timelike. The possibility of making this continuation follows from the fact that the right-hand side and, at least, the integrand on the left-hand side of Eq. (2), are analytic functions of t, k_1^2, and k_2^2. The continuation yields nontrivial information, since, if we first consider an amplitude with t timelike ($\gamma+\gamma\to\pi+\pi$) and follow the steps leading from Eq. (3) to Eq. (2), we do not obtain a sum rule corresponding to the continuation of Eq. (2) but, instead, a completely new sum rule of the type recently discussed by Amati, Jenzo, and Remiddi.[5] Thus we are studying the constraints imposed on a theory which has both a sum rule (2) and sufficient analyticity so that it may be continued. We treat this problem in two parts. In Sec. II we assume the existence of a continuation and find the constraints which make this possible. In subsequent sections we discuss various models which are sufficiently tractable to have such continuations, in order to verify that our constraints actually appear in such examples.

We shall also be able to answer the following questions: (1) How can the left-hand side of Eq. (2) reproduce the analytic properties, in the t variable, of the right-hand side? (2) Why does the right-hand side not depend on the "masses," k_1^2 and k_2^2, of the virtual photons? The first question is not as trivial as it may appear to be, since the only state which contributes to $F_\pi(t)$ has $J=1$ in the t channel, while the states which contribute to $A_1(s,t,k_1^2,k_2^2)$ all have $J\geqslant 2$ in this channel. Thus,

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[1] S. Fubini, Nuovo Cimento 43, 475 (1966); R. Dashen and M. Gell-Mann, Phys. Rev. Letters 17, 340 (1966).

[2] A fairly comprehensive bibliography, together with review of the subject, appears in B. Renner, Rutherford Laboratory Report No. RHEL/R 126, 1966 (unpublished).

[3] A brief account of our results has been given in J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. Letters 18, 32 (1967); see also V. Singh, *ibid.* 18, 36 (1967).

[4] It will be clear in the next section that Eq. (2) is derived for the region of spacelike k_1, k_2, and Δ, with, in fact, the additional restriction $\sqrt{(k_1^2)}+\sqrt{(k_2^2)}>\sqrt{(-t)}$.

[5] D. Amati, R. Jenzo, and E. Remiddi, Phys. Letters 22, 674 (1966).

the sum rule may be seen as correlating the effects of the various partial waves in an apparently complicated manner. In particular, the point $J=1$ (in the complex J plane of the t channel) corresponds to a sense-nonsense[6] transition in $A_1(s,t,\cdots)$.

In the next section we rederive Eq. (2), partly for the sake of completeness, and partly, and more importantly, to sharpen the issues involved. A Regge-pole analysis of the amplitude $A_1(s,t,\cdots)$ is presented here, and it will be shown that the asymptotic limit of this amplitude for large s, if given only by the exchange of moving Regge poles in the t channel, is incompatible with Eq. (2). In fact, the usual Regge theory predicts the superconvergent relation

$$\int ds' \,\mathrm{Im}A_1(s,t,k_1^2,k_2^2)=0 \tag{4}$$

in the region of timelike t, rather than the analytic continuation of Eq. (2). We will show that the existence of a continuation of Eq. (2) requires the existence of a fixed pole at $J=1$ in the angular momentum plane. The following sections will be devoted to the study of some models. Our work in Sec. II yields the conditions which a theory must satisfy if Eq. (2) may be continued. In continuing the left-hand side of Eq. (2), the simplest procedure is first to continue the integrand in t, and then to perform the integration in s, as long as the integral converges. However, the possibility that this method of analytic continuation breaks down must be considered, and this we do in our model. We also illustrate in these sections the origins of the fixed pole at $J=1$ and its role in answering the questions raised above.

We discuss various implications of our results in the final section.

II. SUM RULE AND ANALYSIS OF THE AMPLITUDE

The amplitude $T_{\mu\nu}{}^{\alpha\beta}$ in Eq. (3) is generally not covariant. Instead of it, we choose to discuss the covariant current correlation function as defined by Brown[7]:

$$S_{\mu\nu}{}^{\alpha\beta}(P,K,\Delta)=i(4p_{10}p_{20})^{1/2}\int d^4x\, e^{ik\cdot x} \times\langle\pi_f(p_2)|T\{j_\mu{}^\alpha(x),j_\nu{}^\beta(0)\}-\rho_{\mu\nu}{}^{\alpha\beta}(x)|\pi_i(p_1)\rangle\,, \tag{5}$$

where we have defined

$$P=\tfrac{1}{2}(p_1+p_2)\,,\quad K=\tfrac{1}{2}(k_1+k_2)\,,\quad \Delta=k_2-k_1=p_1-p_2$$

and

$$k_2=p_1+k_1-p_2\,,$$

and $\rho_{\mu\nu}{}^{\alpha\beta}(x)$ satisfies[6]

$$\delta(x_0)[j_0{}^\alpha(x),j_i{}^\beta(0)]=-\partial^\mu\rho_{\mu i}{}^{\alpha\beta}(x)\,,\quad \rho_{00}{}^{\alpha\beta}=\rho_{0i}{}^{\alpha\beta}=0\,.$$

[6] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

[7] L. Brown, Phys. Rev. **150**, 1338 (1966); M. A. B. Bég, Phys. Rev. Letters **17**, 333 (1966); R. P. Feynman (unpublished).

When the current $j_\mu(x)$ is conserved, we have

$$k_1{}^\mu S_{\mu\nu}{}^{\alpha\beta}=i(4p_{10}p_{20})^{1/2}\epsilon^{\alpha\beta\gamma}\langle\pi_f(p_2)|j_\nu{}^\gamma|\pi_i(p_1)\rangle =\frac{1}{(2\pi)^3}2P_\nu F_\pi(t)\epsilon_{\alpha\beta\gamma}\epsilon_{fi\gamma}\,. \tag{6}$$

The covariant amplitude $S_{\mu\nu}{}^{\alpha\beta}$ may be expanded in terms of independent covariants. Suppressing the isospin indices, we write

$$S_{\mu\nu}(P,K,\Delta)=P_\mu P_\nu A_1+P_\mu K_\nu A_2+P_\mu\Delta_\nu A_3 +K_\mu P_\nu B_1+K_\mu K_\nu B_2+K_\mu\Delta_\nu B_3 +\Delta_\mu P_\nu C_1+\Delta_\mu K_\nu C_2+\Delta_\mu\Delta_\nu C_3+g_{\mu\nu}D\,. \tag{7}$$

The invariant scalar amplitudes $A_1,\cdots,D$ are functions of $s=-(P+K)^2$, $t=-\Delta^2$, k_1^2, and k_2^2. Let us now *assume* that the invariant amplitudes $A_1,\cdots,D$ are $O(1/s)$, so that they satisfy unsubtracted dispersion relations for some range of $t\leqslant 0$, and k_1^2, $k_2^2<4\mu^2$. Then Eq. (6) gives

$$\lim_{s\to\infty} sA_1{}^{\alpha\beta}(s,t,k_1^2,k_2^2)=\frac{4}{(2\pi)^3}F_\pi(t)\epsilon_{\alpha\beta\gamma}\epsilon_{fi\gamma} \tag{8}$$

or

$$\frac{2}{\pi}\int_0^\infty ds'\,\mathrm{Im}A_1{}^{\alpha\beta}(s',t,k_1^2,k_2^2)=\frac{4}{(2\pi)^3}F_\pi(t)\epsilon_{\alpha\beta\gamma}\epsilon_{fi\gamma} \tag{2}$$

for $t\leqslant 0$. Equations (8) and (2) are equivalent. Note that only the amplitude corresponding to $T=1$ in the crossed channel is relevant here. (This amplitude is odd under $s\leftrightarrow u$ crossing.) We will write Eq. (8) [and Eq. (2)] as

$$\lim_{s\to\infty}(-s)A_1(s,t,k_1^2,k_2^2)=\frac{4}{(2\pi)^3}F_\pi(t)\,, \tag{9}$$

and understand A_1 to mean the $T=1$ amplitude in the crossed channel. Equation (7) states that, if all the amplitudes are $O(1/s)$ as $s\to\infty$, then $A_1(s,t,\cdots)$ *must be proportional to* $F_\pi(t)/s$ *for large* s. This asymptotic behavior in s is required by the nonzero right-hand side of the matrix element of the commutation relation (1), together with our other assumptions.

It will prove useful to review the complex angular-momentum representation of $A_1(s,t,\cdots)$ in some detail. We consider the partial-wave expansion in the t channel: (two virtual isovector photons) $\leftrightarrow$ (two pions). The amplitude $A_1(s,t,\cdots)$ corresponds to a flip of two units of helicity:

$$f_{0,2}\equiv\langle 0|t|+1,-1\rangle=\tfrac{1}{2}p^2\sin^2\theta\, A_1(s,t,\cdots) =\frac{p^2}{2}\sum_{J=2}^{\infty}(2J+1)\frac{1-e^{-i\pi J}}{2} \times d_{02}{}^J(\theta)f^J(t)\,, \tag{10}$$

where $f_{0,\lambda-\lambda'}\equiv\langle 0|t|\lambda,\lambda'\rangle$ is the t matrix for isovector virtual photons of helicities λ and λ' producing two pions; p is the magnitude of c.m. pion momentum, with $p^2=\frac{1}{4}(t-4\mu^2)$; and θ is the scattering angle in the t channel. [We shall ignore the signature factor $(1-e^{-i\pi J})/2$ whenever it is inessential.] Using the table provided in Appendix A of Ref. 6, we may rewrite Eq. (10) (using a slightly different definition of F^J from that of Refs. 3 and 6) as

$$A_1(s,t,\cdots)=\sum_{J=2}^{\infty}(2J+1)F_J(t,\cdots)\frac{d_{02}{}^J(\theta)}{\sin^2\theta}\times[(J-1)J(J+1)(J+2)]^{1/2}$$

$$=\sum_{J=2}^{\infty}(2J+1)F_J(t,\cdots)P_J''(\cos\theta) \qquad (11)$$

and

$$F_J(t,\cdots)=\frac{1}{(2J-1)(2J+1)(2J+3)}\int dz \times[(2J+3)P_{J-2}(z)-2(2J+1)P_J(z)+(2J-1)P_{J+2}(z)]$$
$$\times A_1(\mu^2-k^2-2(p^2+\mu^2)^{1/2}(k^2-k_1{}^2)^{1/2}+2pkz,t,\cdots), \qquad (12)$$

where k is given by

$$k=\{[t-((-k_1{}^2)^{1/2}+(-k_2{}^2)^{1/2})^2]\times[t-((-k_1{}^2)^{1/2}-(-k_2{}^2)^{1/2})^2]/4t\}^{1/2}.$$

If A_1 has a spectral representation in z such as suggested by, say, the Mandelstam representation, then $F_J(t,\cdots)$ may be given by the Froissart-Gribov[8] definition:

$$F_J(t,\cdots)=\frac{4}{\pi}\int dz \times\frac{(2J+3)Q_{J-2}(z)-2(2J+1)Q_J+(2J-1)Q_{J+2}(z)}{(2J-1)(2J+1)(2J+3)}$$
$$\times \mathrm{Im}A_1(\mu^2-k_1{}^2-2(p^2+\mu^2)^{1/2}(k^2-k_1{}^2)^{1/2}+2pkz,t,\cdots), \qquad (12')$$

for complex J. If F_J is finite at $J=1, 0, -1, -2, \cdots$ then we may extend the sum in Eq. (11)

$$A_1(s,t,\cdots)=\sum_{J=-\infty}^{\infty}(2J+1)F_J(t,\cdots)\mathcal{P}_J''(\cos\theta), \qquad (11')$$

where

$$\mathcal{P}_J(z)=(\pi^{-1}\tan J\pi)Q_{-J-1}(z),$$

(see Ref. 6). The formula (11′) can now be transformed into the Sommerfeld-Watson integral.[8] If the leading singularity of F_J is a moving pole of the form $\gamma(t)\times[J-\alpha(t)]^{-1}$, the asymptotic form of $A_1(s,t,\cdots)$ becomes[9]

$$-\frac{(2\alpha+1)\pi}{\sin\pi\alpha}\gamma(t)\mathcal{P}_\alpha(-z)\to-\frac{(2\alpha+1)\pi}{\sin\pi\alpha}\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)\sqrt{\pi}}\times\gamma(t)\alpha(t)[\alpha(t)-1](-2z)^{\alpha-2}. \qquad (13)$$

The quantity $\mathrm{Im}A_1(s',t,\cdots)$ appears under the integral in Eq. (12′) because a dispersion relation at fixed t is written for $A_1(s,t,\cdots)$ and substituted in Eq. (12). Thus, as the notation implies, $\mathrm{Im}A_1(s',t,\cdots)$ *is the same* quantity which appears in the integrand of the sum rule (2).

Before discussing the compatibility of the Regge behavior expressed by Eq. (13) with the high-energy form of Eq. (8), let us first consider one of the problems involved in generating the t-variable singularities on the left-hand side of Eq. (2). It has been suggested by Fubini and Segrè[10] that, since $F_\pi(t)$ has a pole[11] when $t=m_\rho{}^2$ (where m_ρ is the mass of the ρ meson), the left-hand side of Eq. (2) should develop this pole as a consequence of the integral diverging. This divergence, they argue further, will occur because of the asymptotic form (13), since (13) implies

$$\mathrm{Im}A_1(s,t)\sim\gamma'(t)\alpha(t)[\alpha(t)-1]s^{\alpha(t)-2} \qquad (14)$$

and the integral diverges when the ρ trajectory passes through one at $t=m_\rho{}^2$. That is,

$$\lim_{t\to m_\rho{}^2-}\int_{s_0}^{\infty}ds\, s^{\alpha_\rho(t)-1}\simeq\frac{s_0{}^{\alpha_\rho(t)-1}}{\alpha_\rho(t)-1}.$$

However, the complete asymptotic form Eq. (14) contains a factor $[\alpha(t)-1]$ characteristic of a sense-nonsense transition, and although the integral does diverge, the left-hand side of Eq. (2) approaches some finite limit as t approaches $m_\rho{}^2$ and does not develop the ρ pole. Thus, a pure Regge-pole picture is incompatible with a continuation of Eq. (2) up to $t-m_\rho{}^2$. This result is not surprising: the amplitude $A_1(s,t,\cdots)$ does not have a pole at $t=m_\rho{}^2$, since this would be unphysical; and the factor $[\alpha(\rho)-1]$ is precisely what is needed to cancel this pole in Eq. (13).

In fact, the inconsistency between the Regge-pole picture and the sum rule is much more fundamental than problems involving the ρ pole. As already pointed out, the sum rule predicts that $A_1(s,t,\cdots)\propto F_\pi(t)/s$ for

[8] See, for example, S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963), Chaps. 7 and 8.

[9] For details see Ref. 8.

[10] S. Fubini and G. Segrè, Nuovo Cimento **45**, 641 (1966); G. Furlan and C. Rosseti, Symposium on Weak Interactions, Balatonvilàgos, Trieste report, 1966 (unpublished); D. Amati, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967).

[11] We treat the ρ meson as if it were stable. To be more precise, "the pole at $t=m_\rho{}^2$" or the like should be understood as the pole in the second sheet corresponding to an unstable ρ meson.

large s and a range of t. This is certainly in contradiction to the large-s behavior $A_1(s,t,\cdots)\sim s^{\alpha(t)-2}$ provided by the Regge picture. Another way of stating this is to note that in a region in which $\alpha(t)<1$ (which includes $t<m_\rho^2$), Eqs. (13) and (14) imply

$$\lim_{s\to\infty}(-s)A_1(s)=\lim_{s\to\infty}(-s)\int\frac{ds'}{\pi}\frac{\mathrm{Im}A_1(s',t,\cdots)}{s'-s}$$

$$=\int\frac{ds}{\pi}\,\mathrm{Im}A(s',t,\cdots)=0 \qquad (15)$$

rather than Eq. (2).

The resolution of these difficulties is easy to find. We assert that if the amplitude $A_1(s,t,\cdots)$ has the property that the analytic continuation of Eq. (2) to time-like t is possible *and* that a continuation of the partial-wave amplitude in the t channel may be made to complex J, then this partial-wave amplitude has a fixed pole at $J=1$. The existence of this pole may be inferred immediately from Eq. (12′). The Legendre function $Q_J(x)$ has a pole at negative integral values of J with residue $P_{J+1}(z)$. So

$$\lim_{J\to1}(J-1)F_J(t,\cdots)=\frac{1}{3}\times\frac{4}{\pi}\int dz$$
$$\times\mathrm{Im}A_1(\mu^2-k^2-2(p^2+\mu^2)^{1/2}$$
$$\times(k^2+k_1^2)^{1/2}+2pkz,t,\cdots)$$
$$=+\frac{1}{3}\frac{1}{2pk}\frac{4}{\pi}\int_0^\infty ds$$
$$\times\mathrm{Im}A_1(s,t,\cdots). \qquad (16)$$

Thus, a superconvergence relation [Eq. (15)] is the condition that F_J be regular at $J=1$ (the residue of the pole vanishes), while the sum rule implies

$$\lim_{J\to1}(J-1)F_J(t,\cdots)=+\frac{2}{3}\frac{1}{2pk}\frac{4}{(2\pi)^3}F_\pi(t). \qquad (17)$$

That is, the sum rule is an integral form of the statement that $F_J(t\cdots)$ has a pole at $J=1$ with residue proportional to $F_\pi(t)$.

We now go back to the transition from Eq. (11) to Eq. (12), which requires some modification. We assert that when the isovector photons are off the mass shell, the amplitude $F_J(t,\cdots)$ should be written as

$$F_J(t,\cdots)=\frac{\Gamma_J(t,\cdots)}{(J-1)[J-\alpha(t)]}, \qquad (18)$$

where $\Gamma_J(t,\cdots)$ is nonzero and analytic in the neighborhoods of $J=1$ and $\alpha(t)$. In place of Eq. (11′), we must now write

$$A_1(s,t,\cdots)=\Big(\sum_{J=-\infty}^{0}+\sum_{J=2}^{\infty}\Big)F_J(t,\cdots)\mathcal{P}_J''(\cos\theta)$$
$$\underset{s\to\infty}{\sim}-\lim_{J\to1}z^{-1}\times\tfrac{3}{2}\lim_{J\to1}(J-1)F_J(t,\cdots)$$
$$-\frac{(2\alpha+1)\pi}{\sin\pi\alpha}\frac{\Gamma(\alpha+\frac{1}{2})}{(\sqrt{\pi})\Gamma(\alpha+1)}\alpha(t)$$
$$\times\Gamma_\alpha(t,\cdots)(-2z)^{\alpha-2},\ \left(\frac{1-e^{-i\pi\alpha}}{2}\right) \qquad (19)$$
$$z=\cos\theta\sim s/2pk \quad \text{as} \quad s\to\infty.$$

The last line of Eq. (19) follows from converting the sum over J to a Sommerfeld-Watson integral[12] and taking the limit $z\to\infty$. The amplitude $A_1(s,t,\cdots)$ is seen to approach, as $s\to\infty$,

$$-\tfrac{3}{2}s^{-1}(2pk)\lim_{J\to1}(J-1)F_J(t,\cdots),$$

the fixed pole providing the asymptotic form as long as $\alpha(t)<1$. We also see that the imaginary part of the amplitude goes as

$$\lim_{s\to\infty}\mathrm{Im}A_1(s,t,\cdots)$$
$$=-\tfrac{1}{2}(2\alpha+1)\pi\frac{\Gamma(\alpha+\frac{1}{2})}{(\sqrt{\pi})\Gamma(\alpha+1)}\alpha\Gamma_\alpha(t,\cdots)z^{\alpha-2} \qquad (20)$$

and does not vanish as $\alpha(t)\to1$, so that the mechanism suggested by Fubini and Segrè now works. Here again the fixed pole at $J=1$ of F_J saves the day, and allows the continuation of the sum rule up to $t=m_\rho^2$.

In concluding this section, it is worthwhile to note that $t=m_\rho^2$, with $\alpha(m_\rho^2)=1$, is not a singularity of $A_1(s,t,\cdots)$. If it were, we would have a physically unacceptable result, since $J=1$ is a "sense-nonsense" point of A_1. That $t=m_\rho^2$ is not a singularity of A_1 can be seen from Eq. (19). As $t\to m_\rho^2$, singularities of the term proportional to z^{-1} and the leading Regge-pole contribution cancel exactly.

III. CURRENT CONSERVATION

The remaining sections of this paper will be devoted to showing that the type of behavior discussed above actually holds in a model of strongly interacting mesons. Here we wish to point out that, for the case of a conserved current, we can prove that there exists a line in (k_1^2,k_2^2,t) space along which Eq. (2) can be continued, so that the pure Regge behavior leading to Eq. (15) must be modified.

[12] To be more precise, we convert the sum into a Sommerfeld-Watson contour integral encircling the entire real axis counterclockwise plus a contour integral encircling $J=1$ clockwise.

We write the expansion for the absorptive part of $S_{\mu\nu}$, defined by

$$\mathrm{Im}S_{\mu\nu}{}^{\alpha\beta}(P,K,\Delta)=(4p_{10}p_{20})^{1/2}\int d^4x\, e^{ik\cdot x} \times\langle\pi_f(p_2)|[j_\mu{}^\alpha(x),j_\nu{}^\beta(0)]|\pi_i(p_i)\rangle \quad (21)$$

as in Eq. (6),

$$\mathrm{Im}S_{\mu\nu}(P,K,\Delta)=P_\mu P_\nu a_1+P_\mu K_\nu a_2+P_\mu\Delta_\nu a_3 +K_\mu P_\nu b_1+K_\mu K_\nu b_2+K_\mu\Delta_\nu b_3 +\Delta_\mu P_\nu c_1+\Delta_\mu K_\nu c_2+\Delta_\mu\Delta_\nu c_3+g_{\mu\nu}d. \quad (22)$$

If $\partial_\mu j^\mu(x)=0$, then we have

$$k_1{}^\mu\, \mathrm{Im}S_{\mu\nu}=P_\nu[-\nu a_1+k_1\cdot Kb_1+k_1\cdot\Delta c_1] +K_\nu[k_1\cdot Pa_2+k_1\cdot Kb_2+k_1\cdot\Delta c_2+d] +\Delta_\nu[k_1\cdot Pa_3+k_1\cdot Kb_3+k_1\cdot\Delta c_3-\tfrac{1}{2}d]=0.$$

The coefficient of P_ν of this equation, written in terms of the invariants t, $k_1{}^2$, $k_2{}^2$, and $\nu=-k_1\cdot P$, is

$$-\nu a_1+[\tfrac{1}{4}k_2{}^2+\tfrac{1}{4}t+\tfrac{3}{4}k_1{}^2]b_1 +[\tfrac{1}{2}k_2{}^2-\tfrac{1}{2}k_1{}^2+\tfrac{1}{2}t]c_1=0, \quad (23)$$

so that if we take

$$k_1{}^2=0,\quad k_2{}^2=-t,$$

then we have

$$\nu a_1(\nu,\,-k_2{}^2,\,0,\,k_2{}^2)=0,$$

so that

$$a_1(\nu,\,-k_2{}^2,\,0,\,k_2{}^2)\propto\delta(\nu), \quad (24)$$

if there exists an intermediate state in Eq. (21) degenerate in mass with the final pion. If no such state exists, then $a_1(\nu,\,-k_2{}^2,\,0,\,k_2{}^2)=0$ [it is easy to verify that neither b_1 nor c_1 is singular at the point where their coefficients in Eq. (23) vanish]. Of course the intermediate state in equation is the pion itself, and so from Eq. (21) we can compute $a_1(\nu,k_2{}^2,0,k_2{}^2)$ exactly, Eq. (24) ensuring that the continuum contribution vanishes. Then, we can do the integral appearing in Eq. (2) exactly, and we obtain,

$$\frac{1}{\pi}\int ds' a_1(s',k_2{}^2,0,k_2{}^2)=\frac{4}{(2\pi)^3}F_\pi(k_2{}^2)F_\pi(0).$$

Since we have assumed current conservation, $F_\pi(0)=1$ and Eq. (2) is seen to be exactly true for *all* values of $k_2{}^2$, positive and negative.

The point of this demonstration is that for a conserved current we can find a line in the variable t, including both timelike and spacelike values, where the continuum contribution to Eq. (2) vanishes and the single-particle pole saturates the sum rule. Thus we may state with certainty that the pure Regge-pole picture, which leads to Eq. (15), must be modified.

IV. EXAMPLE IN FIELD THEORY

We will now discuss the validity of Eqs. (16) and (17) in a field theory. The field theory we have in mind is that of self-coupled scalar pions (say, $\lambda\epsilon_{\alpha\beta\gamma}\phi^\alpha\phi^\beta\phi^\gamma$ theory). We assume that the $\pi\pi$ scattering amplitude (even when some of the external lines are off the mass shell) is analytic[13] in the region $\mathrm{Re}J>1-\epsilon$, $\epsilon>0$.

In such a theory, the current $j_\mu{}^\alpha(x)$ is given by

$$j_\mu{}^\alpha(x)=\epsilon^{\alpha\beta\gamma}\phi_\beta(x)\partial_\mu\phi_\gamma(x). \quad (25)$$

We consider the T matrix for the process: (two virtual isovector photons) $\rightarrow$ (two pions). We shall call a graph for this T matrix reducible if it can be split up into two disconnected parts by drawing a line which cuts only two pion lines, such that two external pion lines are in one part and two external lines are in the other, and each of the two resulting parts is a connected graph. Let us call the sum total of irreducible graphs the irreducible kernel $I_{\mu\nu}$. We then have[14]

$$T_{\mu\nu}(\Delta,K,P)=I_{\mu\nu}(\Delta,K,P)+\frac{1}{2\pi}\int d^4Q \times\frac{I_{\mu\nu}(\Delta,K,Q)T(\Delta,Q,P)}{[(\frac{1}{2}\Delta+Q)^2+\mu^2-i\epsilon][(\frac{1}{2}\Delta-Q)^2+\mu^2-i\epsilon]}. \quad (26)$$

Here Δ is the center-of-mass momentum of the system; μ is the mass of the pion; the momenta of the isovector photons are $\frac{1}{2}\Delta+P$, $\frac{1}{2}\Delta-P$, and those of pions $\frac{1}{2}\Delta+K$, $\frac{1}{2}\Delta-K$; $T(\Delta,Q,P)$ is the pion-pion scattering amplitude off the mass shell. $t=-\frac{1}{4}\Delta^2$ is the usual (energy)^2 variable in this channel, and we have suppressed the isospin indices: all amplitudes refer to those of $T=1$ in the t channel. The final pions may be taken to be on the mass shell, so that $\Delta\cdot P=0$, $(\frac{1}{2}\Delta)^2+P^2=-\mu^2$.

We can decompose $I_{\mu\nu}$ and $T_{\mu\nu}$ just as in Eq. (7):

$$I_{\mu\nu}(\Delta,K,P)=P_\mu P_\nu I_1+\cdots. \quad (27)$$

We now extract the coefficients of $P_\mu P_\nu$ from Eq. (26). This can be achieved most readily in a special frame of reference. We choose the frame in which $\Delta=(0,\sqrt{t})$ and $K=(k\hat{e}_3,k_0)$ and the vector $\mathbf{P}$ lies in the 31-plane. In this frame we write $Q=(\mathbf{q},w)$. Then

$$A_1(\Delta,K,P)=I_1(\Delta,K,P)+\frac{1}{(2\pi)}\int d^4Q\left(\frac{q}{p}\right)^2 e^{2i\phi}\frac{\sin^2\theta}{\sin^2\alpha} \times\frac{I_1(\Delta,K,Q)T(\Delta,Q,P)}{[(\frac{1}{2}\Delta+Q)^2+\mu^2-i\epsilon][(\frac{1}{2}\Delta-Q)^2+\mu^2-i\epsilon]}, \quad (28)$$

where θ and ϕ are the polar and azimuthal angles of q, and α is the polar angle of $\mathbf{p}$. Let us assume that $I_1(\Delta,K,Q)$ has a spectral representation in $\hat{K}\cdot\hat{Q}=\cos\theta=z$

$$I_1(\Delta,K,Q)=\int\frac{dz'}{z'-z}\rho(z',t;k,k_0;q,w), \quad (29)$$

[13] The Bethe-Salpeter amplitude in ladder approximation is analytic in the entire J plane. See B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962); G. Tiktopoulos, *ibid.* **133**, B1231 (1964). We assume here the full amplitude is analytic for $\mathrm{Re}J>1-\epsilon$. Proof for this is lacking, but so is a counter example. We ignore the probable inconsistency of cubic-interaction field theories.

[14] We shall deal with the mass-renormalized, but otherwise unrenormalized version of the theory.

so that I_1 may be expanded as

$$I_1=\sum_{J=2}^{\infty}(2J+1)I^J(t;k,k_0;q,w)\frac{d_{02}{}^J(\theta)}{\sin^2\theta}[(J-1)J(J+1)(J+2)]^{1/2}, \tag{30}$$

with

$$I^J(t,\cdots)=\frac{1}{\pi}\int dz'\rho(z',t,\cdots)\frac{(2J+3)Q_{J-2}(z')-2(2J+1)Q_J(z')+(2J-1)Q_{J+2}(z')}{(2J-1)(2J+1)(2J+3)}. \tag{31}$$

We define the partial-wave $\pi\pi$ amplitude:

$$T_J(t;q,w;p,0)=\frac{1}{2}\int_{-1}^{1}dz\,P_J(z)T(\Delta,Q,P), \tag{32}$$

where we have written $P=(\mathbf{p},0)$. Inserting Eqs. (11), (30), and (32) in Eq. (28), we finally obtain

$$\begin{aligned}F_J(t;k,k_0;\cdots)&=I^J(t;k,k_0;\cdots)\\&\quad+2\int_{\infty}^{\infty}dw\int_0^{\infty}q^2dq\left(\frac{q}{p}\right)^2 I^J(t;k,k_0;q,w)\\&\quad\times F^{-1}(q,w,t)T_J(t;q,w,\cdots),\\F(q,w,t)&=[q^2+\mu^2-(w+\tfrac{1}{2}\sqrt{t})^2-i\epsilon]\\&\quad\times[q^2+\mu^2-(w-\tfrac{1}{2}\sqrt{t})^2-i\epsilon].\end{aligned} \tag{33}$$

Equation (33) can be defined for complex J: the function I_J is defined by the Froissart-Gribov definition (31), and we use the usual definition of T_J for complex J.

In lowest order in the coupling constant λ, there are two graphs contributing to $I_{\mu\nu}$ [see Figs. 1(a) and (b)]. In second order, aside from self-energy and vertex correction graphs, there is one graph, Fig. 1(c). The lowest-order graphs give

$$I_1^{(0)}(\Delta,K,P)=-\frac{4}{(2\pi)^3}\frac{1}{2}\left[\frac{1}{(K-P)^2+\mu^2}-\frac{1}{(K+P)^2+\mu^2}\right], \tag{34}$$

so that

$$I^{J(0)}(t;k,k_0;q,w)=-\frac{4}{(2\pi)^3}\left(\frac{1}{2kq}\right)\left\{\frac{(2J+3)Q_{J-2}(x_+)-2(2J+1)Q_J(x_+)+(2J-1)Q_{J+2}(x_+)}{(2J-1)(2J+1)(2J+3)}+(x_+\leftrightarrow x_-)\right\}, \tag{35}$$

where $x_\pm=[k^2+q^2+\mu^2-i\epsilon-(k_0\mp w)^2]/2kq$. $I^{J(0)}$ is singular at $J=1$, since $Q_{J-2}(x)\to(J-1)^{-1}$ as $J\to 1$. The second-order graph, Fig. 1(c), gives $I^{J(2)}$ which is not singular at $J=1$. To see this we compute the high-s limit of Fig. 1(c). It is known that this diagram goes as $\ln^2 s/s^2$ as $s\to\infty$, so that $I^{J(2)}$ is in fact regular at $J=1$.[15] We conjecture that, in general, $I^{J(2n)}$ goes as $s^{-(n+1)}$, disregarding logarithmic factors. This means that the power-series development in λ^2 of the irreducible kernel is convergent as $s\to\infty$, and that the singular part of I^J at $J=1$ comes entirely from $I^{J(0)}$:

$$\lim_{J\to 1}I^J(t;k,k_0;q,w)=-\frac{4}{(2\pi)^3}\left(\frac{1}{2kq}\right)^2\frac{2}{3}\frac{1}{J-1}. \tag{36}$$

Therefore, it follows from Eq. (33) that

$$\begin{aligned}&-(2kp)\tfrac{3}{2}\lim_{J\to 1}(J-1)F_J(t,k,k_0;p,0)\\&=\frac{4}{(2\pi)^3}\left[1+2\int_{-\infty}^{\infty}dw\int_0^{\infty}q^2dq\left(\frac{q}{p}\right)\times F^{-1}(q,w,t)T_1(t;q,w;p,0)\right].\end{aligned} \tag{37}$$

One recognizes immediately that the quantity in the square bracket on the right-hand side of Eq. (37) is exactly $F_\pi(t)$, which is in our theory[16]

$$P^\mu F_\pi(t)=P^\mu+\frac{1}{2\pi}\int d^4Q Q^\mu\times\frac{T(\Delta,Q,P)}{[(Q+\frac{1}{2}\Delta)^2+\mu^2-i\epsilon][(Q-\frac{1}{2}\Delta)^2+\mu^2-i\epsilon]}.$$

Thus, we have seen that F_J is singular at $J=1$ and Eq. (17) is verified.

Near $J=\alpha(t)$, $T_J(t,\cdots)$ behaves as[17]

$$T_J(t,\cdots)\simeq\frac{t(t,\cdots)}{J-\alpha(t)}. \tag{38}$$

Substituting Eq. (38) in Eq. (33), we see that

$$F_J(t;k,k_0;p,0)\underset{J\to\alpha(t)}{\sim}\frac{1}{(\alpha-1)[J-\alpha(t)]}\Gamma_\alpha(t;k,k_0;p,0) \tag{39}$$

[15] See, for example, P. Federbush and M. Grisaru, Ann. Phys. (N. Y.) 23, 262 (1963). The other second-order graphs, being vertex and self-mass insertions, go as $1/s^2$ as well, disregarding logarithmic factors.

[16] In accord with the remark made in Ref. 14, $F_\pi(t)$ here is the unrenormalized form factor and does not satisfy $F_\pi(0)=1$. The condition $F_\pi(0)$ is restored as a consequence of vertex and wave-function renormalizations.

[17] The form of T_J in Eq. (38) can be guessed at on general grounds and verified in a model. See B. W. Lee and R. F. Sawyer, Ref. 13.

and

$$\Gamma_\alpha(t;k,k_0;p,0)$$
$$=2(\alpha-1)\int_{-\infty}^{\infty}dw\int_0^{\infty}q^2dq\left(\frac{q}{p}\right)^2 I^\alpha(t;k,k_0;q,w)$$
$$\times F^{-1}(q,w,t)t(t;q,w;p,0). \quad (40)$$

Therefore, the pole at $J=\alpha(t)$ gives rise to the asymptotic form Eq. (20):

$$\lim_{s\to\infty}\mathrm{Im}A_1(s,t,\cdots)$$
$$=-(2\alpha+1)\frac{\Gamma(\alpha+\frac{1}{2})\pi}{(\sqrt{\pi})\Gamma(\alpha+1)}\alpha\Gamma_\alpha(t,\cdots)\left(\frac{s}{2kp}\right)^{\alpha-2}\tfrac{1}{2}.$$

As $t\to m_\rho^2$ such that $\alpha(m_\rho^2)=1$, we have

$$\lim_{t\to m_\rho^2}\frac{2}{\pi}\int_{s_0}^{\infty}ds'\,\mathrm{Im}A_1(s,t,\cdots)$$
$$=+\tfrac{3}{2}(2kp)\frac{1}{\alpha(t)-1}\Gamma_1(m_\rho^2;k,k_0;p,0).$$

Now as $\alpha\to 1$, I^α becomes singular, as in Eq. (36), so that Γ_1 is given by [see Eq. (40)]

$$\Gamma_1(t,\cdots)=-\frac{4}{(2\pi)^3}\frac{2}{3}\frac{1}{(2kp)}2\int_{\infty}^{\infty}dw\int_0^{\infty}q^2dq\left(\frac{q}{p}\right)$$
$$\times F^{-1}(q,w,t)t(t;q,w;p,0).$$

Hence

$$\lim_{t\to m_\rho^2}\frac{2}{\pi}\int_{s_0}^{\infty}ds'\,\mathrm{Im}A_1(s,t,\cdots)$$
$$=\frac{4}{(2\pi)^3}2\int_{\infty}^{\infty}dw\int_0^{\infty}q^2dq\left(\frac{q}{p}\right)$$
$$\times F^{-1}(q,w,t)\frac{t(t;q,w;p,0)}{1-\alpha(t)}. \quad (41)$$

The right-hand side is just $4(2\pi)^{-3}$ times $F_\pi(t)$ in the Regge-pole representation Eq. (31) of T_1 [see Eq. (37)], which should be a good approximation in the neighborhood of $t=m_\rho^2$.

In our argument, we have inferred the singularities of F_J in the complex J plane from those of I^J and T_J. It is possible that there are other singularities, arising from the q or w integration in Eq. (33). We have verified that no such singularities arise in the ladder approximation[3] of T_J, at least in the region $\mathrm{Re}J>1-\epsilon$, $\epsilon>0$.

The reason why the left-hand side of Eq. (8) does not depend on k_1^2, k_2^2 (or k, k_0) is now clear. It is because the residue of $(2pk)I^J$ at $J=1$ is independent of these quantities. In Eq. (2), the way in which the left-hand side becomes independent of k_1^2 and k_2^2 is somewhat

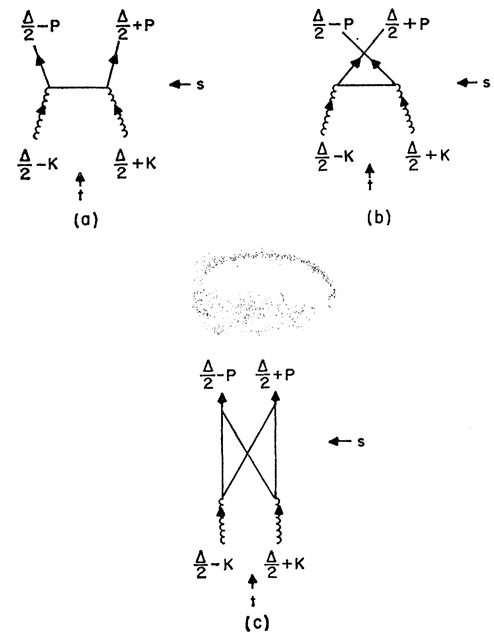


FIG. 1. Some lower-order Feynman diagrams for the irreducible kernel $I_{\mu\nu}(\Delta,K,P)$. Solid lines are pion lines; wavy lines, current.

hard to see, since the low-energy part and the high-energy part contribute equally to the integral.

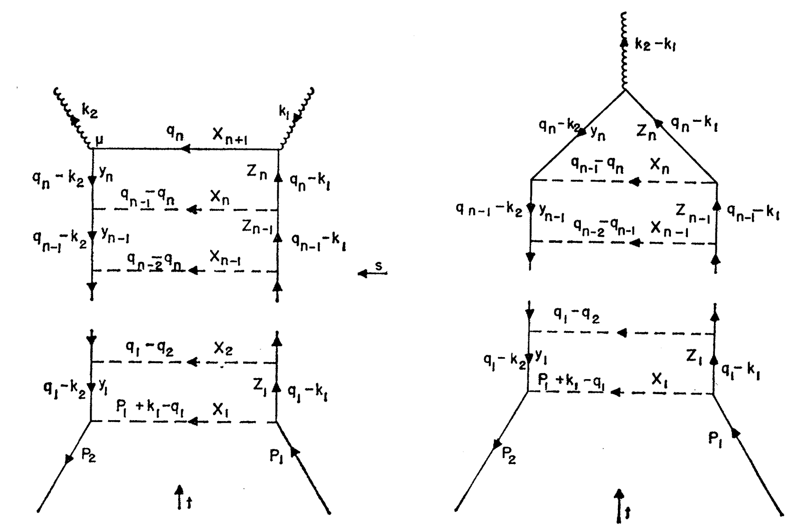
Lastly let us discuss why the amplitude $A_1(s,t,k_1^2k_2^2)$ has a fixed pole at $J=1$. Equation (28) shows that A_1 is *linear* in I_1, the irreducible kernel, which has this singularity. This situation arises because we are dealing with the current correlation function.[6] That is to say, we are considering the process: (two virtual photons) $\to$ (two pions), to *lowest* order in the photon-meson coupling. On the other hand, the amplitude for the process, [two physical vector particles (say, ρ-mesons)] $\to$ (two pions), cannot be linear in its Born approximation and cannot have a fixed pole at $J=1$. Let us consider this case more carefully. We shall denote the $\pi\pi$ channel and $\rho\rho$ channel[18] by $i=1$ and 2, respectively. The unitarity condition for complex J reads

$$t_{ij}(s,J)-t_{ij}^*(s,J^*)=2i\sum_{k=1,2,\cdots}t_{ik}(s,J)\rho_k^{(s)}t_{kj}(s,J^*), \quad (42)$$

where ρ_k is the appropriate phase-space factor of the kth channel. Assume that the amplitudes $t_{11}(s,J)$ are analytic in J, $J>1-\epsilon$. Equation (42) tells us that it is inconsistent for $t_{12}(s,t)$ to have a simple fixed pole at $J=1$. For, if it did, then t_{11} would have a double pole at $J=1$ and, using Eq. (42) again, we would conclude that t_{12} must have a triple pole. But for the first process the unitarity condition reads as

$$t_{0j}(s,J)-t_{0j}^*(s,J^*)=2i\sum_{k\neq 0}t_{0k}(s,J)\rho_k t_{kj}^*(s,J^*), \quad (43)$$

[18] Of helicity $\lambda-\lambda'=\pm 2$.



(a) (b)

FIG. 2. (a) The $(n+1)$-rung Feynman diagram for the amplitude $S_{\mu\nu}$. The solid lines are the pions, the dashed lines are scalar isoscalar mesons, and the wiggly lines represent the currents. The quantities x_i, y_i, z_i are the Feynman parameters introduced in Eq. (A1) so that the integrals over the internal momenta q_i may be performed. (b) The Feynman diagram for the vertex corresponding to Fig. 2(a). This diagram is not a function of k_1 and k_2 separately, but only of the difference k_2-k_1, as may be verified by a simple calculation. We have chosen labels for internal lines so that the comparison with Fig. 2(a) is simple.

where $i=0$ stands for the channel made up of two virtual isovector photons. Equation (43) is linear in the amplitudes t_{0k}, and it is entirely consistent for t_{0k} to have a simple fixed pole at $J=1$ if the t_{kj}'s $(k, j\neq 0)$ do not.

V. CONCLUDING REMARKS

A few remarks are in order:

(1) It is interesting that the non-Regge behavior we have found applies to an amplitude that is not directly measurable, i.e., scattering of charged photons. Our arguments specifically do not apply to the scattering of real photons, nor in their present form to the photoproduction of (e^+,ν) or (μ^+,ν), since in the latter case there are extra amplitudes present which cannot be analyzed in terms of two-body processes.

(2) The high-energy limit of the amplitude A_1 we have deduced here does not have any detectable consequence in high-energy scattering of strongly interacting particles. In the approximation of treating the ρ mesons as stable particles, the T matrix for $\pi\rho$ scattering is given, to within a well-defined multiplicative factor, by

$$\lim_{k_1^2,k_2^2\to -m_\rho^2} (k_1^2+m_\rho^2)(k_2^2+m_\rho^2)S_{\mu\nu}(P,K,\Delta)\,, \quad (44)$$

where $S_{\mu\nu}$ is defined in Eq. (5). The asymptotic behavior of $A_1(s,t,\cdots)$ we have deduced, i.e., $\sim F_\pi(t)/z$, gives no information for $\pi\rho$ scattering, as this leading term is independent of k_1^2 and k_2^2, and gives a vanishing contribution in Eq. (44).[19]

In the same vein, we caution against using a naive version of PCAC (partially conserved axial-vector current) in deducing high-energy pion-hadron scattering.[20,21] For, here again, the axial vector current correlation function has a quite different asymptotic behavior from that of the physical pion-hadron amplitude.

(3) While the current correlation function does not have quite the same asymptotic behavior as the T matrix of strongly interacting particles, our study (in Sec. III) indicates that if that T matrix satisfies an unsubtracted dispersion relation (by the Regge mechanism, for instance), then so does the current correlation function. This conclusion is in support of the independent observation by Dashen and Frautschi,[22] who have used it in establishing the algebraic structure of (bootstrapping) self-consistent currents.

APPENDIX: EXAMPLE IN PERTURBATION THEORY

In this appendix we will prove that the sum rule (2) is true, order by order, for ladder graphs in perturbation theory. This section is complementary to Sec. IV, since here we will work throughout in the physical region of the s channel, while in Sec. IV we worked in the physical region of the t channel.

For simplicity we shall consider a slightly different model from that in Sec. IV. We take a theory of pions (with mass μ) interacting strongly with scalar, isoscalar mesons (with mass m). We shall ignore the explicit complications of renormalizations; we have checked these and they are correct. We shall content ourselves with showing that A_1, computed from the Feynman

[19] Stated in the context of Eq. (2), this forms the basis of the superconvergent sum rules discussed by V. de Alfaro, S. Fubini, G. Furlan, and C. Rosetti, Phys. Letters **21**, 576 (1966).

[20] As opposed to low-energy scattering; as shown by S. Weinberg and others [see S. Weinberg, Phys. Rev. Letters **16**, 879 (1966)], the Adler-Weissberger relation is a low-energy theorem.

[21] We are indebted to Dr. H. T. Nieh for discussions on this point.

[22] R. F. Dashen and S. C. Frautschi, Phys. Rev. **145**, 1287 (1966).

diagram given in Fig. 2(a), when inserted in the left-hand side of Eq. (2), reproduces the functional dependence of $F_\pi(t)$, computed from the diagram given in Fig. 2(b).

We compute the scattering amplitude corresponding to the diagram of Fig. 2(a) by the usual techniques[23] and obtain

$$S_{\mu\nu}\sim(3n)!\int\frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_n}{(2\pi)^4}\int\prod_{i=1}^{n}(dx_idy_idz_i)dx_{n+1}$$
$$\times\delta\left(1-\sum_{i=1}^{n}(x_i+y_i+z_i)-x_{n+1}\right)$$
$$\times\frac{(2q_n-k_2)_\mu(2q_n-k_1)_\nu}{Q^{3n+1}},\quad\text{(A1)}$$

where[24]

$$Q=\sum_{i,j=1}^{n}\alpha_{ij}q_i\cdot q_j+2\sum_{i=1}^{n}k_i\cdot b_i+c,\quad\text{(A2)}$$

with

$$\alpha_{ii}=x_i+y_i+z_i+x_{i+1},$$
$$\alpha_{i,i+1}=\alpha_{i+1,i}=-x_{i+1},$$
$$\alpha_{ij}=0\quad\text{for}\quad j\neq i,\quad i\pm1,$$
$$b_1=-x_1(p_1+k_1)-k_2y_1-k_1z_1,$$
$$b_i=-k_2y_i-k_1z_i,\quad i\neq1,$$
$$c=(k_2^2+\mu^2)\sum_{i=1}^{n}y_i+(k_1^2+\mu^2)\sum_{i=1}^{n}z_i+m^2\sum_{i=1}^{n}x_i+sx_1.$$

The change of variables

$$q_i'=\sum_j P_{ij}q_j+R_i,$$

where

$$P_{ij}=\delta_{ij}+\delta_{i+1,j}\alpha_{i,i+1}\Delta_{i-1}/\Delta_i,$$
$$R_i=\lambda_i\Delta_{i-1}/\Delta_i,$$
$$\lambda_i=\sum_{j=1}^{i}b_{i-j+1}\left[\prod_{k=1}^{j-1}(-\alpha_{i-k,i-k+1})\Delta_{i-k+1}/\Delta_{i-k}\right],$$
$$\Delta_p=\det|\alpha_{ij}|,\quad i,j\leqslant p,$$

diagonalizes Q, and it is easy to see, after doing the loop integrations for the coefficient of $P_\mu P_\nu$, that A_1 is given by

$$A_1(s,t,\cdots)\sim3n!\int\prod_{i=1}^{n}(dx_idy_idz_i)dx_{n+1}$$
$$\times\delta\left(1-\sum_{i=1}^{n}(x_i+y_i+z_i)+x_{n+1}\right)$$
$$\times\left(\prod_{i=1}^{n}x_i\right)^2(\Delta_n)^{n-1}/D^{n+1}(s,t,\cdots).\quad\text{(A3)}$$

Here D is given by either of the two equivalent forms

$$D=\det\begin{vmatrix}\alpha_{ij}&b_j\\b_i&c\end{vmatrix},$$

or

$$D=c\Delta_n-\sum_{i=1}^{n}\left[\prod_{j\neq i}\Delta_j/\Delta_{j-1}\right]\lambda_i\cdot\lambda_i.$$

The denominator D contains the variable s linearly. The use of the second form of D, together with

$$b_1\cdot b_1=sx_1(x_1+y_1+z_1)+\cdots,$$
$$2b_1\cdot b_i=sx_1(y_i+z_i)+\cdots,$$
$$c=sx_1+\cdots,$$

and the identity

$$-\frac{1}{\Delta_i}\left\{\sum_{j=1}^{i}(y_j+z_j)\prod_{k=j}^{i}\frac{(-\alpha_{k,k+1})}{\Delta_k/\Delta_{k-1}}+x_1\frac{(-\alpha_1)(-\alpha_2)\cdots(-\alpha_{i-1,i})}{\Delta_{i-1}}\right\}=\frac{1}{\Delta_i}x_{i+1}-\frac{1}{\Delta_{i-1}},$$

yields

$$D(s,t,\cdots)=s\prod_{i=1}^{n+1}x_i+\cdots,\quad\text{(A4)}$$

where we have explicitly displayed all the s dependence of $D(s,t,\cdots)$.

A calculation of $F_\pi(t)$, the coefficient of P_μ from Fig. 2(a) yields

$$F_\pi(t)\sim(3n-1)!\int\prod_{i=1}^{n}(dx_idy_idz_i)$$
$$\times\delta\left(1-\sum_{i=1}^{n}(x_i+y_i+z_i)\right)$$
$$\times\left(\prod_{i=1}^{n}x_i\right)(\Delta_n')^{n-2}/D'^n(t\cdots),\quad\text{(A5)}$$

where the prime indicates that x_{n+1} is to be set equal to zero.[25]

[23] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Company, Inc., New York, 1965).

[24] In this section we leave the index μ off of four-vectors.

[25] It is easy to verify from Eq. (A5) that $F_\pi(t)$ is not a function of k_1^2 or k_2^2 separately but only of $t=(k_2-k_1)^2$.

We define a function $F(s,t,\cdots;\lambda)$ by

$$F(s,t,\cdots;\lambda)=3n!\int\prod_{i=1}^{n}(dx_i dy_i dz_i)dx_{n+1}$$
$$\times\delta\left(1-\sum_{i=1}^{n}(x_i+y_i+z_i)+x_{n+1}\right)$$
$$\times\left(\prod_{i=1}^{n}x_i\right)^2(\Delta_n)^{n-1}/[D(s,t,\cdots)+\lambda],\quad\text{(A6)}$$

so that

$$A_1(s,t,\cdots)\sim(-1)^n\frac{1}{n!}\frac{d^n}{d\lambda^n}F(s,t,\cdots;\lambda)\Big|_{\lambda=0}.\quad\text{(A7)}$$

A standard analysis[23] reveals that $A_1(s,t,\cdots)$ has a cut in the s variable starting at the normal threshold, $s_0=(nm+\mu)^2$, and from Eqs. (A7) and (A6)

$$\text{Im}A_1(s,t,\cdots)\sim(-1)^n\frac{1}{n!}\frac{d^n}{d\lambda^n}\text{Im}F(s,t,\cdots;\lambda)\Big|_{\lambda=0},$$

where

$$\text{Im}F(s,t,\cdots;\lambda)=3n!\int\prod_{i=1}^{n}(dx_i dy_i dz_i)dx_{n+1}$$
$$\times\delta\left(1-\sum_{i=1}^{n}(x_i+y_i+z_i)+x_{n+1}\right)$$
$$\times\left(\prod_{i=1}^{n}x_i\right)^2(\Delta_n)^{n-1}\delta(D(s,t,\cdots)+\lambda).\quad\text{(A8)}$$

Now

$$\int_{s_0}^{\infty}ds'\,\text{Im}A_1(s',t,\cdots)\sim(-1)^n\frac{1}{n!}\frac{d^{n-1}}{d\lambda^{n-1}}$$
$$\times3n!\int\prod_{i=1}^{n}(dx_i dy_i dz_i)dx_{n+1}$$
$$\times\delta\left(1-\sum_{i=1}^{n}(x_i+y_i+z_i)+x_{n+1}\right)$$
$$\times\left(\prod_{i=1}^{n}x_i\right)^2(\Delta_n)^{n-1}\int_{s_0}^{\infty}ds'\frac{1}{x_i}\frac{\partial}{\partial s'}$$
$$\times\delta\left(\prod_{i=1}^{n+1}x_i s'+\cdots+\lambda\right)\Big|_{\lambda=0},\quad\text{(A9)}$$

where we have used Eq. (A4), so

$$\int_{s_0}^{\infty}ds'A_1(s',t,\cdots)\sim(-1)^n\frac{1}{n!}\frac{d^{n-1}}{d\lambda^{n-1}}$$
$$\times3n!\int\prod_{i=1}^{n}(dx_i dy_i dz_i)dx_{n+1}$$
$$\times\delta\left(1-\sum_{i=1}^{n}(x_i+y_i+z_i)+x_{n+1}\right)$$
$$\times\frac{\left(\prod_{i=1}^{n}x_i\right)}{x_{n+1}}\Big/(\Delta_n)^{n-1}$$
$$\times\delta\left(\prod_{i=1}^{n+1}x_i s'+\cdots+\lambda\right)\Big|_{s_0}^{\infty}\Big|_{\lambda=0}.\quad\text{(A10)}$$

It is easy to verify that the right-hand side of Eq. (A10) vanishes at the normal threshold (this is because phase space vanishes at s_0), so that we may use the second delta function in Eq. (A10) to do the integration over x_{n+1}, setting $s'=\infty$ afterward. Since, from Eq. (A4),

$$x_{n+1}=D'\Big/s'\prod_{i=1}^{n}x_i+\cdots,$$

we obtain

$$\int_{s_0}^{\infty}ds'A_1(s',t,\cdots)\sim(-1)^n\frac{1}{n!}\frac{d^{n-1}}{d\lambda^{n-1}}$$
$$\times3n!\int\prod_{i=1}^{n}(dx_i dy\, dz_i)\delta\left(1-\sum_{i=1}^{n}(x_i+y_i+z_i)\right)$$
$$\times\left(\prod_{i=1}^{n}x_i\right)(\Delta_n')^{n-1}\frac{1}{D'(t\cdots)+\lambda}\Big|_{\lambda=0},\quad\text{(A11)}$$

which, after doing the λ differentiations, is identical to Eq. (A5). Thus, we see that doing the integration over s in Eq. (A11) has the effect of putting the Feynman parameter x_{n+1} equal to zero and turning Fig. 2(a) into Fig. 2(b).

Having established the validity of the sum rule for the n-loop ladder, it is clear that we can continue it, order by order, into the region of timelike t, since we have explicit expressions available for all the quantities involved and there is nothing to block the continuation.